THE CAC 40 IN 2015: THE SHADOW OF A QE BUBBLE

AN ANALYSIS OF THE DYNAMIC OF THE FRENCH STOCK MARKET AND OF THE EUROPEAN QUANTITATIVE EASING IMPACT ON IT

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ABSTRACT

We assess on the existence of a speculative bubble on the main French stock index, the CAC 40, and study the hypothesis that this bubble is caused by the Quantitative Easing policy of the European Central Bank. We first present the state of the art on the question of the bubble diagnosis, and then explain deeply our methodology based on the use of the Robust Log Periodic Power Law model estimated by the non-linear least squares (and a complementary study of the residuals). We estimate the model using two different methods, the Levenberg-Marquardt Algorithm and the Sequential Quadratic Programming Algorithm, and evaluate a possible Ornstein-Uhlenbeck structure in the residuals. This estimation leads to diagnose the probable presence of a bubble on the CAC 40. Hence we propose an estimation of the date of the bubble onset using an indicator that we designed for this purpose. The Consolidated Bubble Indicator measure the evolution through a rolling window of the relative goodness of fit of the Robust LPPL model, and of an alternative model. A very clear increase in this indicator seems to indicate that the bubble starts to grow together with the announcement of the QE. We continue our researches with the estimates of a simple linear regression and a non-parametric regression between the CBI and a proxy for the QE, the european government bonds spread between different maturities. We present our economic interpretation of these estimates. We conclude that the QE may have a mainly positive link with the probable bubble on the CAC 40, and present the limits and possible developments of our thesis.

SUMMARY

INTRODUCTION

The starting point of our thesis is a simple figure: 24%. This is the growth rate of the main French stock index, the CAC 40, in three months and half, between 2015/01/01 and 2015/04/15[1].

An index should represent the ability of the firms which composed it to generate value (and thus dividends) in the future. Then the following question can be asked: what can motivate such a dramatic increase in the index firm's prospects? On the considered period, there is three elements which are really impacting the economic context, the low oil prices, the euro depreciation and the quantitative easing announced the 2015/01/22 by the European Central Bank.[2]

Of course, the two first elements can well explain the change in expectations. Their impact for the European and thus the French economy is considered as huge by the consensus. But if we have a more precise look at the prices of these variables, they are not a satisfying explanation for the sudden rise of the CAC 40 since the beginning of 2015. This is shown by figure 1.

As we can see on this figure, the declining trends of the EUR/USD and the Crude Oil prices start well before the beginning of the CAC 40 rise. Thus, they can't be a satisfying explanation of the dramatic movement of the CAC 40 since the beginning of 2015. The same argument also holds for the low (and sometimes negative) interest rates.

Indeed, some can also argue that the low level of the interest rates have a positive impact on the economic environment. And even if it's true, the same answer that we proposed for the question of the price of the crude oil and the EUR/USD exchange rate remains can be opposed, knowing that the low interest rates are not anymore a new phenomenon in 2015.

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[1] (5,254.35-4,252.29)/4,252.29=0.2357. Source: Yahoo Finance, http://finance.yahoo.com/q/hp?s=%5EFCHI&a=00&b=1&c=2015&d=03&e=15&f=2015&g=d&z=66&y=0

[2] See the website of the ECB, http://www.ecb.europa.eu/press/tvservices/webcast/html/webcast_150122.en.html

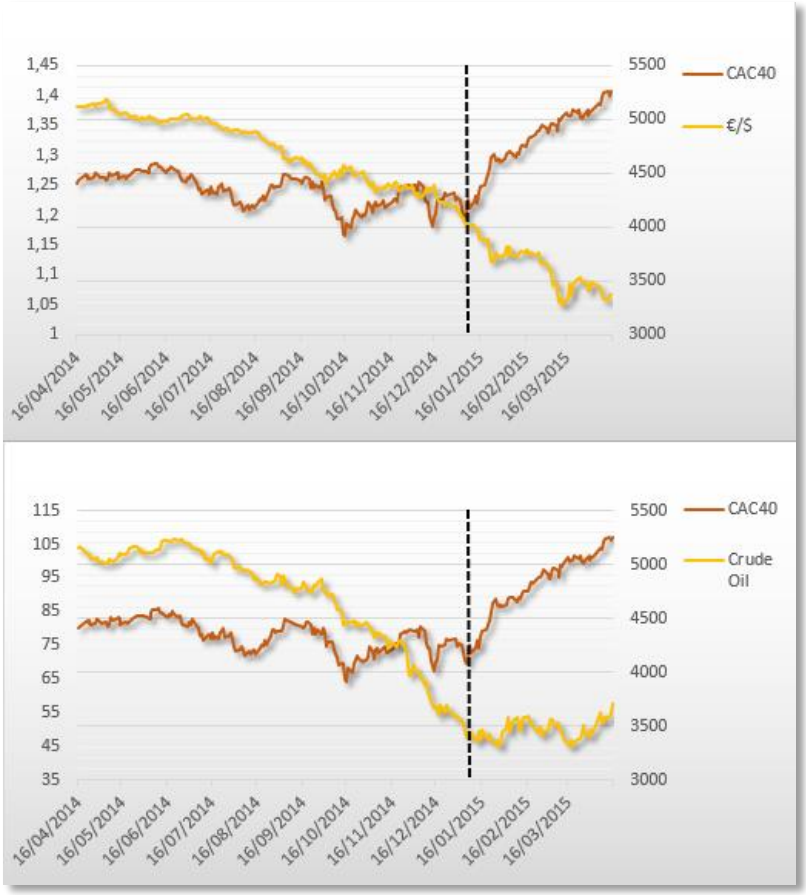

Figure 1 - Evolutions of CAC 40, Crude Oil prices and €/$ Exchange Rate[3]

Hence, in the absence of other explanation, the only justification which stays is the effect of the Quantitative Easing (QE) policy of the European Central Bank (ECB).

The QE announced by Mario Draghi, the ECB president, is basically an expensive monetary policy which consists to buy each month €60 billion of assets, mainly government bonds. Of course, the main objectives of this policy are to fight against the onset of the deflation in the euro area and to devaluate the European currency to restore the competitiveness (even if this second objective is not assumed). But there is also another goal for using this unconventional policy. It's well known that the banks are forced by the Basel III regulation to buy low risk assets, such as government bonds (but also other types of bonds, bills, notes, etc…). The vicious effect of these restrictions is that the banks doesn't play well

[3] Source of the data: http://fr.investing.com/

anymore their role of lender to the real economy, as the latter is risky. Hence, it's not incoherent to think that the QE is also designed to force the banks to take risks, as this is normally their role in the economy. In order to reach this objective, the ECB wants to drain the sovereign bond markets, and because it also offers negative interest rates on the deposits, the ECB try to eliminate all the riskless options of investment for the banks. And even if this is not the original purpose of the ECB, the banks are pushed to the stock market.

In the present thesis, we will try to prove that the QE generates this movement of the investments, from the bond to the stock market. But such a movement is interesting because of its consequences, and considering the figure that we evocate at the beginning of this introduction, we strongly think to a speculative bubble on French the stock market.

So, after finishing to outlining our reflections in this small introduction, we will discuss first of what is a bubble and how to detect it. After this review of the literature which will support the methodology that we will choose, we will lead an empirical analysis. Considering the results of our attempt to diagnose the bubble, we will try to determine its starting date, and then we will study directly the link between the quantitative easing and the bubble. The last part of this paper will be dedicated to conclude about the existence or the absence of a bubble caused by the QE on the CAC 40, and about the strengths and weaknesses of our analysis.

TOOLS FOR BUBBLE DIAGNOSIS

THE STANDARD APPROACH

In this section, we are going to present the different tools used to detect the financial bubbles which exist in the literature. But before to present how to detect it, a good first step is to define such a phenomenon. The classic definition of a bubble is that this is a situation when the market price of a financial asset is not anymore equal to the net present value of its cash flows. This is formalized by the following equation which became a central idea in the reflection about speculative bubbles with Blanchard and Watson (1982):

$$P_t^* = \sum_{i=0}^{\infty} \theta^{i+1} E(d_{t+i}|\Omega_t) \qquad with\ \theta = (1+r)^{-1}$$

Equation 1 - Market Fundamental Value

Where P is the market price, θ is the discount factor, d the dividends and Ω_t the set of all available information at date t.

Of course this equation was presented in the context of stock pricing but it can be generalized to every financial asset (or market index) just by replacing the dividends by the cash flows generated by the asset in question. In this approach, which is the standard one, one of the main hypothesis is the agent's rationality (we will consider hereafter that this hypothesis is fulfilled and thus we will restrict us to the idea of (at least partly) rational bubbles ; for the reader interested, some explanations about fully irrational bubbles can be found in Vissing-Jorgensen (2003)).

So, following the standard approach, a bubble is just a situation on the market when equation 1 is not fulfilled. In this situation the price of a stock is not (perfectly) link with the expected discounted dividends, and so is not link with the fundamentals of the underlying firm which determine the firm ability to generate value (and thus dividends). In such a case, this is the expected resale price which drive the market price of the stock more than the expected dividends.

Keeping in mind this definition based on equation 1, many tools have been used to detect these situations. To quote only some of them, we have the Variance Bounds Tests

(see Shiller (1981) and LeRoy and Porter (1981)), the West's test (1981) and the Integration/Cointegration based tests (e.g. Diba and Grossman (1988)). Unfortunately, all these tools have been unable to provide clear results about the existence of bubbles. Usually, these tools construct some expectations of the dividends based on the previous observations. It's problematic in the sense that if the data generating process changes, and it changes with the dividend policy of the firms, then the expectations are not valid anymore. Many other criticisms can be addressed to these tools (to see a complete inventory, see Gürkaynak (2005)). For example, these artificial expectations are only approximations of the fundamental value, which is not observable. Thus, a deviation of the price from the - artificially built - fundamental value can always be considered as an error in the estimation of such a value rather than a true deviation, i.e. a bubble. This problem is well summarize by Gürkaynak (2005): *"econometric tests of asset price bubbles shows that, despite recent advances, econometric detection of asset price bubbles cannot be achieved with a satisfactory degree of certainty. For each paper that finds evidence of bubbles, there is another one that fits the data equally well without allowing for a bubble. We are still unable to distinguish bubbles from time-varying or regime switching fundamentals, while many small sample econometrics problems of bubble tests remain unresolved."*

PRESENTATION OF THE LPPL MODELS

In response to this impasse where the science was stuck, some theories were developed around the notion of Log Periodic Power Law (LPPL models). The foundation of these theories is the article by Johansen, Ledoit and Sornette (2000), which is the first to present an LPPL model (named JLS from the name of the authors). The idea of this model comes from physics, and it was originally designed to fit different processes which arrived to a critical point. The framework in which this theory emerges is well described in Sornette (2009), and in Geraskin and Fantazzini (2013). The main point of these theories is that in a situation where a bubble growth, the price index which is in question is driven by a power law decorated with log periodic oscillations. This is formalized by the following equation (in discrete time):

$$log\left[\frac{p_c}{p(t)}\right] \approx \frac{(t_c - t)^{\beta}}{\sqrt{1 + \left(\frac{t_c - t}{\Delta_t}\right)^{2\beta}}}\left\{B_0 + B_1 cos\left[\omega \log(t_c - t) + \frac{\Delta_\omega}{2\beta} log\left(1 + \left(\frac{t_c - t}{\Delta_t}\right)^{2\beta}\right) + \phi\right]\right\}$$

$$\equiv \ln(p_t) = A + B(t_c - t)^m + C(t_c - t)^m \cos(\omega\, ln(t_c - t) - \phi)$$

Equation 2 - The original LPPL model (JLS)

The first part of the equation describe the power law (which represent the general trend of the price, i.e. the rational expectation). This power law is a pattern which usually well describe the movement of the prices before that large losses occurs (i.e. just before a bubble burst). Johansen and Sornette (2006) found that the probability distribution of these extreme events, called dragon-kings (or meaningful outliers), is different of the probability distribution of 99% of the returns. Hence, it's relevant to consider that a particular process can drive the prices in such an extreme context. In economic terms, the existence of this power law is well justify, because the burst of the bubble is stochastic, so the more we are close in time of the –uncertain- burst of the bubble, the more the risk is high and then well remunerated by the market. Simple reasoning but efficient representation.

The second part of equation 2 model the log periodic oscillations of the model. These observations are called log-periodic because there is a periodicity in the logarithm of the variable (t_c-t). From an economic point of view, this is less obvious that such a structure is justified. Sornette (2009) and Geraskin and Fantazzini (2013) suggested an interpretation based on the fact that there could exists some self-organizing and hierarchical behavior of the (noise) traders which leads to larger movements of the price, and this because of the herding behavior that such a structure of interactions among traders implies.

In equation 2, the parameters can be describe has follows:

- A is the expected value of the natural logarithm of the price at the critical time,
- B is the increase in the natural logarithm over the time unit before the crash,
- C is the proportionnal magnitude of the oscillations around the exponential growth,
- m is a parameter which describe the decrease in the oscillations,
- t_c is the expected time of the end of the bubble,
- ω is the frequency of the oscillations during the bubble,
- and ϕ is a (meaningless) phase parameter.

The LPPL models have been well supported by the Financial Bubble Experiment, which was made using LPPL models to real-time diagnose financial bubble and predict the moment when they will disappear[4] (see Sornette & al, 2010a, 2010b and Woodard, Sornette and Fedorovsky (2011)). The results were amazingly goods, this is this efficiency of the model which explains why the class of LPPL model is nowadays the most used to fit or diagnose bubbles.

CRITICISMS AND IMPROVMENTS OF THE LPPL MODELS

Of course, some criticisms have been addressed to the JLS model. For example, Chang and Feigenbaum (2006), found that using a modified LPPL model applied to the returns of a time serie gives quite weak results. They found that the LPPL model was outperformed by the Black-Scholes model for their analysis of the returns, in terms of marginal likelihoods. This finding, coming from a Bayesian approach, was questioning the JLS model.

An appropriate answer have been proposed by Lin, Ren and Sornette (2009). They incorporated in the original LPPL model a mean reverting component (via an Ornstein-Uhlenbeck structure for the residuals), accompanied of a stochastic conditional return. If the first new component of the model clearly respond to the critic of Chang and Feigenbaum (2006), the second add a realistic feature because it allows for the continuous update of the investors beliefs.

This new version of the JLS model have been called the "volatility-confined LPPL model". Even if for some aspects of the original LPPL model it represents some significant improvements, this model with mean-reverting residuals have been criticized by Liberatore (2011), who concludes *"In short, due to noise, there is no trace of LPPL during bubbles"*.

Of course each model stays an approximation of the reality and have its own defaults, and it would be unconscious to think that a model can gives perfect results which can't be questioned. But the results based on the JLS model presented in the Financial Bubble

[4] A bubble don't always burst, the end of the bubble can also be a stabilization of the price, the LPPL model is yet unable to predict what will happen after the end of the bubble.

Experiment I, II and III are excellent , and thus we consider that, while being conscious of its limits, we can use this tool to obtain correct results in our diagnosis of the CAC 40 bubble.

Furthermore, the LPPL fitting as a diagnosis can be completed by the tests proposed by Lin, Ren and Sornette (2009). This tests have been successfully used by Geraskin and Fantazzini (2013) and by Jiang and al (2010). They mainly consists in unit-root tests on the residuals of the LPPL model, such as the Augmented Dickey Fuller test and the Philippe-Perron test. The goal is to show that the residuals are stationary, and thus able to follow an AR(1) process which is typical of the supposed Ornstein-Uhlenbeck structure for the residuals. Hence, discovering stationarity in the residuals or directly obtain a correct fit with an AR(1) modelling is a strong argument to confirm the presence of a bubble after a LPPL estimation.

METHODOLOGY FOR DIAGNOSIS

OPTIMIZATION WITH SLAVED PARAMETERS

As we saw in the previous part, the LPPL structure provides a good fit of the prices when a bubble is present. Our approach will be to try to fit the CAC 40 index on the period where we suspect a bubble, from January, 1st to May, 19th 2015. We will use daily data of the closing price. The explanatory variable, namely the time, will be an index starting from 1 and going to 95 as there is 95 observations (i.e. trading days) over the period. All the data is extracted from the website http://fr.investing.com/ that we thank in this occasion.

In order to fit the CAC 40, we are going to use the robust version of the JLS model which is presented in Filimonov and Sornette (2013). The main contribution of this paper is to reduce the number of non-linear parameters. Their starting point is the model presented in equation 2. In this equation, we have 3 linear parameters, A, B and C, and 4 non-linear parameters, m, t_c, ω and ϕ. The estimation method used is the non-linear least square method, which proceed by searching the estimated parameters which minimize the sum of squared residuals (thanks to the Levenberg–Marquardt algorithm in the original article). Hence, we have to minimize the following equation:

$$S(t_c, m, \omega, \phi, A, B, C) = \sum_{t=1}^{T} [\ln(p_t) - A - B(t_c - t)^m - C(t_c - t)^m \cos(\omega \ln(t_c - t) - \phi)]^2$$

Equation 3 - Sum of Squared Residuals of the JLS model

Where t is the time index and T is the maximum value of this index, 95 in our case. But we meet now the matter that an optimization problem in 7 dimensions is a bit hard to solve, that's why we slave the linear parameters to the non-linear ones. We use the following fact:

$$\min_{t_c, m, \omega, \phi, A, B, C} S(t_c, m, \omega, \phi, A, B, C) \equiv \min_{t_c, m, \omega, \phi} S_1(t_c, m, \omega, \phi)$$

$$where \qquad S_1(t_c, m, \omega, \phi) = \min_{A, B, C} S(t_c, m, \omega, \phi, A, B, C)$$

Equation 4 - Slaving of the linear parameters

So we can rewrite the original optimization problem as:

$$\{\hat{A}, \hat{B}, \hat{C}\} = \underset{A,B,C}{\operatorname{argmin}}\, S(t_c, m, \omega, \phi, A, B, C) = \underset{A,B,C}{\operatorname{argmin}} \sum_{t=1}^{T} [y_t - A - Bf_t - Cg_t)]^2$$

$$where\ y_t = \ln(p_t), \qquad f_t = (t_c - t)^m, \qquad and\ \ g_t = (t_c - t)^m \cos(\omega \ln(t_c - t) - \phi)$$

Equation 5 - Modified optimization problem with 3 slaved parameters

This modified version of the original problem is very interesting for us, because knowing that the variables A,B,C, are linear, with the conditions m≠0, ω≠0, and t_c > T, our problem has a unique solution which is obtained from the first order condition.

The first order conditions provide us the following matrix:

$$\begin{pmatrix} T & \Sigma f_t & \Sigma g_t \\ \Sigma f_t & \Sigma {f_t}^2 & \Sigma f_t g_t \\ \Sigma g_t & \Sigma f_t g_t & \Sigma {g_t}^2 \end{pmatrix} \begin{pmatrix} \hat{A} \\ \hat{B} \\ \hat{C} \end{pmatrix} = \begin{pmatrix} \Sigma y_t \\ \Sigma y_t f_t \\ \Sigma y_t g_t \end{pmatrix}$$

Equation 6 – Matrix of the solutions for 3 slaved parameters

And we can solve the underlying linear system in the general case to have A, B and C expressed as function of the non-linear parameters.

THE ROBUST LPPL MODEL

The matter that we meet now is that an optimization problem with 4 parameters is still difficult to solve. The most famous methodologies to deal with this issue are the Taboo Search or the Genetic Algorithm. But, as it is emphasize by Filimonov and Sornette (2013), the algorithm is very likely to be trapped in a local minima, because there exists plenty of them. They propose the following trick, based on an expansion of the cosine term:

$$\ln(p_t) = A + B(t_c - t)^m + C(t_c - t)^m \cos(\omega \ln(t_c - t) - \phi)$$

$$\Leftrightarrow \ln(p_t) = A + B(t_c - t)^m + C_1(t_c - t)^m \cos(\omega \ln(t_c - t)) + C_2(t_c - t)^m \sin(\omega \ln(t_c - t))$$

$$where\ C_1 = C.\cos(\phi), \ \ and\ C_2 = C.\sin(\phi)$$

Equation 7 - Robust LPPL model

Of course, following the same methodology as before, this new model leads to a simpler optimization problem, as there is only 3 non-linear parameters which left to be optimized. The solution of the first order condition is now given by the new matrix:

$$\begin{pmatrix} T & \Sigma f_t & \Sigma g_t & \Sigma h_t \\ \Sigma f_t & \Sigma {f_t}^2 & \Sigma f_t g_t & \Sigma f_t h_t \\ \Sigma g_t & \Sigma f_t g_t & \Sigma {g_t}^2 & \Sigma g_t h_t \\ \Sigma h_t & \Sigma f_t h_t & \Sigma g_t h_t & \Sigma h_t^2 \end{pmatrix} \begin{pmatrix} \hat{A} \\ \hat{B} \\ \widehat{C_1} \\ \widehat{C_2} \end{pmatrix} = \begin{pmatrix} \Sigma y_t \\ \Sigma y_t f_t \\ \Sigma y_t g_t \\ \Sigma y_t h_t \end{pmatrix}$$

$$where\ \mathrm{y}_t = \ln(p_t)\,, \qquad f_t = (t_c - t)^m, \qquad g_t = (t_c - t)^m \cos(\omega \ln(t_c - t)),$$

$$and \quad h_t = (t_c - t)^m \sin(\omega \ln(t_c - t))$$

Equation 8 - Matrix of the solutions for 4 slaved parameters

And obviously, this robust definition of the LPPL model leads to the following cost function:

$$S(t_c, m, \omega, A, B, C_1, C_2)$$
$$= \sum_{t=1}^{T} [\ln(p_t) - A - B(t_c - t)^m - C_1(t_c - t)^m \cos(\omega \ln(t_c - t))$$
$$- C_2(t_c - t)^m \sin(\omega \ln(t_c - t))]^2$$

Equation 9 - Sum of Squared Residuals of the Robust LPPL model

The solution of the matrix given in equation 8 is not simple. We can invert the matrix, but it implies some constraints on A, B, C_1 and C_2 for the determinant of the matrix to be different of 0. Another option is to rewrite this equation as a linear system of 4 equations with 4 unknowns. Because no one of the values inside of equation 8 is known *a priori*, we have to find the general solution of the system. This operation have been done using a formal calculus software (Maxima). As the solutions obtained are a bit towering, they are presented in Annex 1.

Using these estimations of our 4 linear parameters, we get a new function for the sum of squared residuals which have only 3 parameters. In order to minimize this cost function with respect to these parameters, we use the Sequential Quadratic Programming method. This iterative method is well known to produce good results in the field of non-linear optimization. It is implemented in Matlab under the form of a minimization algorithm, which seems suitable for our problematic. The Matlab code used to get the results of the present thesis is available in Annex 4 (the code presented in this Annex generated most of the results presented in the present thesis).

SPECIFICATION CONSTRAINTS

According to the literature, and especially with Geraskin and Fantazzini (2013) and Filimonov and Sornette (2013), some constraints on the parameters must be set in order to have a well specified model. These constraints are the following:

$$A > 0, \quad B < 0, \quad |C| < 1, \quad 0.1 \leq m \leq 0.9, \quad 6 \leq \omega \leq 13, \quad 0 \leq \phi \leq 2\pi$$

Equation 10 - Constraints on the parameters of the JLS model

They can be rewrite for the Robust LPPL model as:

$$A > 0, \quad B < 0, \quad |C_1| < 1, \quad |C_1| < 1, \quad 0.1 \leq m \leq 0.9, \quad 6 \leq \omega \leq 13$$

Equation 11 - Constraints of the parameters of the Robust LPPL model

Of course, even if this is not explicitly written, t_c must be bigger than T if the bubble doesn't burst yet. So we will make sure that these constraints will be fulfilled for the estimated values of the parameters. If these constraints are not set at the beginning of the estimation process, we obtain estimates of an unrealistic model. For example, a negative estimated value for A implies a negative price at the date of termination of the bubble.

STUDY OF THE RESIDUALS

After this attempt to fit the CAC 40 index by the Robust LPPL model, we will conduct some assessment on our results. We will complete a qualitative assessment of the goodness of fit by a complementary test, such that the recommend tests on the residuals. Many articles settle for unit-root tests such as ADF and Philippe-Perron tests. The aim of these tests is to show that the residuals are stationary, and thus that an AR(1) structure (and so a Ornstein-Uhlenbeck process) for the residuals is possible. Hence, we will run ADF tests and KPSS tests.

We will go further in the evaluation of the AR(1) structure by analyzing the autocorrelogram and the partial autocorrelogram of the residuals and by estimating the AR(1) model whose quality will be evaluated by the coefficient of determination.

RESULTS OF THE DIAGNOSIS

ROBUST LPPL MODELLING

In spite of the claim of Filimonov and Sornette (2013), the optimization problems exhibit a lot of local minimum. We choose carefully the initial vector of parameters, which was [0.5;200;10] respectively for m, t_c and ω for the first guess. Using the SQP algorithm, we found correct values of the parameters, but we try to change the initial vector of parameters to check of their robustness. And the estimated values of the parameters changed a lot with the initial guess on the parameters. Because we still wanted to find the parameters which minimize the sum of squared residuals, we chose to explore the 3-dimensionnal space where our cost function is defined. For each initial values of the parameters, we report the sum of squared residuals (SSR).

In our approach we initially tried to use an absolute measure of the goodness of fit to evaluate the quality of our LPPL modelling. The coefficient of determination (R^2) well answered to this ambition, as it gives a measure included in the interval [0;1]. Of course we discovered that quickly that this measure is not suitable for non-linear models such as ours, but as it was a part of our research work, we chose to keep the measures obtained as witnesses of the inefficiency of the R^2. We obtained the results presented in the following table:

				m							
				0.1		*0.4*		*0.7*		*0.9*	
				SSR	R^2	SSR	R^2	SSR	R^2	SSR	R^2
tc	*120*	**ω**	*6*	0.018573	0.80402	0.018977	0.73155	0.018448	0.94282	0.018569	0.33148
			8	0.070461	0.79745	0.070461	0.79745	0.070461	0.79745	0.070461	0.79745
			10	0.070461	0.79745	0.070461	0.79745	0.070461	0.79745	0.070461	0.79745
			13	0.018958	0.72513	0.018804	0.90493	0.018581	0.18509	NA	NA
	250	**ω**	*6*	0.018237	0.67893	0.018375	0.90456	0.018437	0.87798	0.018431	0.91724
			8	0.018439	0.86532	0.018334	0.83566	0.018446	0.76248	0.01825	0.54685
			10	0.018433	0.90981	0.018437	0.878	0.018244	0.70869	0.018433	0.90944
			13	0.018439	0.85881	0.018427	0.93527	0.018419	0.95067	0.018429	0.92793
	500	**ω**	*6*	0.018262	0.65709	0.018252	0.67303	0.018246	0.68829	0.018243	0.69809
			8	0.018467	0.70368	0.018455	0.71694	0.018312	0.95123	0.018326	0.80953
			10	0.01875	0.77173	0.018314	0.94415	0.018289	0.6793	0.018243	0.69793
			13	0.018248	0.62108	0.018243	0.69768	0.018243	0.69752	0.018243	0.69768
	1000	**ω**	*6*	NA	NA	0.024761	0.92514	NA	NA	NA	NA
			8	NA	NA	0.018121	0.7624	0.018105	0.76831	**0.018104**	**0.77222**
			10	NA	NA	0.018162	0.57796	0.018158	0.59244	0.018159	0.60155
			13	0.018287	0.79826	0.018267	0.80395	0.018266	0.8096	0.018265	0.81334

Figure 2 - Table of the SSE for different values of the initial parameters

The values of the parameters are the initial values that we set. In black, we have the values for the sum of the squared residuals that we obtained, and in blue the values for the coefficients of determination. The couple in red correspond to the initial values of the parameters which gives the lowest SSR of the table. The values in white with a blue background are the values of the selected model. This table report a sample of the values tested for the initial parameters because this work of iterative research was done with a lot of different values.

This change in the estimation due to the initial parameters emphasize that our results are not really robust. But if our attempt to find manually a correct estimation cannot be completely trust, because of the very high number of iterations, it would lead to a correct estimation of the global minimum of the cost function.

These results are a bit surprising because we do not expect to converge to a lot of different local minima. Of course the fact that we have a lot of different values for the SSR implies a lot of different estimations of the parameters.

However, regarding to our initial objective, this is not a problem at all. Of course our estimation of the date of the end of the bubble, t_c, must be considered carefully. Of course, our estimated parameters would provide an uncertain forecast. But our aim in this part of our thesis is to diagnose the presence of a bubble on the CAC 40. Recall that the main idea is that if the data are well fit by a LPPL model it indicates the presence of a bubble, these results are good. This is the case because, for many different local minima, which imply very different estimated parameters, we obtain a correct fit.

Thus, if the values of the parameters are not so important to obtain an acceptable fit of the data, this is really the structure of the LPPL model which allows to obtain such a good fit. These results are thus strongly support the hypothesis of a bubble.

Because we need to keep a model, we choose the one which have the lowest SSR. Hence, we obtained the following estimations of our parameters:

$$A = 8.2181, \quad B = -1.5648e - 005, \quad C_1 = -0.0028113,$$

$$C_2 = 1.7114e - 006, \qquad m = 0.78888, \qquad \omega = 6.7722, \quad t_c = 500.07$$

The plot of the Robust LPPL estimation of the log of the CAC 40 using the SQP Algorithm is available in Annex 2.

In order to be sure that our results are correct, we also run an estimation using the Levenberg–Marquardt algorithm. Our results were really close to the previous ones for each parameters (and they also fulfilled all the constraints):

$$A' = 8.2151, \quad B' = -1.9351e - 007, \quad C_1{}' = -0.0028216,$$

$$C_2{}' = -2.0049e - 008, \qquad m' = 0.79917, \qquad \omega' = 6.8656, \quad t_c{}' = 460.88$$

The Levenberg-Marquardt algorithm does not represent a significant improvement. Moreover, it confirms what we saw using the SQP algorithm, because the two parameters m and ω are robust to the change in the initial vector of parameters.

The initial value of t_c is very important (if it is not chosen carefully the algorithm get trapped in a lot of local minima). For initial values of t_c over 460, we always converge to this estimated value which seems to be a global minimizer of the cost function (this is not the case for initial values below 460, which goes to many different estimations of t_c).

This estimated value of the termination date would imply that the burst of the bubble will occur around the autumn 2016.

The Levenberg-Marquardt algorithm does not gives a really better fit. Hence we will continue to work with the results of our previous estimation. As you can see on the following graph, the two estimations gives very close fits which are similar. The plot of the Robust LPPL

estimation of the log of the CAC 40 using the Levenberg–Marquardt Algorithm alone is available in Annex 3.

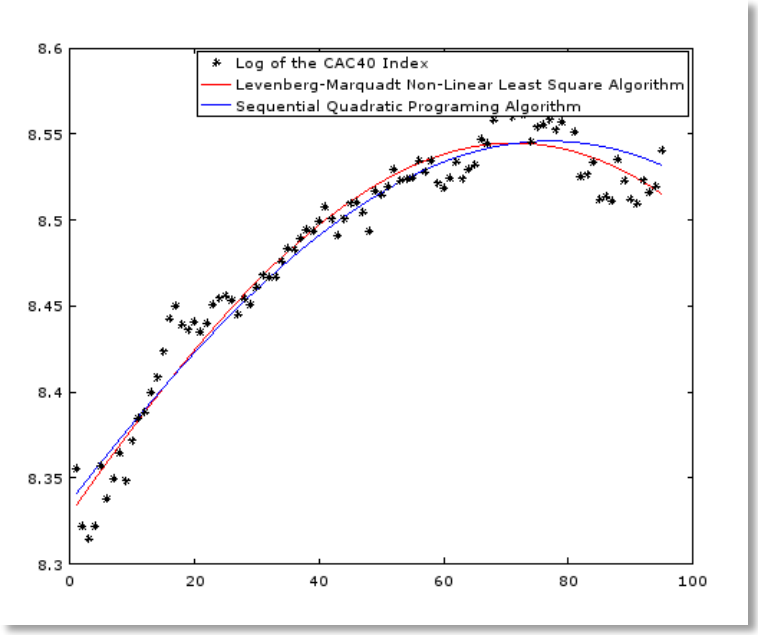

Figure 3 - Comparison between the fit of the log of the CAC 40 by the two algorithms

STUDY OF THE RESIDUALS

As it was explained in the methodological part, we want to go further and we will try to assess the possibility of the presence of an Ornstein-Uhlenbeck process in the residuals. In order to reach that goal, we launched some unit root tests on the residuals of the LPPL model, the ADF test and the KPSS test. Below are presented the results obtained in OxMetrics:

```
TESTS :
=======
-----------------------------------------------

KPSS Test without trend and 2 lags
H0: LPPL_res is I(0)

KPSS Statistics: 0.268514
Asymptotic critical values of Kwiatkowski et al. (1992), JoE, 54,1, p. 159-178

   1%   5%   10%
0.739 0.463 0.347
```

Figure 4 - Results of the KPSS test on the LPPL residuals, without trend

```
TESTS :
=======
-----------------------------------------------

ADF Test with 2 lags
No intercept and no time trend
H0: LPPL_res is I(1)

ADF Statistics: -2.89882

Asymptotic critical values, Davidson, R. and MacKinnon, J. (1993)

     1%       5%       10%
 -2.56572  -1.94093  -1.61663

OLS Results
              Coefficient    t-value
y_1             -0.215220   -2.8988
dy_1            -0.109209   -1.0125
dy_2            -0.118936   -1.2047
RSS             0.008312
OBS             92.000000
Information Criteria (to be minimized)
Akaike       -6.408246  Shibata       -6.410284
Schwarz      -6.326014  Hannan-Quinn   -6.375056
---------------
```

Figure 5 - Results of the ADF test on the LPPL residuals, without trend

In the ADF test, we accept at 99% the hypothesis that our residuals are integrated of order 1, and in the KPSS test we are unable to accept the stationarity even at a 90% confidence interval. So, both tests exhibit the same results of non-stationarity.

But the ADF is known to have a low power when the data generating process is an AR(1) process with a coefficient close to one. This technical point is very important as it can discredits the result of the test. Hence, we will carefully not conclude directly about the stationarity and run the tests again to see if the tests could show that the residuals are trend stationary. We obtained the following results:

```
TESTS :
=======
-----------------------------------------------

KPSS Test with a trend and 2 lags
H0: LPPL_res is I(0)

KPSS Statistics: 0.172814
Asymptotic critical values of Kwiatkowski et al. (1992), JoE, 54,1, p. 159-178

  1%   5%   10%
0.216 0.146 0.119
```

Figure 6 - Results of the KPSS test on the LPPL residuals, with a trend

```
TESTS :
=======
-----------------------------------------------

ADF Test with 2 lags
Intercept and time trend
H0: LPPL_res is I(1)

ADF Statistics: -3.05889

Asymptotic critical values, Davidson, R. and MacKinnon, J. (1993)

     1%      5%      10%
 -3.96104  -3.41127  -3.12748

OLS Results
            Coefficient   t-value
y_1           -0.230979   -3.0589
dy_1          -0.109486   -1.0130
dy_2          -0.112695   -1.1377
Constant       0.000429   0.42487
Trend         -0.000046   -1.1996
RSS           0.008158
OBS           92.000000
Information Criteria (to be minimized)
Akaike      -6.382431  Shibata      -6.387943
Schwarz     -6.245378  Hannan-Quinn   -6.327115
---------------
```

Figure 7 - Results of the ADF test on the LPPL residuals, with a trend

In these new tests, we accept the hypothesis of a trend stationary data generating process at a 95% confidence interval according to the KPSS test, and we reject the hypothesis of integration of order 1 thanks to the ADF test.

Considering these mixed results, it's difficult to decide on the presence or not of a unit root. If we are unable to tell if the Ornstein-Uhlenbeck process is feasible using a unit-root test, we can still directly estimate an AR(1) model and see how are looking the results. The unit-root will be directly shown by a coefficient AR(1) equal or very close to 1.

Before to do that, we can have a look at the autocorrelogram and the partial autocorrelogram of the residuals. The kind of pattern that we are looking for is a gradually cuts off of the tails in the autocorrelation function and a cut off after 1 lag in the partial autocorrelation function. This particular structure is typical of the AR(1) processes.

Done in Matlab (see Annex 4), the ACF is obtained by computation of the autocovariances, which are just divided by the variance to get the autocorrelation vector. The PACF is computed using a basic linear regression of the twenty firsts lags over the dependent variable Y_t (the residuals of the LPPL model). Here we use a 99% confidence

interval (based on a normal assumption). The plot presented on the next page contains both the ACF and the PACF.

We can directly notice that these graphs are absolutely not compatible with a unit root, because in such a case the first lag of the PACF reach almost 100% while the ACF is declining very slowly. Moreover, the structure of the autocorrelogram and the partial autocorrelogram clearly indicates an AR(1) process.

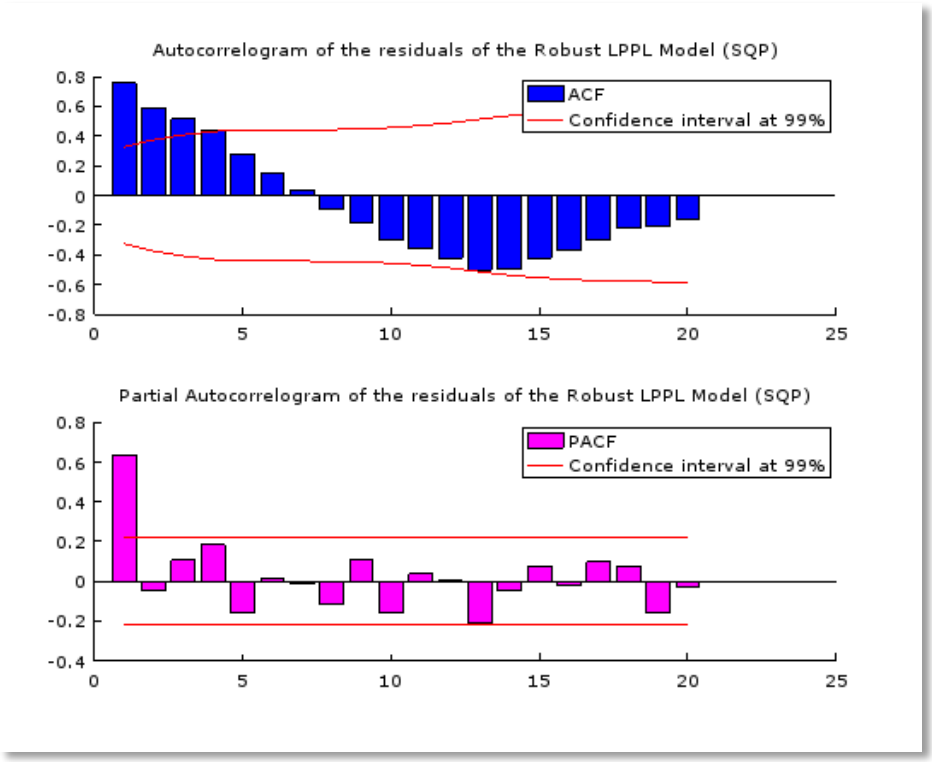

Figure 8 - ACF and PACF of the residuals of the LPPL model

Hence, the hypothesis of an AR(1) model seems now relevant, that's why we exploit it. We define our AR(1) model as follows:

$$Y_t = c + \varphi Y_{t-1} + \varepsilon_t$$

Equation 12 - AR(1) model for the residuals of the LPPL model

We can estimate our AR(1) model simply using the Yule-Walker equations. By solving these equations, we obtain the following estimated parameters:

- *Estimated autoregressive parameter:* $\varphi = \frac{\gamma_1}{\gamma_0} = 0.74041$
- *Estimated variance:* $Var(\varepsilon) = \gamma_0 * (1 - \varphi^2) = 1.0525e - 004$

- *Estimated constant* $c = \left(\frac{1}{T}\sum_{t=1}^{T} Y_t\right) * (1-\varphi) = -1.2154e-014$

We can notice here the very small value of the variance, which is a good clue of the quality of our model. Furthermore, the value of the coefficient of determination that we computed is 0.54794, which is a very acceptable level of goodness of fit.

A graphical representation of the residuals of the LPPL model and the associated AR(1) model confirm that the model described in equation 12 is well designed.

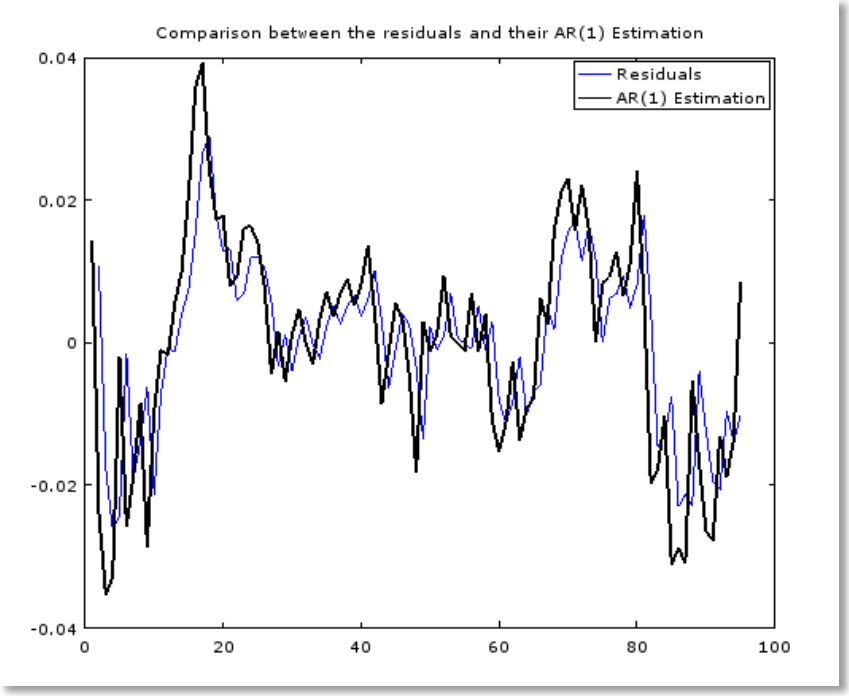

Figure 9 - Comparison between the residuals and their AR(1) Estimation

We obtained an excellent fit for the LPPL estimations using both the Sequential Quadratic Programming Algorithm and the Levenberg-Marquardt Non-Linear Least Square Algorithm. From our qualitative graphical evaluation, we conclude that the (Robust) LPPL model fits well the log of the CAC 40 index.

The unit root tests on the residuals of the Robust LPPL estimation obtained with the SQP algorithm doesn't gives trustworthy results. However, the analysis of the autocorrelogram and the partial autocorrelogram exhibits patterns which are not compatible with a unit root. The coefficient given by the partial autocorrelation function for the first lag is clearly different from 1. Nevertheless, these patterns are consistent with an AR(1) process. The estimation of this model leads to a coefficient of determination equal to

0.54794, which is a correct result. We can conclude that the test for the existence of an Ornstein-Uhlenbeck process in the residuals is positive.

Thus, we get positive signals from the two tools that we selected to assess on the existence of a bubble. In the limit of the quality of these tools, our diagnosis leads to the following conclusion: there is a speculative bubble on the main French stock index, the CAC 40, on the period from January the 1st to May, the 19th.

STARTING DATE OF THE BUBBLE

DEFINITION OF THE CONSOLIDATED BUBBLE INDICATOR

The diagnosis of the bubble have been a first step to assess our initial statement. But in order to confirm our hypothesis, we need to have a look to the link which could exists between the existence of the bubble and the quantitative easing policy. Hence, we chose to proceed in two steps: the first is to determine if the date of the onset of the bubble correspond to the announcement of the QE, the second is to evaluate the link between these two variables by running a regression.

Determining if the two dates match requires to know when the two phenomenon start. For the question of the quantitative easing, it's not difficult. The possible impact of the QE on the entire economy, and furthermore on the CAC 40, would start near to the announcement of the QE by the European Central Bank, the 2015/01/22. This date must be considered as more or less the beginning of the QE effects, but not precisely our starting point. Indeed, it seems not completely impossible that the decision about the QE have been taken a few days or weeks before the announcement, and thus that someone(s) had access to this information and used it in a way having a non-negligible impact on the French stock market. Of course, we are not interesting in proving an insider dealing, and there is no way to consider that the present work can argue in favor or in disfavor of this hypothesis, but as economists, we have to take into account that some possible behaviors could have important consequences on our work.

So, if we can have a (more or less) precise idea about the starting date of the QE, we still need to determine what is the starting date of the bubble. Hence, more than a qualitative analysis of the fit, we need a tool which indicates precisely if we are or not in a bubble regime. Thus, we decide to build a bubble indicator, based on the use of a rolling window. We proceed as follows: we use a new sample of data[5], between 2014/01/01 and 2015/06/08.

[5] As previously, these data were extracting from http://fr.investing.com/, they are the daily closing prices of the CAC40 between 2014/01/01 and 2015/06/08. Over this period, there were 364 trading days.

Using different widths, we launch an estimation of the Robust LPPL model over the different sub-samples of the rolling window. Each time we estimate the LPPL model, we also estimate an alternative model. Here, the point is to have a model which correspond to "normal conditions" on the stock market. Hence, we decide to use the random walk model, which represent the well-known efficiency hypothesis of the market. In normal time, we thus assume that the CAC 40 is driven by the following process:

$$\ln(p_t) = \ln(p_{t-1}) + \varepsilon_t$$

Equation 13 - Random Walk model for the rolling window

In this model, ε_t is assumed to be an identically and independently distributed white noise. Of course, the variable $\ln(p_t)$ used to build our model have not been randomly selected but comes from the fact that this is also the variable explained by the LPPL model.

After estimating these two models, we used some goodness of fit measures in order to assess the respective and relative quality of our two alternative models. If the coefficient of determination can't be used because of the non-linearity of the LPPL model, there is a lot of different measures which include advantages and drawbacks. Here, the selection of only one measure leads to an undesirable additional subjectivity for our bubble indicator. Hence, we chose to use five different measures of the goodness of fit of our models, which are the Mean Squared Errors (MSE), the Mean Absolute Percentage Errors (MAPE), the Akaike Information Criterion (AIC), the Bayesian Information Criterion (BIC), and the log-likelihood (LogL).

On the Bayesian and the Akaike criterions, we made a choice which can affect the original measures given by them. Their formulas are as follows:

$$\mathrm{AIC} = 2\mathrm{k} - 2\ln(\hat{L})$$

Equation 14 - Definition of the Akaike Information Criterion

$$\mathrm{BIC} = \mathrm{k} * \ln(\mathrm{N}) - 2\ln(\hat{L})$$

Equation 15 - Definition of the Bayesian Information Criterion

In these two equations, k is the number of free parameters to be estimated, N is the number of observations, and $\hat{L}$ is the value of the maximized likelihood. This last variable is the value of the likelihood obtained when the estimated parameters maximize it. In our case,

the models have been already estimated using other methods (the Robust LPPL model used the Levenberg-Marquardt algorithm and the Random Walk model doesn't need any estimation since we do not care about the variance in this analysis). Hence, we will not use the likelihood obtained from an estimation of the parameters based on the maximum likelihood method, but instead the likelihood of our estimations, the ones that we want to compare.

For each window, we compute the estimates of our two models and the five associated goodness of fit measures. Then, the bubble indicator is define as follow, for a given window "v" and a given goodness of fit measure "GoF":

$$BI_{v,GoF} = \frac{GoF_v^{RW}}{GoF_v^{RW} + GoF_v^{LPPL}}$$

Equation 16 - Bubble Indicator

Of course having five goodness of fit measures leads to five different bubble indicators. Thus, if the different aspects of the reality that they exhibit are interesting, we need to aggregate the results in only one indicator, the Consolidated Bubble Indicator (CBI):

$$CBI_v = \frac{(BI_{v,MSE} + BI_{v,MAPE} + 0.5 * BI_{v,AIC} + 0.5 * BI_{v,BIC} + BI_{v,LogL})}{4}$$

Equation 17 - Consolidated Bubble Indicator

As you can see in equation 17, the CBI is nothing more than the weighted average of each of our five indicators. The AIC and the BIC have a smaller weight, because they integrate another dimension than the goodness of fit. Indeed, they add a penalty for each parameter estimated, and we are not sure that such a penalty for the Robust LPPL model is relevant in our problem (in spite of the Occam's razor statement).

The last question which stay to solve is the definition of the size of the rolling window. Unfortunately there is no golden rule to define it and this width depends completely of the matter treated. Our approach will be to consider different widths in order to light the different aspects of the problem that the rolling window can explain. Hence, we will choose to consider the CBI with a size of the window of 120, 90, 75, 60 and 45 trading days. These values doesn't come from nowhere, as we estimate 3 non-linear parameters in the Robust LPPL model (the 4 linear parameters are not considered as estimated in the Levenberg-

Marquardt algorithm, because as long as they are slaved they don't enter into the optimization problem as minimizors), we need at least 3*15 observations to estimate these parameters. The upper value, 120, have been chosen because of an empirical finding: if we increase again the size of the window, we only obtain a straight line because of the smoothing effect of the width increase.

ESTIMATION OF THE CONSOLIDATED BUBBLE INDICATOR

After running our estimations with the five width for our rolling windows, we obtained the following results:

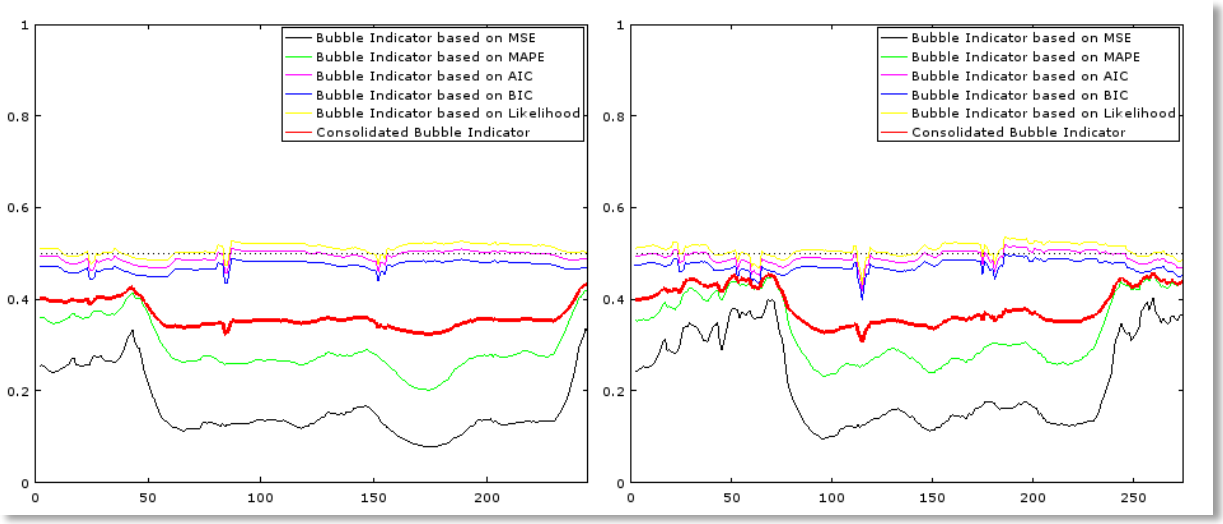

Figure 10 - Consolidated Bubble Indicator, for a width of 120 and 90 trading days

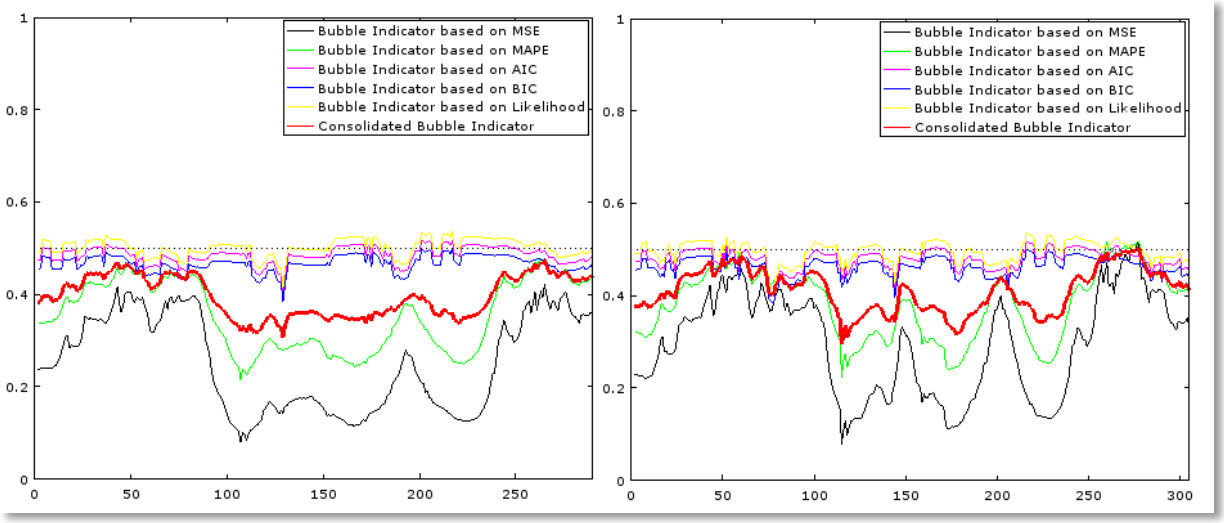

Figure 11 - Consolidated Bubble Indicator, for a width of 75 and 60 trading days

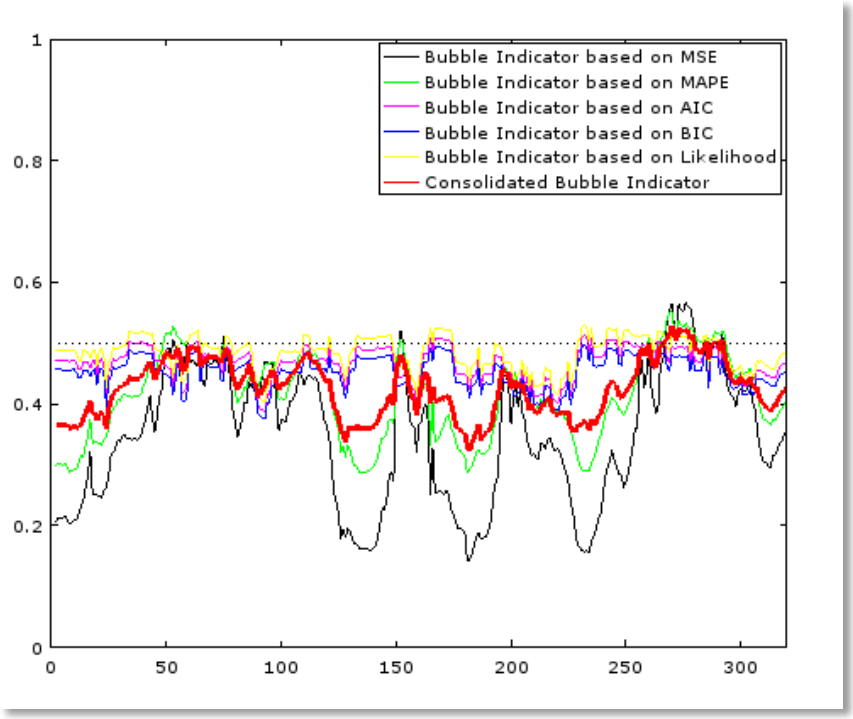

Figure 12 - Consolidated Bubble Indicator, for a width of 45 trading days

Considering these five plots, we can make several remarks. First of all, we see that the CBI, even if it's just an average, is really important to clarify what the five other curves express. We also see that we have similar movements in the two classes of goodness of fit measures: the ones based on the errors, and the ones based on likelihood.

Another more important finding is the shape of the curves with respect to the window width. As it was planned, the variations increase with the decrease in the size of the window. So, even if in most of the cases, the frontier to the bubble regime (the 0.5 line) is not crossed, we see that when we decrease the size of the window until 45 trading days, we obtain a switch to the bubble regime.

From an economic point of view, this is not clear that we are in a bubble regime since the beginning of the year regarding the results of the CBI, since the cross of the frontier is not frank on every graphs. However, this can be due to an important limit of our indicator: the definition of the random walk as the "normal time" regime.

Indeed the efficient market hypothesis of Fama is not completely inconsistent with the LPPL model, because the rational traders of the LPPL framework have access to all the information and make the market tending to its efficient level. Recalling that we obtained a clear signal of bubble in the beginning of the present thesis, the alternative model must provide very poor results on the period from 2015/01/01 to 2015/05/01. It would be the case if the alternative model, the random walk, is well selected.

And we get exactly the inverse situation, with a coefficient of determination equal to 0.96784 for our estimation of the random walk model on this period (note that as the random walk is a linear model, hence the R^2 is suitable to assess the quality of the model). The following figure also tends to confirm the random walk model.

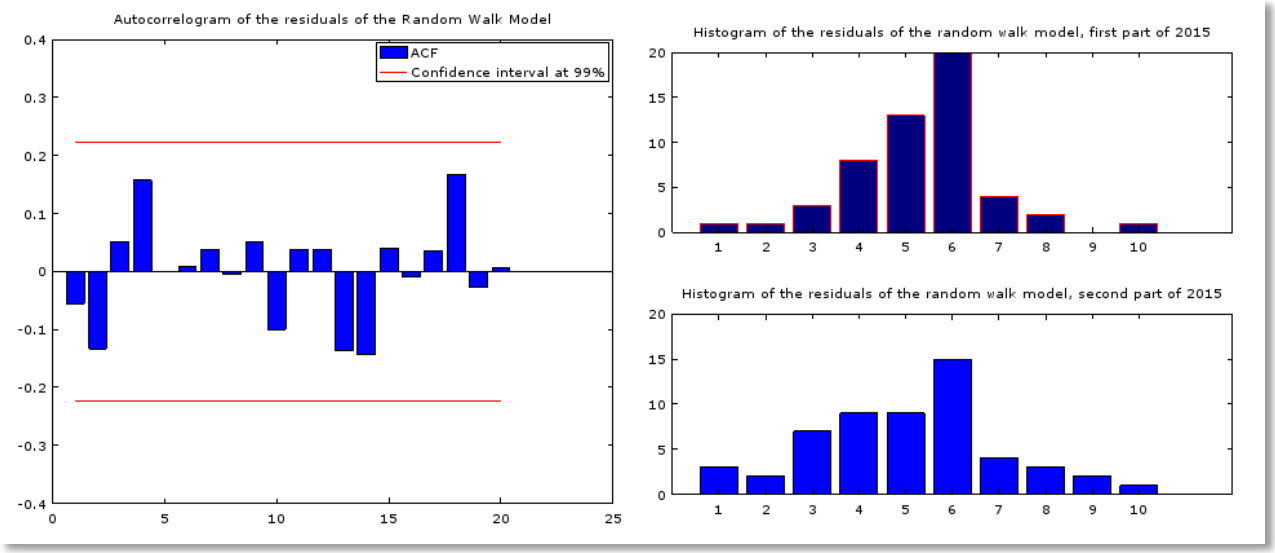

Figure 13 - Graphical assessment of the I.I.D assumption on the random walk residuals

More than having a correct goodness of fit, a model must respect the hypothesis on which it is build. The key assumption in the basic random walk model as it is define by equation 13 is that the error term is an identically and independently distributed white noise. Figure 13 show the ACF of the residuals on its left part, and indicates that there is no autocorrelation in the residuals (with a confidence interval at 99%). The histograms presented on the left are the histograms of the residuals for two periods. These two periods are the first part and the second part of our sample (2015/01/01 to 2015/05/01), which have been just divided in two equal parts. We can see that the distribution of the residuals doesn't

seems to change from one period to another. Finally, the mean of the residuals over the period is 0.0012324, a value very close to 0. These elements strongly suggests that the error term is effectively a white noise. All these findings gives credibility to the fact that the efficient market hypothesis (and thus the random walk model) and the presence of a speculative bubble (and thus the LPPL model) are compatible.

This is a strong limit for the Consolidated Bubble Indicator. To be perfect, the CBI must aggregate bubbles indicators using a No-Bubble-at-All model against the LPPL model, which left to be defined. The fact that the Fama efficiency could occur (at least in its semi-strong form) during a bubble period, and thus that the "normal time" model remains partially valid in the bubble period implies a down drift of the CBI. Even if it's not possible to quantify and correct it, it must be taken into account into our next reflexions. So if we consider this fact, the shy cross of the frontier by the CBI can be interpreted as a strong signal of switch to a bubble regime.

Moreover, in every situations, we note a remarkable increase of the CBI between the values 200 and 250 of our time index, which correspond to the dates 2014/10/13 and 2014/12/22. More precisely, the beginning of the increase which is well drawn in figure 10 is the point 232 (2014/11/26). The date at which we switch into a bubble regime according to figure 12 is 2015/01/19 (point 267 in the time index). Recalling that the announcement date of the QE is 2015/01/22, just 3 days after the switch into the bubble regime, we conclude that the onset of the bubble seems to happen (again, more or less) in the same period than the bubble on the CAC 40.

LINK BETWEEN THE QE AND THE BUBBLE

DEFINITION OF A VARIABLE AS A QE PROXY

The simplest way to decide on the existence of a link between the speculative bubble on the CAC 40 and the quantitative easing is to run a regression with some variables representative of these two phenomenon.

For the bubble, in absence of a better solution, the CBI can be used.

For the QE, of course the amount spent for buying government bonds is the variable which best represent the direct effect of this policy. But we also need to take into account the indirect effects due to the agents expectations. Hence, we need a variable which directly react to the announcements concerning the QE, and if possible which also captures the reactions to the purchases when they are done.

The literature use to consider the spread between the yields of government bonds with different maturities as a proxy for the quantitative easing. To justify this stance, the paper written by Joyce and al (2011) on the British quantitative easing is a good explanation. This kind of proxy for the QE have been widely used from this date, for example in Kapetanios and al (2012) and in Berkmen (2012), who follows the same approach to expatiate on the Japanese quantitative easing.

Thus we choose to follow this approach and we use the euro bonds spreads between the 3 months and the 10 year maturities provided by the European Central Bank[6]. Of course we use daily data on the same period as the one where we compute the CBI, from 2014/04/17 to 2015/06/08.

[6] Precise informations about the dataset, as it was provided by the ECB: "Dataset name: Financial market data - yield curve; Frequency: Business; Reference area: Euro area (changing composition); Currency: Euro; Financial market provider: ECB; Financial market instrument: Government bond, nominal, all issuers whose rating is triple A; Financial market provider identifier: Svensson model - continuous compounding - yield error minimisation". Name of the variable used, as it was provided by the ECB: "Yield curve spot rate, spread between the 10-year and 3-month maturity".

The next figure is the plot of this variable over the period from 2015/01/01 to 2015/06/08. As you can see, this variable seems to be sensitive to the QE.

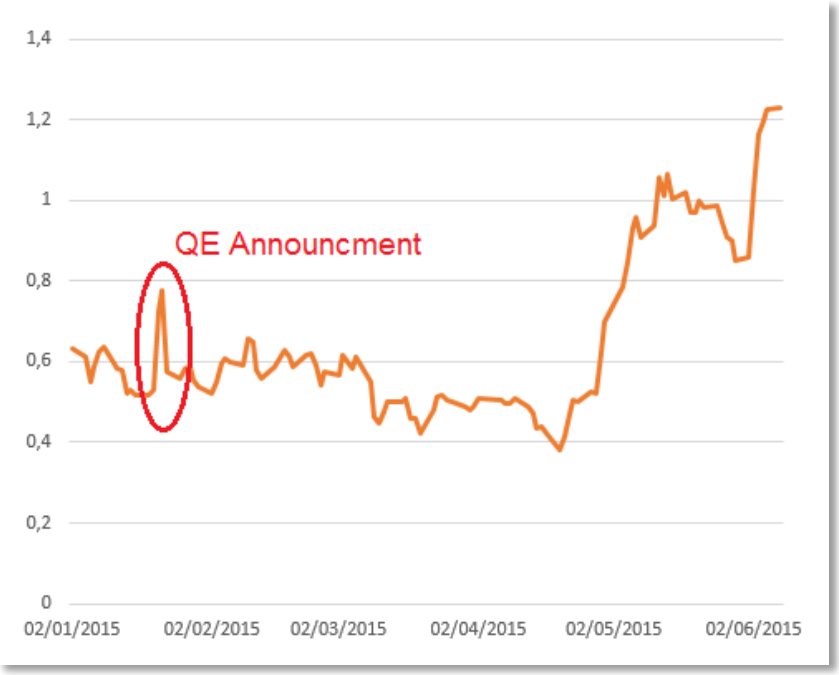

Figure 14 - Spread between the yield curves of euro bonds with maturities of 3 months and 10 years

SIMPLE LINEAR REGRESSION

We run a linear regression using the ordinary least squares, with our QE proxy as the explanatory variable. Our model is:

$$CBI = a + \beta * QE + \varepsilon$$

Equation 18 - Simple Linear Regression for the QE/CIB link

We obtained 0.362646 and 0.033354 as estimated values respectively for a and β. This estimated value for β indicates that it exists a positive linear link between the size of the spreads and the bubble on the CAC 40.

But, as it's well explained in Joyce and al (2011), the QE is supposed to have a negative impact on the spreads. It seems coherent, as an announcement of an increase in the QE reflect that all the prices of the government bonds will fall (if the government use this increase in the demand to do massive issuances of government bonds). The values of the long-term bonds must be more impacted as they are more sensitive to the prices

fluctuations. Thus, an increase in the purchases of the QE should generates a decrease of the spreads, which leads to – following our linear regression – a decrease of the bubble. More than odd, this result is unrealistic.

If this first finding using regressions is interesting, it remains that the use of a simple linear regression to decide on such a question is a half-baked method. An excellent argument to confirm that stance is the value of the coefficient of determination, which is only 0.070960 in our case. This poor value leads us to be septic on the conclusions that such an estimation can allow.

Of course we could have a look to the link between the CIB and the square of the spreads, to search for a quadratic relation for example. We could also assess on the stability of our estimation by adding some control variables, or by evaluating the stability of our previous estimations using a rolling window.

But it seems difficult to give a strong and robust interpretation of the link between the bubble and the QE using this naïve model. About that, the graph which represent our estimation (plotted over the values of the QE proxy) is eloquent.

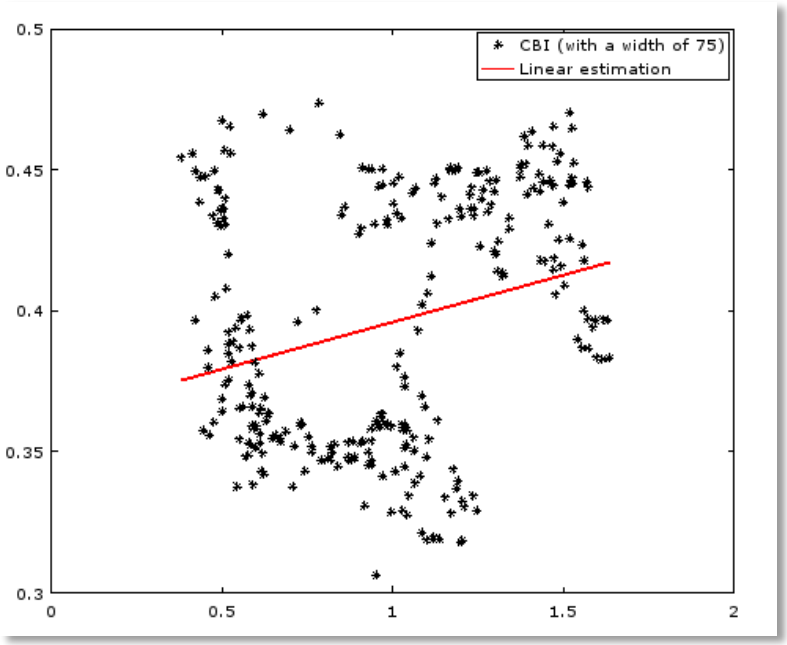

Figure 15 - Simple linear estimation of the relation CIB/QE

NADARAYA-WATSON ESTIMATION

Hence, our choice will be to improve our model using a non-parametric estimation. Many estimator have been designed to do kernel regressions, but we choose one of the first ones (and of the most efficient ones), the Nadaraya-Watson estimator, described in Nadaraya (1964) and Watson (1964). In the non-parametric regressions, we try to directly estimate the relation E(Y|X)=f(X), where f is an unknown function. To do that, the Nadaraya-Watson approach is to estimate f as a locally weighted average, using for weighting function a kernel. Thus the Nadaraya-Watson estimator is defined in the general case as:

$$\hat{f}_h(x) = \frac{\sum_{i=1}^{n} K_h(x - x_i) y_i}{\sum_{i=1}^{n} K_h(x - x_i)}$$

Equation 19 - Definition of the Nadaraya-Watson estimator

The kernel used K_h is not very important, and have a very limited impact on the estimation.

But the value of the bandwidth h determine largely the accuracy of the estimation. If h is too small, it overfit the data and the estimation doesn't really exhibit a clear relation between the variables, but if h is too big, the estimation is too smoothed and the complexity of the relation is omitted.

A standard way to choose the value of h is to use the Silverman's h, as it have been shown by Silverman (1998), using a Gaussian kernel (in the univariate case), the optimal bandwidth can be approximated by:

$$h_{Silverman} \approx 1.06 \hat{\sigma} n^{-1/5}$$

Equation 20 - Silverman's rule of thumb

In this equation, $\hat{\sigma}$ is the empirical standard deviation of the sample.

Usually, we saw that the Silverman's h, even if it's theoretically the optimal value for the bandwidth, leads to very precise fits. Thus, we will also present others estimations using a larger bandwidth to make the relation more readable between the variables.

We run 3 estimations and obtained the results presented on the next page.

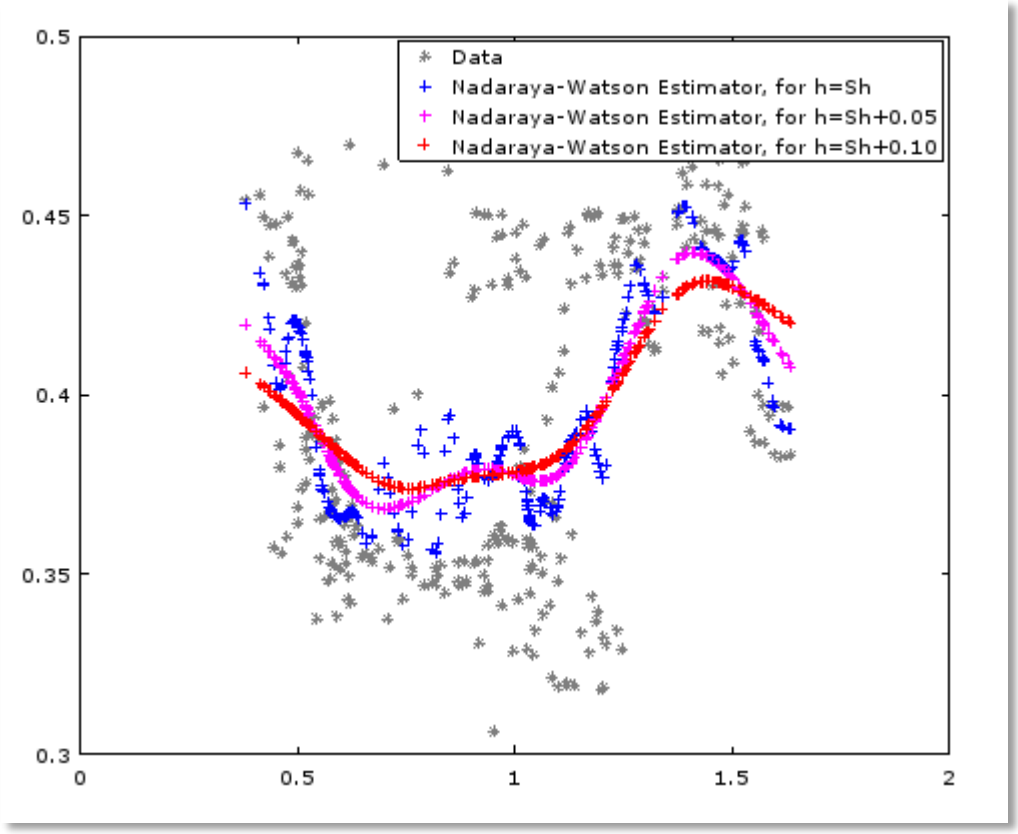

Figure 16 - Nadaraya-Watson Non-parametric Estimations

Thanks to these estimations, we can see that the relation between the CBI and our QE proxy is more complex than what we think before. Anyway, we get an indisputable proof of the existence of a link between the spread and the CBI.

The relation is negative when the spread is between 0.4 and 0.75, it become positive when it is between 0.75 and 1.4, and then start to be negative again. Here, we have to go further than the conclusion of Joyce and al (2011), *"we find that asset purchases financed by the issuance of central bank reserves […] may have depressed medium to long-term government bond yields by about 100 basis points*".

If the spread between different maturities of government bonds is really a good proxy for assessing on the quantitative easing impact, the impact of QE could be more complex than simply negative in all the cases. Indeed, from the starting date of the QE (the 2015/03/09), there is no considerable decrease in the spreads, and moreover, we have a remarkable peak at the end of the period. Such a shape of the spread curve suggest that the quantitative easing could have (mainly) a positive impact on the spreads. This opinion can be supported by the huge positive perturbation caused by the QE announcement.

A positive link between the QE and the government bonds spreads implies that the agents consider in their expectations that the governments are not going to take all the benefit of the huge purchases (implied by the QE) by an unlimited increased of their bond issuances. Hence, if the offer of bonds is restricted, and it is easy to understand that in a world where the austerity policies are expanded it could be the case, then mechanically the price (and thus the spread) have to increase if the demand (the QE purchase) increase.

Such a positive link between QE and spreads indicates that, in the first part of the graph, there is a negative link between the QE and the bubble, but when the spreads go upper than 0.75, the relation becomes positive, before to revert again when the spread is around 1.4.

In economic terms, these results can be interpreted as follows:

- When the QE start to produce its effects by the expectations channel, the impact of the QE on the stock market is negative, because the agents expect that the interest rates will rise and thus moves their money from the stock to the bond market (we are between 0.4 and 0.75).
- But the first effect of the QE is to dried the government bond market when the purchases of the ECB begin, hence the money which was invested in this market moves from the bond to the stock market, and thus this sudden excess of liquidity starts to make a bubble growing. We have in this period (between 0.75 and 1.4) a positive relation between the QE and the bubble.
- After a while, the money invested in the stock market came back in the government bond market, because the massive purchases of the ECB fulfilled the expectations of increase in the interest rates. The relation between the QE and the bubble becomes negative again (after 1.4)

Of course this interpretation is based on the fact that the spreads are almost always increasing (or just slowly decreasing) over the considered period.

If we consider that such a story is convincing, it accredit the assumption of a positive link between the QE and the spreads. This position allows another interpretation of the

results of the simple linear regression. Indeed, the positive value for β implies now a positive link between the QE and the bubble.

On the base of the assumption of a positive link between the QE and the spreads, we obtain a plausible interpretation of the relation between the QE and the bubble, for the simple linear regression and also for the Nadaraya-Watson non-parametric regression. If this interpretation can be contested the graphs of the regressions are eloquent: since the government bond spreads reflect the impact of the QE on the economy, the link between the QE and the bubble exists.

CONCLUSIONS

In the present thesis, we first saw that, according to the tools that we used, a bubble seems to be present on the CAC 40 in the first semester of the year 2015. Of course, graphically, this is difficult to denigrate the quality of the Robust LPPL fit. We also obtain coherent results on the assessment of the presence of an Ornstein-Uhlenbeck structure in the residuals.

But this diagnosis is limited by the fact that we do not have any quantitative measure of the quality of the fit. If the originally selected device was the R^2, it was clearly a bad idea to use it in our case. Another model used as a benchmark (e.g. the Black-Scholes model), together with goodness of fit measures such as AIC and BIC to make comparisons can improve this assessment on the quality of the LPPL modelling. Using the Volatility-Confined LPPL model could also be an improvement. Finally, being aware of these limits, we choose to conclude that the presence of a speculative bubble on the CAC 40 in the beginning of the year 2015 seems probable.

Defining an alternative model which is not the random walk (e.g. a GARCH(1,1)), is an important correction which should be done to the CBI, which is unable to show clearly the bubble regime as it is not constructed with a No-Bubble-at-All model. Anyway, the CBI of course capture the variations of the quality of the fit of the Robust LPPL model, and thus reflects the importance of the speculative behaviors on the market. Hence, the increase of these behaviors in the first semester, and thus the increase of the probability of existence of the bubble, are not really questionable.

Finally, the two regressions of the spread between the yields of different maturities of european government bonds on the CBI exhibit a clear link between the quantitative easing and the speculative bubble on the CAC 40. This is particularly the case with the Nadaraya-Watson regression, which doesn't allows to say that there is no link. But the interpretation of this link stays subjective, and the direction of the causality is not established. Running regressions with control variables is also an improvement which should be done.

Considering all these elements, it's likely that there is a speculative bubble on the CAC 40 which is (at least partly) feed by the quantitative easing of the European Central Bank. Such a pervert effect of this unconventional monetary policy must be taken into account by the policy makers in the future.

LIST OF TABLES

TABLE OF EQUATIONS

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ANNEXES

ANNEX 1: SOLUTION OF THE LINEAR SYSTEM OF EQUATIONS FOR A, B, C1 AND C2

We first gives names to all the elements of the matrix, because we can't write them directly as sums.

$N=T$; $q=\Sigma(f_t)$; $j=\Sigma(g_t)$; $l=\Sigma(h_t)$; $s=\Sigma(y_t)$; $e=\Sigma(f_t^2)$; $k=\Sigma(f_t*g_t)$; $m=\Sigma(f_t*h_t)$; $v=\Sigma(y_t*f_t)$; $r=\Sigma(g_t^2)$ $o=\Sigma(g_t*h_t)$; $w=\Sigma(y_t*g_t)$; $p=\Sigma(h_t^2)$; $z=\Sigma(y_t*h_t)$

The following expressions are solutions of the linear system provided by Maxima:

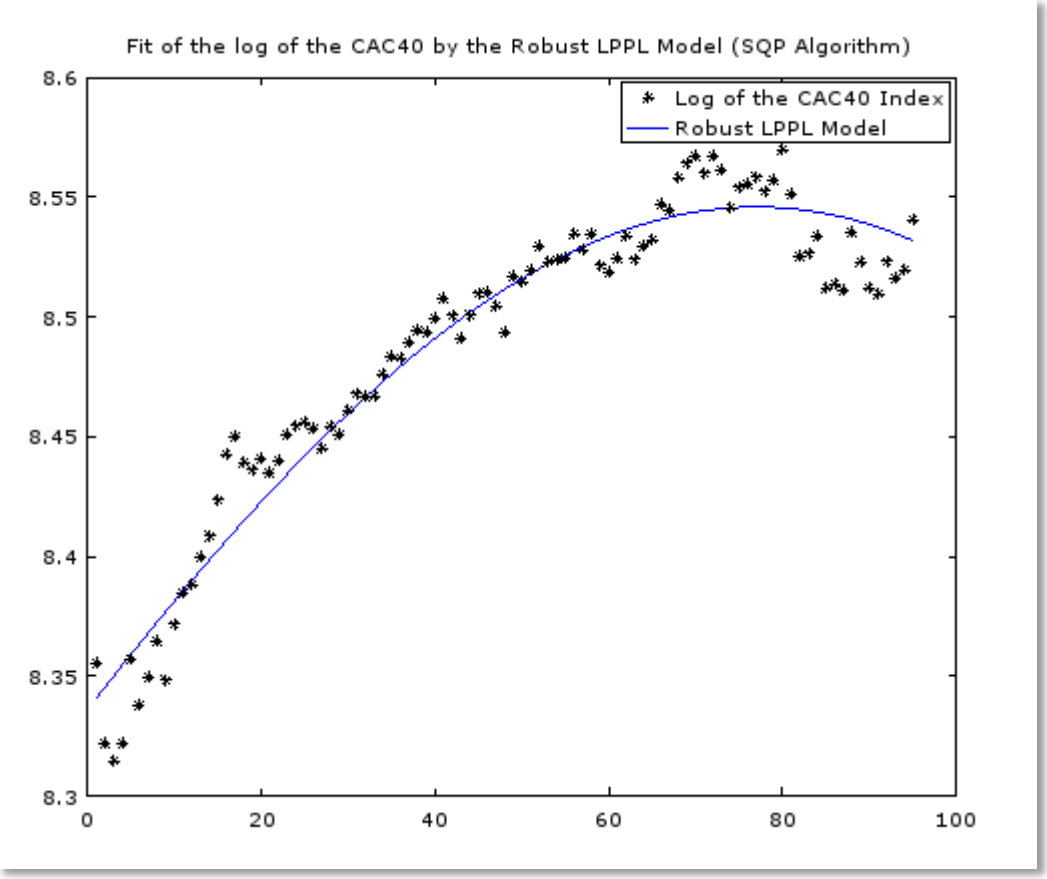
A=(l*(e*(r*z-o*w)-k^2*z+k*m*w+(k*o-m*r)*v)+q*(m*(o*w-r*z)+k*(o*z-p*w)+(p*r-o^2)*v)+j*(e*(p*w-o*z)+k*m*z-m^2*w+(m*o-k*p)*v)+(e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*s)/((e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*N+q*(l*(2*k*o-2*m*r)+j*(2*m*o-2*k*p))+q^2*(p*r-o^2)+l^2*(e*r-k^2)+j^2*(e*p-m^2)+j*l*(2*k*m-2*e*o))

B=-((m*(o*w-r*z)+k*(o*z-p*w)+(p*r-o^2)*v)*N+q*(l*(r*z-o*w)+j*(p*w-o*z))+j^2*(m*z-p*v)+j*l*(-k*z-m*w+2*o*v)+l^2*(k*w-r*v)+(q*(o^2-p*r)+l*(m*r-k*o)+j*(k*p-m*o))*s)/((e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*N+q*(l*(2*k*o-2*m*r)+j*(2*m*o-2*k*p))+q^2*(p*r-o^2)+l^2*(e*r-k^2)+j^2*(e*p-m^2)+j*l*(2*k*m-2*e*o))

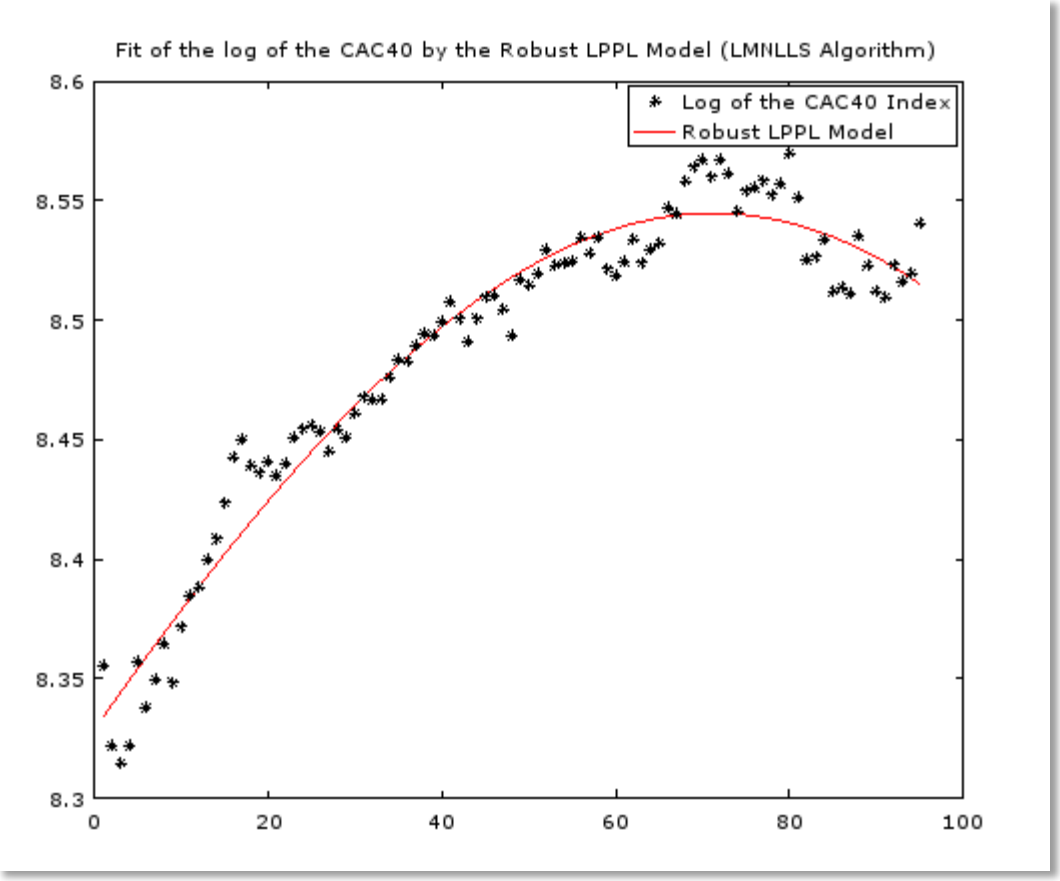
C1=-((e*(p*w-o*z)+k*m*z-m^2*w+(m*o-k*p)*v)*N+q*(j*(p*v-m*z)+l*(-k*z+2*m*w-o*v))+q^2*(o*z-p*w)+j*l*(e*z-m*v)+l^2*(k*v-e*w)+((k*p-m*o)*q+j*(m^2-e*p)+l*(e*o-k*m))*s)/((e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*N+q*(l*(2*k*o-2*m*r)+j*(2*m*o-2*k*p))+q^2*(p*r-o^2)+l^2*(e*r-k^2)+j^2*(e*p-m^2)+j*l*(2*k*m-2*e*o))

C2=((e*(o*w-r*z)+k^2*z-k*m*w+(m*r-k*o)*v)*N+q*(j*(-2*k*z+m*w+o*v)+l*(k*w-r*v))+q^2*(r*z-o*w)+j^2*(e*z-m*v)+j*l*(k*v-e*w)+(q*(k*o-m*r)+l*(e*r-k^2)+j*(k*m-e*o))*s)/((e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*N+q*(l*(2*k*o-2*m*r)+j*(2*m*o-2*k*p))+q^2*(p*r-o^2)+l^2*(e*r-k^2)+j^2*(e*p-m^2)+j*l*(2*k*m-2*e*o))

ANNEX 2: ROBUST LPPL ESTIMATION OF THE LOG OF THE CAC 40 USING THE SQP ALGORITHM



ANNEX 3: ROBUST LPPL ESTIMATION OF THE LOG OF THE CAC 40 USING THE LEVENBERG–MARQUARDT ALGORITHM



ANNEX 4: MATLAB CODE FOR THE LPPL ESTIMATIONS

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                                                           %
% THE CAC 40 IN 2015: THE SHADOW OF A QE BUBBLE                                                      %
% MASTER THESIS                                                                     %
%                                                                           %
%-------------------------------------------------------------------------------------------------------------------------------%
% MATLAB CODE FOR ESTIMATIONS                                                                  %
%-------------------------------------------------------------------------------------------------------------------------------%
%                                                                           %
% THOMAS BARRAU                                                                       %
% MASTER OF SCIENCE - ECONOMICS AND FINANCE                                                    %
% CA'FOSCARI UNIVERSITY                                                               %
% SCHOOLYEAR 2014/2015                                                                 %
%                                                                           %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                                                           %
% Bubble Diagnosis - Robust LPPL Model Estimation                                                %
%                                                                           %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

clc
clf
close all

% Warning: The following code have been written to work under Octave 3.8.0, with the Optimization package.

% Warning: In the matlab code the steps can be organize differently than in the thesis.
% The original path of reasoning is the one describe in the thesis, the order of the steps followed
% have been changed for convenience reasons.

% IMPLEMENTATION OF THE DATA

% Objective variable
% y=[4252.29      4111.36      4083.5   4112.73  4260.19  4179.07  4228.24  4290.28  4223.24  4323.2
                  4379.62      4394.93  4446.02  4484.82  4552.8   4640.69  4675.13  4624.21  4610.94
                  4631.43      4604.25  4627.67  4677.9   4696.3   4703.3   4691.03  4651.08  4695.65
                  4679.38      4726.2   4759.36  4751.95  4753.99  4799.03  4833.28  4830.9   4862.3
                  4886.44      4882.22  4910.62  4951.48  4917.32  4869.25  4917.35  4963.51  4964.35
                  4937.2       4881.95  4997.75  4987.33  5010.46  5061.16  5028.93  5033.42  5037.18
                  5087.49      5054.52  5088.28  5020.99  5006.35  5034.06  5083.52  5033.64  5062.22
                  5074.14      5151.19  5136.86  5208.95  5240.46  5254.12  5218.06  5254.35  5224.49
                  5143.26      5187.59  5192.64  5211.09  5178.91  5201.45  5268.91  5173.38  5039.39
                  5046.49      5081.97  4974.07  4981.59  4967.22  5090.39  5027.87  4974.65  4961.86
                  5029.31      4993.82  5012.31  5117.3];
y=[4252.29        4111.36      4083.50  4112.73  4260.19  4179.07  4228.24  4290.28  4223.24  4323.20
                  4379.62      4394.93  4446.02  4484.82  4552.80  4640.69  4675.13  4624.21  4610.94
                  4631.43      4604.25  4627.67  4677.90  4696.30  4703.30  4691.03  4651.08  4695.65
                  4679.38      4726.20  4759.36  4751.95  4753.99  4799.03  4833.28  4830.90  4862.30
                  4886.44      4882.22  4910.62  4951.48  4917.32  4869.25  4917.35  4963.51  4964.35
                  4937.20      4881.95  4997.75  4987.33  5010.46  5061.16  5028.93  5033.42  5037.18
                  5087.49      5054.52  5088.28  5020.99  5006.35  5034.06  5083.52  5033.64  5062.22
                  5074.14      5151.19  5136.86  5208.95  5240.46  5254.12  5218.06  5254.35  5224.49
                  5143.26      5187.59  5192.64  5211.09  5178.91  5201.45  5268.91  5173.38  5039.39
                  5046.49      5081.97  4974.07  4981.59  4967.22  5090.39  5027.87  4974.65  4961.86
                  5029.31      4993.82  5012.31  5117.30  5133.30  5146.70  5142.89  5117.17  5083.54
                  5182.53      5137.83  5007.89  5025.30  5004.46  5034.17  4987.13  4920.74  4857.66];
y=log(y);
% Explanatory variable (time)
nbobs=length(y)
x=t=[1:nbobs];

%-------------------------------------------------------------------------------------------------------------------------------
```

```
% Estimation using the Levenberg-Marquardt Non_Linear Regression
%-----------------------------------------------------------------------------------------------------------------------------------------------------------

% SOLVING THE LINEAR SYSTEM FOR THE LINEAR PARAMETERS A, B, C1, C2:

% Defining the elements of the matrix

% p(1)=m
% p(2)=tc
% p(3)=w

% fi=(p(2)-x).^p(1);
% yi=log(y);
% gi=((p(2)-x).^p(1)).*cos(p(3)*log(p(2)-x));
% hi=((p(2)-x).^p(1)).*sin(p(3)*log(p(2)-x));

% N=nbobs;
% q=sum(fi);
% j=sum(gi);
% l=sum(hi);
% s=sum(yi);
% e=sum(fi.^2);
% k=sum(fi.*gi);
% m=sum(fi.*hi);
% v=sum(yi.*fi);
% r=sum(gi.^2);
% o=sum(gi.*hi);
% w=sum(yi.*gi);
% p=sum(hi.^2);
% z=sum(yi.*hi);

% Solving this system of 4 equations with 4 unknowns leads to formal calculous, because m, tc and w can't be define a priori.
% Thus Matlab is unable to give a general solution.
% Hence the following expressions are solutions of the linear system provided by Maxima:

%       A=(l*(e*(r*z-o*w)-k^2*z+k*m*w+(k*o-m*r)*v)+q*(m*(o*w-r*z)+k*(o*z-p*w)+(p*r-o^2)*v)+j*(e*(p*w-o*z)+k*m*z-m^2*w+(m*o-
k*p)*v)+(e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*s)/((e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*N+q*(l*(2*k*o-2*m*r)+j*(2*m*o-
2*k*p))+q^2*(p*r-o^2)+l^2*(e*r-k^2)+j^2*(e*p-m^2)+j*l*(2*k*m-2*e*o));
%            B=-((m*(o*w-r*z)+k*(o*z-p*w)+(p*r-o^2)*v)*N+q*(l*(r*z-o*w)+j*(p*w-o*z))+j^2*(m*z-p*v)+j*l*(-k*z-m*w+2*o*v)+l^2*(k*w-
r*v)+(q*(o^2-p*r)+l*(m*r-k*o)+j*(k*p-m*o))*s)/((e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*N+q*(l*(2*k*o-2*m*r)+j*(2*m*o-
2*k*p))+q^2*(p*r-o^2)+l^2*(e*r-k^2)+j^2*(e*p-m^2)+j*l*(2*k*m-2*e*o));
%     C1=-((e*(p*w-o*z)+k*m*z-m^2*w+(m*o-k*p)*v)*N+q*(j*(p*v-m*z)+l*(-k*z+2*m*w-o*v))+q^2*(o*z-p*w)+j*l*(e*z-m*v)+l^2*(k*v-
e*w)+((k*p-m*o)*q+j*(m^2-e*p)+l*(e*o-k*m))*s)/((e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*N+q*(l*(2*k*o-2*m*r)+j*(2*m*o-
2*k*p))+q^2*(p*r-o^2)+l^2*(e*r-k^2)+j^2*(e*p-m^2)+j*l*(2*k*m-2*e*o));
%          C2=((e*(o*w-r*z)+k^2*z-k*m*w+(m*r-k*o)*v)*N+q*(j*(-2*k*z+m*w+o*v)+l*(k*w-r*v))+q^2*(r*z-o*w)+j^2*(e*z-m*v)+j*l*(k*v-
e*w)+(q*(k*o-m*r)+l*(e*r-k^2)+j*(k*m-e*o))*s)/((e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*N+q*(l*(2*k*o-2*m*r)+j*(2*m*o-
2*k*p))+q^2*(p*r-o^2)+l^2*(e*r-k^2)+j^2*(e*p-m^2)+j*l*(2*k*m-2*e*o));

% DEFINITION OF THE MODEL

F=@(x,p) A+B*((p(2)-x).^p(1))+C1*((p(2)-x).^p(1)).*cos(p(3)*log(p(2)-x))+C2*((p(2)-x).^p(1)).*sin(p(3)*log(p(2)-x));

% VECTOR OF INITIAL PARAMETERS
m=0.8; % p(1)=m
tc=500; % p(2)=tc
w=6.5; % p(3)=w
% Definition of the vector of initial parameters
init=[m;tc;w];

% TOLERANCE
tolerance = .01;

% NUMBER OF ITERATIONS
max_iterations = 100;

weights = ones(1,nbobs);
dp = [.001; .001; .001]; % bidirectional numeric gradient stepsize
dFdp = "dfdp"; % function for gradient (numerical)

% Determination of the bounds
options.bounds=[0.1,0.9;109,Inf; 6,13];
```

```
% WARNING: The value of the upper bound of tc is unfortunately never accepted by the software when it's higher than 200.
% This problem have been avoid using the following trick: We fix a very acceptable value of the bound for the software, e.g. 96, and
% then we set a very high initial value. Thus, the software will not take into account the upper bound of tc,
% and this leads us to a situation where there is no upper bound, exactly what we were looking for.

% COMPUTATION OF THE LEVENBERG-MARQUARDT NON-LINEAR REGRESSION
global verbose;
verbose = false;
[f1, p, cvg, iter, corp, covp, covr, stdresid]=leasqr (x, y, init, F, tolerance, max_iterations, weights, dp, dFdp, options);

figure(1)
hold on
plot(x,y,'k*')
plot(x,f1,'r-')
title('Fit of the log of the CAC 40 by the Robust LPPL Model (LMNLLS Algorithm) ')
legend('Log of the CAC 40 Index','Robust LPPL Model')
hold off

% Computation of the residuals

residuals=f1-y';

% Extraction of the non-linear estimated parameters
p1=p(1);
p2=p(2);
p3=p(3);

% Computation of the linear parameters
f=(p2-x).^p1;
ly=y;
g=((p2-x).^p1).*cos(p3*log(p2-x));
h=((p2-x).^p1).*sin(p3*log(p2-x));

N=nbobs;
q=sum(f);
j=sum(g);
l=sum(h);
s=sum(ly);
e=sum(f.^2);
k=sum(f.*g);
m=sum(f.*h);
v=sum(ly.*f);
r=sum(g.^2);
o=sum(g.*h);
w=sum(ly.*g);
p=sum(h.^2);
z=sum(ly.*h);

% Presentation of the 7 estimated parameters
m=p1
tc=p2
omega=p3
A=(l*(e*(r*z-o*w)-k^2*z+k*m*w+(k*o-m*r)*v)+q*(m*(o*w-r*z)+k*(o*z-p*w)+(p*r-o^2)*v)+j*(e*(p*w-o*z)+k*m*z-m^2*w+(m*o-
k*p)*v)+(e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*s)/((e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*N+q*(l*(2*k*o-2*m*r)+j*(2*m*o-
2*k*p))+q^2*(p*r-o^2)+l^2*(e*r-k^2)+j^2*(e*p-m^2)+j*l*(2*k*m-2*e*o))
B=-((m*(o*w-r*z)+k*(o*z-p*w)+(p*r-o^2)*v)*N+q*(l*(r*z-o*w)+j*(p*w-o*z))+j^2*(m*z-p*v)+j*l*(-k*z-m*w+2*o*v)+l^2*(k*w-
r*v)+(q*(o^2-p*r)+l*(m*r-k*o)+j*(k*p-m*o))*s)/((e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*N+q*(l*(2*k*o-2*m*r)+j*(2*m*o-
2*k*p))+q^2*(p*r-o^2)+l^2*(e*r-k^2)+j^2*(e*p-m^2)+j*l*(2*k*m-2*e*o))
C1=-((e*(p*w-o*z)+k*m*z-m^2*w+(m*o-k*p)*v)*N+q*(j*(p*v-m*z)+l*(-k*z+2*m*w-o*v))+q^2*(o*z-p*w)+j*l*(e*z-m*v)+l^2*(k*v-
e*w)+((k*p-m*o)*q+j*(m^2-e*p)+l*(e*o-k*m))*s)/((e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*N+q*(l*(2*k*o-2*m*r)+j*(2*m*o-
2*k*p))+q^2*(p*r-o^2)+l^2*(e*r-k^2)+j^2*(e*p-m^2)+j*l*(2*k*m-2*e*o))
C2=((e*(o*w-r*z)+k^2*z-k*m*w+(m*r-k*o)*v)*N+q*(j*(-2*k*z+m*w+o*v)+l*(k*w-r*v))+q^2*(r*z-o*w)+j^2*(e*z-m*v)+j*l*(k*v-
e*w)+(q*(k*o-m*r)+l*(e*r-k^2)+j*(k*m-e*o))*s)/((e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*N+q*(l*(2*k*o-2*m*r)+j*(2*m*o-
2*k*p))+q^2*(p*r-o^2)+l^2*(e*r-k^2)+j^2*(e*p-m^2)+j*l*(2*k*m-2*e*o))

% Computation of a goodness of fit measure, the coefficient of determination
SStot=sum((y-mean(y)).^2);
SSres=sum(residuals.^2);
R2=1-SSres/SStot

% Computation of the forecast up to tc:
%T=floor(tc)-1;
```

```
%tt=[1:T];
%forecast=A+B*((p2-tt).^p1)+C1*((p2-tt).^p1).*cos(p3*log(p2-tt))+C2*((p2-tt).^p1).*sin(p3*log(p2-tt));

% Plot of the forecast:
%figure(666)
%hold on
%plot(exp(y),'k*')
%plot(exp(forecast),'g-')
%title('Forecast ofthe  CAC 40 by the Robust LPPL Model (LMNLLS Algorithm) ')
%legend('CAC 40 Index','Forecast of the Robust LPPL Model')
%hold off

% Unfortunately the forecast gives results which seems wrong, thus we will left the forecast for further researchs.

%-----------------------------------------------------------------------------------------------------------------------------------
% Estimation using the Sequential Quadratic Programming Algorithm
%-----------------------------------------------------------------------------------------------------------------------------------

% ESTIMATION OF THE PARAMETERS

% We minimize directly the sum of the squared residuals, which is defined in the following function of the parameters:

function phi=cost_func(p)
 % In this function, this is the value of the 3 parameters m,tc and omega which change.
 % Hence you have to put the data inside of the function as if it was some parameters.

 % Objective variable
 % y=[4252.29          4111.36          4083.5          4112.73          4260.19          4179.07          4228.24          4290.28          4223.24          4323.2
                       4379.62          4394.93          4446.02          4484.82          4552.8          4640.69          4675.13          4624.21          4610.94
                       4631.43          4604.25          4627.67          4677.9          4696.3          4703.3          4691.03          4651.08          4695.65
                       4679.38          4726.2          4759.36          4751.95          4753.99          4799.03          4833.28          4830.9          4862.3
                       4886.44          4882.22          4910.62          4951.48          4917.32          4869.25          4917.35          4963.51          4964.35
                       4937.2          4881.95          4997.75          4987.33          5010.46          5061.16          5028.93          5033.42          5037.18
                       5087.49          5054.52          5088.28          5020.99          5006.35          5034.06          5083.52          5033.64          5062.22
                       5074.14          5151.19          5136.86          5208.95          5240.46          5254.12          5218.06          5254.35          5224.49
                       5143.26          5187.59          5192.64          5211.09          5178.91          5201.45          5268.91          5173.38          5039.39
                       5046.49          5081.97          4974.07          4981.59          4967.22          5090.39          5027.87          4974.65          4961.86
                       5029.31          4993.82          5012.31          5117.3];
y=[4252.29          4111.36          4083.50          4112.73          4260.19          4179.07          4228.24          4290.28          4223.24          4323.20
                       4379.62          4394.93          4446.02          4484.82          4552.80          4640.69          4675.13          4624.21          4610.94
                       4631.43          4604.25          4627.67          4677.90          4696.30          4703.30          4691.03          4651.08          4695.65
                       4679.38          4726.20          4759.36          4751.95          4753.99          4799.03          4833.28          4830.90          4862.30
                       4886.44          4882.22          4910.62          4951.48          4917.32          4869.25          4917.35          4963.51          4964.35
                       4937.20          4881.95          4997.75          4987.33          5010.46          5061.16          5028.93          5033.42          5037.18
                       5087.49          5054.52          5088.28          5020.99          5006.35          5034.06          5083.52          5033.64          5062.22
                       5074.14          5151.19          5136.86          5208.95          5240.46          5254.12          5218.06          5254.35          5224.49
                       5143.26          5187.59          5192.64          5211.09          5178.91          5201.45          5268.91          5173.38          5039.39
                       5046.49          5081.97          4974.07          4981.59          4967.22          5090.39          5027.87          4974.65          4961.86
                       5029.31          4993.82          5012.31          5117.30          5133.30          5146.70          5142.89          5117.17          5083.54
                       5182.53          5137.83          5007.89          5025.30          5004.46          5034.17          4987.13          4920.74          4857.66];
 % Explanatory variable (time)
nbobs=length(y);
x=t=[1:nbobs];

p1=p(1); % This step is done to avoid double indexation in the for loop
p2=p(2);
p3=p(3);

% Computation of A, B, C1 and C2:
f=(p2-x).^p1;
ly=log(y);
g=((p2-x).^p1).*cos(p3*log(p2-x));
h=((p2-x).^p1).*sin(p3*log(p2-x));

N=nbobs;
q=sum(f);
j=sum(g);
l=sum(h);
s=sum(ly);
e=sum(f.^2);
k=sum(f.*g);
m=sum(f.*h);
```

```
 v=sum(ly.*f);
 r=sum(g.^2);
 o=sum(g.*h);
 w=sum(ly.*g);
 p=sum(h.^2);
 z=sum(ly.*h);

 A=(l*(e*(r*z-o*w)-k^2*z+k*m*w+(k*o-m*r)*v)+q*(m*(o*w-r*z)+k*(o*z-p*w)+(p*r-o^2)*v)+j*(e*(p*w-o*z)+k*m*z-m^2*w+(m*o-
k*p)*v)+(e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*s)/((e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*N+q*(l*(2*k*o-2*m*r)+j*(2*m*o-
2*k*p))+q^2*(p*r-o^2)+l^2*(e*r-k^2)+j^2*(e*p-m^2)+j*l*(2*k*m-2*e*o));
 B=-((m*(o*w-r*z)+k*(o*z-p*w)+(p*r-o^2)*v)*N+q*(l*(r*z-o*w)+j*(p*w-o*z))+j^2*(m*z-p*v)+j*l*(-k*z-m*w+2*o*v)+l^2*(k*w-
r*v)+(q*(o^2-p*r)+l*(m*r-k*o)+j*(k*p-m*o))*s)/((e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*N+q*(l*(2*k*o-2*m*r)+j*(2*m*o-
2*k*p))+q^2*(p*r-o^2)+l^2*(e*r-k^2)+j^2*(e*p-m^2)+j*l*(2*k*m-2*e*o));
 C1=-((e*(p*w-o*z)+k*m*z-m^2*w+(m*o-k*p)*v)*N+q*(j*(p*v-m*z)+l*(-k*z+2*m*w-o*v))+q^2*(o*z-p*w)+j*l*(e*z-m*v)+l^2*(k*v-
e*w)+((k*p-m*o)*q+j*(m^2-e*p)+l*(e*o-k*m))*s)/((e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*N+q*(l*(2*k*o-2*m*r)+j*(2*m*o-
2*k*p))+q^2*(p*r-o^2)+l^2*(e*r-k^2)+j^2*(e*p-m^2)+j*l*(2*k*m-2*e*o));
 C2=((e*(o*w-r*z)+k^2*z-k*m*w+(m*r-k*o)*v)*N+q*(j*(-2*k*z+m*w+o*v)+l*(k*w-r*v))+q^2*(r*z-o*w)+j^2*(e*z-m*v)+j*l*(k*v-
e*w)+(q*(k*o-m*r)+l*(e*r-k^2)+j*(k*m-e*o))*s)/((e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*N+q*(l*(2*k*o-2*m*r)+j*(2*m*o-
2*k*p))+q^2*(p*r-o^2)+l^2*(e*r-k^2)+j^2*(e*p-m^2)+j*l*(2*k*m-2*e*o));

 % Initiamization of the squared residuals vector
 resqr=zeros(size(x));

 % Computation of the squared residuals
  for u=1:nbobs
    resqr(u)=(log(y(u))-A-B*((p2-u)^p1)-C1*((p2-x(u)).^p1).*cos(p3*log(p2-x(u)))-C2*((p2-x(u)).^p1).*sin(p3*log(p2-x(u))))^2;
  endfor

 % Computation of the result of the cost function, the sum of squared residuals
 phi=sum(resqr);

endfunction

% After defining the cost function, let's minimize it:
% Vector of initial parameters
m=0.1; % p(1)=m
tc=400; % p(2)=tc
omega=13; % p(3)=omega
init=[m;tc;omega];

% Definition of the constraints on the parameters
lb=[0.1;nbobs;6];
ub=[0.9;100000;13];

% Minimization of the cost function
[p, phi, info, iter, nf, lambda] = sqp (init,@cost_func,[],[],lb,ub)

% COMPUTATION OF THE RESULTS OF THE ROBUST LPPL MODEL:

% Extraction of the non-linear estimated parameters
p1=p(1);
p2=p(2);
p3=p(3);

% Computation of the linear parameters
f=(p2-x).^p1;
ly=y;
g=((p2-x).^p1).*cos(p3*log(p2-x));
h=((p2-x).^p1).*sin(p3*log(p2-x));

N=nbobs;
q=sum(f);
j=sum(g);
l=sum(h);
s=sum(ly);
e=sum(f.^2);
k=sum(f.*g);
m=sum(f.*h);
v=sum(ly.*f);
r=sum(g.^2);
o=sum(g.*h);
w=sum(ly.*g);
```

```
p=sum(h.^2);
z=sum(ly.*h);

% Presentation of the 7 estimated parameters
m=p1
tc=p2
omega=p3
A=(l*(e*(r*z-o*w)-k^2*z+k*m*w+(k*o-m*r)*v)+q*(m*(o*w-r*z)+k*(o*z-p*w)+(p*r-o^2)*v)+j*(e*(p*w-o*z)+k*m*z-m^2*w+(m*o-
k*p)*v)+(e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*s)/((e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*N+q*(l*(2*k*o-2*m*r)+j*(2*m*o-
2*k*p))+q^2*(p*r-o^2)+l^2*(e*r-k^2)+j^2*(e*p-m^2)+j*l*(2*k*m-2*e*o))
B=-((m*(o*w-r*z)+k*(o*z-p*w)+(p*r-o^2)*v)*N+q*(l*(r*z-o*w)+j*(p*w-o*z))+j^2*(m*z-p*v)+j*l*(-k*z-m*w+2*o*v)+l^2*(k*w-
r*v)+(q*(o^2-p*r)+l*(m*r-k*o)+j*(k*p-m*o))*s)/((e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*N+q*(l*(2*k*o-2*m*r)+j*(2*m*o-
2*k*p))+q^2*(p*r-o^2)+l^2*(e*r-k^2)+j^2*(e*p-m^2)+j*l*(2*k*m-2*e*o))
C1=-((e*(p*w-o*z)+k*m*z-m^2*w+(m*o-k*p)*v)*N+q*(j*(p*v-m*z)+l*(-k*z+2*m*w-o*v))+q^2*(o*z-p*w)+j*l*(e*z-m*v)+l^2*(k*v-
e*w)+((k*p-m*o)*q+j*(m^2-e*p)+l*(e*o-k*m))*s)/((e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*N+q*(l*(2*k*o-2*m*r)+j*(2*m*o-
2*k*p))+q^2*(p*r-o^2)+l^2*(e*r-k^2)+j^2*(e*p-m^2)+j*l*(2*k*m-2*e*o))
C2=((e*(o*w-r*z)+k^2*z-k*m*w+(m*r-k*o)*v)*N+q*(j*(-2*k*z+m*w+o*v)+l*(k*w-r*v))+q^2*(r*z-o*w)+j^2*(e*z-m*v)+j*l*(k*v-
e*w)+(q*(k*o-m*r)+l*(e*r-k^2)+j*(k*m-e*o))*s)/((e*(o^2-p*r)+m^2*r+k^2*p-2*k*m*o)*N+q*(l*(2*k*o-2*m*r)+j*(2*m*o-
2*k*p))+q^2*(p*r-o^2)+l^2*(e*r-k^2)+j^2*(e*p-m^2)+j*l*(2*k*m-2*e*o))

% Computation of the results of the robust LPPL model
f2=A+B*((p2-x).^p1)+C1*((p2-x).^p1).*cos(p3*log(p2-x))+C2*((p2-x).^p1).*sin(p3*log(p2-x));

% Plot of the data and the fitted LPPL model
figure(2)
hold on
plot(x,y,'k*')
plot(x,f2,'b-')
title('Fit of the log of the CAC 40 by the Robust LPPL Model (SQP Algorithm) ')
legend('Log of the CAC 40 Index','Robust LPPL Model')
hold off

% Computation of the residuals
residuals2=ly-f2;

% Computation of a goodness of fit measure, the coefficient of determination
SStot=sum((ly-mean(ly)).^2);
SSres=sum(residuals2.^2);
R22=1-SSres/SStot

% Comparison between the two fits

figure(3)
hold on
plot(x,y,'k*')
plot(x,f1,'r-')
plot(x,f2,'b-')
title('Comparison between the fit of the log of the CAC 40 by the two algorithms')
legend('Log of the CAC 40 Index','Levenberg-Marquadt Non-Linear Least Square Algorithm', 'Sequential Quadratic Programing Algorithm')
hold off

%-----------------------------------------------------------------------------------------------------------------------------------
% Estimation of an AR(1) model using the SQR Algorithm estimation of the Robust LPPL Model
%-----------------------------------------------------------------------------------------------------------------------------------

% Computation of the ACF and PACF of the residuals

function gama = autocovec_emp(Y,K); % Compute vector of autocovariance

T = length(Y);
gama=zeros(K,1); % This vector (of preallocation) allows us to put all the previsions in one vector
for k = 1:K
gama(k) = mean(Y(k+1:T).*Y(1:T-k))-mean(Y(1:T-k)).*mean(Y(k+1:T)); % From Koenig's equation of the covariance
endfor
endfunction

% Computation of the empirical autocovariances

format none
residuals2=residuals2';
format short
```

```
K=20;
gama_emp_res=autocovec_emp(residuals2,K);

% Computation of the empirical autocorrelations

var_res=mean(residuals2.*residuals2)-mean(residuals2)^2; % Empirical variance of the residuals
ACF_res=gama_emp_res/var_res;

% Computation of the PACF

Reslag=lagmatrix(residuals2,[1:K]);
PACF_res=ols (residuals2,Reslag);

% Plot of the ACF/PACF

% Definition of the lower/upper bounds

% Level of significance (using the quantile alpha of the SN distribution):
alpha=0.99; % Level of significance=(1-alpha)*2
quantile=norminv(alpha);

PACF_lconf = -(quantile/sqrt(length(residuals2)))*ones(K); % Definition of lower confidence interval
PACF_upconf = (quantile/sqrt(length(residuals2)))*ones(K); % Definition of upper confidence interval

% ACF bounds for looking at correlation:
ACF_lconf=-(quantile/sqrt(length(residuals2)))*ones(K);
ACF_upconf=(quantile/sqrt(length(residuals2)))*ones(K);

% ACF bounds for determinating ARIMA orders:
ACF_lconf=zeros(length(K));
ACF_upconf=zeros(length(K));
for k=1:K
ACF_lconf(k)=-(quantile*sqrt((1+2*sum(ACF_res(1:k).^2))/length(residuals2)));
ACF_upconf(k)=quantile*sqrt((1+2*sum(ACF_res(1:k).^2))/length(residuals2));
endfor

figure(4)

subplot(2,1,1)
hold on
bar(ACF_res, 'b')
plot(ACF_lconf, 'r')
plot(ACF_upconf, 'r')
title('Autocorrelogram of the residuals of the Robust LPPL Model (SQP)')
legend('ACF','Confidence interval at 99%')
hold off

subplot(2,1,2)
hold on
bar(PACF_res, 'm')
plot(PACF_lconf, 'r')
plot(PACF_upconf, 'r')
title('Partial Autocorrelogram of the residuals of the Robust LPPL Model (SQP)')
legend('PACF','Confidence interval at 99%')
hold off

% Estimation of the ARMA (1,0) model for the residuals

# On the base of the Yule-Walker equations

Y=residuals2;
gama_emp1=autocovec_emp(Y,1);
gama_emp0=var_res;

# Estimated autoregressive parameter
phi_est=gama_emp1/gama_emp0

# Estimated variance
var_est=gama_emp0*(1-phi_est^2)

# Estimated constant
c_est=mean(Y)*(1-phi_est)
```

```
# Plot of the AR(1) process, without any error term

est_AR0=c_est+phi_est*Y(1:nbobs-1);

figure(5)
hold on
plot([2:nbobs],est_AR0,'b-')
plot(Y,'k-')
title('Comparison between the residuals and their AR(1) Estimation')
legend('Residuals','AR(1) Estimation')
hold off

% Residuals of the AR(1) modelling of the residuals of the Robust LPPL model

est_res=Y(2:nbobs)-(c_est+phi_est*Y(1:nbobs-1));

% Computation of the coefficient of determination of this model
SStot=sum((Y-mean(Y)).^2);
SSres=sum(est_res.^2);
R2_3=1-SSres/SStot

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                                                      %
% Bubble Diagnosis - Determination of the starting date of the bubble                  %
%                                                                      %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% IMPLEMENTATION OF THE DATA

% Objective variable
% For the period from 01/01/2014 to 08/06/2014

y=[4227.28          4247.65         4227.54  4262.68  4260.96  4225.14  4250.60  4263.27  4274.20  4332.07
                    4319.27         4327.50  4322.86  4323.87  4324.98  4280.96  4161.47  4144.56  4185.29
                    4156.98         4180.02  4165.72  4107.75  4117.45  4117.79  4188.10  4228.18  4237.13
                    4283.32         4305.50  4312.80  4340.14  4335.17  4330.71  4341.10  4355.49  4381.06
                    4419.13         4414.55  4396.91  4396.39  4408.08  4290.87  4395.90  4391.25  4417.04
                    4366.42         4370.84  4349.72  4306.26  4250.51  4216.37  4271.96  4313.26  4308.06
                    4327.91         4335.28  4276.34  4344.12  4385.15  4379.06  4411.26  4391.50  4426.72
                    4430.86         4449.33  4484.55  4436.08  4424.83  4442.68  4413.49  4365.86  4384.56
                    4345.35         4405.66  4431.81  4484.21  4451.08  4479.54  4443.63  4460.53  4497.68
                    4487.39         4458.17  4462.69  4428.07  4446.44  4507.24  4477.28  4493.65  4505.02
                    4501.04         4444.93  4456.28  4469.76  4452.35  4469.03  4478.21  4493.15  4526.93
                    4529.75         4531.63  4530.51  4519.57  4515.89  4503.69  4501.00  4548.73  4581.12
                    4589.12         4595.00  4555.11  4554.40  4543.28  4510.05  4536.07  4530.37  4563.04
                    4541.34         4515.57  4518.34  4460.60  4439.63  4436.99  4422.84  4461.12  4444.72
                    4489.88         4468.98  4405.76  4342.53  4359.84  4301.26  4316.50  4350.04  4305.31
                    4369.06         4316.12  4335.31  4304.74  4369.52  4376.32  4410.65  4330.55  4344.77
                    4365.58         4312.30  4246.14  4202.78  4217.22  4232.88  4207.14  4149.83  4147.81
                    4197.70         4162.16  4194.79  4205.43  4174.36  4230.65  4254.45  4240.79  4292.93
                    4252.80         4342.11  4393.41  4395.26  4366.04  4381.04  4379.73  4378.33  4421.87
                    4494.94         4486.49  4474.93  4452.37  4450.79  4440.90  4441.70  4428.63  4409.15
                    4431.41         4464.70  4461.22  4442.55  4359.35  4413.72  4355.28  4394.75  4358.07
                    4416.24         4365.27  4242.67  4281.74  4286.52  4209.14  4168.12  4141.45  4073.71
                    4078.70         4088.25  3939.72  3918.62  4033.18  3991.24  4081.24  4105.09  4157.68
                    4128.90         4096.74  4112.67  4110.64  4141.24  4233.09  4194.03  4130.19  4208.42
                    4227.68         4189.89  4222.82  4244.10  4179.88  4187.95  4202.46  4226.10  4262.38
                    4266.19         4234.21  4347.23  4368.44  4382.31  4373.42  4382.34  4390.18  4377.33
                    4388.30         4391.86  4323.89  4419.48  4375.48  4263.94  4227.91  4225.86  4108.93
                    4005.38         4093.20  4111.91  4249.49  4241.65  4254.43  4314.97  4295.85  4317.93
                    4245.54         4272.75  4252.29  4111.36  4083.50  4112.73  4260.19  4179.07  4228.24
                    4290.28         4223.24  4323.20  4379.62  4394.93  4446.02  4484.82  4552.80  4640.69
                    4675.13         4624.21  4610.94  4631.43  4604.25  4627.67  4677.90  4696.30  4703.30
                    4691.03         4651.08  4695.65  4679.38  4726.20  4759.36  4751.95  4753.99  4799.03
                    4833.28         4830.90  4862.30  4886.44  4882.22  4910.62  4951.48  4917.32  4869.25
                    4917.35         4963.51  4964.35  4937.20  4881.95  4997.75  4987.33  5010.46  5061.16
                    5028.93         5033.42  5037.18  5087.49  5054.52  5088.28  5020.99  5006.35  5034.06
                    5083.52         5033.64  5062.22  5074.14  5151.19  5136.86  5208.95  5240.46  5254.12
                    5218.06         5254.35  5224.49  5143.26  5187.59  5192.64  5211.09  5178.91  5201.45
```

```
                    5268.91    5173.38    5039.39    5046.49    5081.97    4974.07    4981.59    4967.22    5090.39
                    5027.87    4974.65    4961.86    5029.31    4993.82    5012.31    5117.30    5133.30    5146.70
                    5142.89    5117.17    5083.54    5182.53    5137.83    5007.89    5025.30    5004.46    5034.17
                    4987.13    4920.74    4857.66];
y=log(y);

% Explanatory variable (time)
T=length(y);
x=[1:T];

%-----------------------------------------------------------------------------------------------------------------------------------
% Construction of the Consolidated Bubble Indicator
%-----------------------------------------------------------------------------------------------------------------------------------

% Size of the window
S=75;
S=S-1;

% Preallocation

MSE_RW=zeros(T-S-1:1);
MSE_LPPL=zeros(T-S-1:1);
MAPE_RW=zeros(T-S-1:1);
MAPE_LPPL=zeros(T-S-1:1);
AIC_RW=zeros(T-S-1:1);
AIC_LPPL=zeros(T-S-1:1);
BIC_RW=zeros(T-S-1:1);
BIC_LPPL=zeros(T-S-1:1);
lL_RW=zeros(T-S-1:1);
lL_LPPL=zeros(T-S-1:1);

for t=2:T-S

 % Preallocation
 e1=zeros(length(S-1));
 e2=zeros(length(S-1));

 % ESTIMATION AND ERRORS OF THE RANDOM WALK MODEL:
 e1=y(t:t+S)-y(t-1:t+S-1);

 % ESTIMATION AND ERRORS OF THE LPPL MODEL:
  % LPPL function
  F=@(x,p) A+B*((p(2)-x).^p(1))+C1*((p(2)-x).^p(1)).*cos(p(3)*log(p(2)-x))+C2*((p(2)-x).^p(1)).*sin(p(3)*log(p(2)-x));
  % Vector of initial parameters
  m=0.9; % p(1)=m
  tc=800; % p(2)=tc
  w=6.5; % p(3)=w
  init=[m;tc;w];
  % Tolerance
  tolerance = .01;
  % Maximum number of iterations
  max_iterations = 100;
  % Other options
  weights = ones(1,S+1);
  dp = [.001; .001; .001]; % bidirectional numeric gradient stepsize
  dFdp = "dfdp"; % function for gradient (numerical)
  % Determination of the bounds
  options.bounds=[0.1,0.9;364,Inf; 6,13];
  % WARNING: the lower bound of tc should be t+S, but it generates a bug to put this kind of value in the bounds, thus write here the
numerical value of T.
  % Computation of the Levenberg-Marquardt non-linear regression
  global verbose;
  verbose = false;
  [f3, p, cvg, iter, corp, covp, covr, stdresid]=leasqr (x(t:t+S), y(t:t+S), init, F, tolerance, max_iterations, weights, dp, dFdp, options);
  % Computation of the errors
  e2=f3-y(t:t+S)';

  % COMPUTATION OF THE MEAN-SQUARED ERRORS
  MSE_RW(t)=mean((y(t:t+S)-y(t-1:t+S-1)).^2);
  MSE_LPPL(t)=mean((f3-y(t:t+S)').^2);

  % COMPUTATION OF THE MEAN ABSOLUTE PERCENTAGE ERROR
```

```
 MAPE_RW(t)=mean((abs(y(t:t+S)-y(t-1:t+S-1)))./y(t:t+S));
 MAPE_LPPL(t)=mean(abs(f3-y(t:t+S)')./y(t:t+S)');

 % COMPUTATION OF THE AIC
  % Computation of the RW Max likelihood:
  edges=linspace(min(y(t-1:t+S-1)),max(y(t-1:t+S-1)),15);
  hist=histc(y(t-1:t+S-1),edges);
  hist(hist==0) = [];
  L_RW=prod(hist./sum(hist));

  % Storage of the RW log-likelihood:
  lL_RW(t)=-log(L_RW);

  % Determination of AIC_RW
  AIC_RW(t)=2*0-2*log(L_RW);

  % Determination of BIC_RW:
  BIC_RW(t)=log(length(y(t-1:t+S-1)))*0-2*log(L_RW);

  % Computation of the LPPL Max likelihood:
  edges2=linspace(min(f3),max(f3),15);
  hist2=histc(f3,edges2);
  hist2(hist2==0) = [];
  L_LPPL=prod(hist2./sum(hist2));

  % Storage of the LPPL log-likelihood:
  lL_LPPL(t)=-log(L_LPPL);

  % Determination of AIC_LPPL
  AIC_LPPL(t)=2*3-2*log(L_LPPL);

  % Determination of BIC_LPPL
  BIC_LPPL(t)=log(length(f3))*3-2*log(L_LPPL);

endfor

% Optionnal verification of the size of the output
Verif1=size(MSE_RW);
Verif2=size(MSE_LPPL);
Verif3=size(MAPE_RW);
Verif4=size(MAPE_LPPL);
Verif5=size(AIC_RW);
Verif6=size(AIC_LPPL);
Verif7=size(BIC_RW);
Verif8=size(BIC_LPPL);
Verif7=size(lL_RW);
Verif8=size(lL_LPPL);

% COMPUTATION OF THE BUBBLE INDEXES
r=MSE_RW./(MSE_RW+MSE_LPPL);
k=MAPE_RW./(MAPE_RW+MAPE_LPPL);
o=AIC_RW./(AIC_RW+AIC_LPPL);
p=BIC_RW./(BIC_RW+BIC_LPPL);
q=lL_RW./(lL_RW+lL_LPPL);
CBI=(r+k+o.*0.5+p.*0.5+q)/4;

figure(6)
hold on
%dates=[02/01/2014    03/01/2014    06/01/2014    07/01/2014    08/01/2014    09/01/2014
                      10/01/2014    13/01/2014    14/01/2014    15/01/2014    16/01/2014
                      17/01/2014    20/01/2014    21/01/2014    22/01/2014    23/01/2014
                      24/01/2014    27/01/2014    28/01/2014    29/01/2014    30/01/2014
                      31/01/2014    03/02/2014    04/02/2014    05/02/2014    06/02/2014
                      07/02/2014    10/02/2014    11/02/2014    12/02/2014    13/02/2014
                      14/02/2014    17/02/2014    18/02/2014    19/02/2014    20/02/2014
                      21/02/2014    24/02/2014    25/02/2014    26/02/2014    27/02/2014
                      28/02/2014    03/03/2014    04/03/2014    05/03/2014    06/03/2014
                      07/03/2014    10/03/2014    11/03/2014    12/03/2014    13/03/2014
                      14/03/2014    17/03/2014    18/03/2014    19/03/2014    20/03/2014
                      21/03/2014    24/03/2014    25/03/2014    26/03/2014    27/03/2014
                      28/03/2014    31/03/2014    01/04/2014    02/04/2014    03/04/2014
                      04/04/2014    07/04/2014    08/04/2014    09/04/2014    10/04/2014
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29/04/2015 30/04/2015 04/05/2015 05/05/2015 06/05/2015
07/05/2015 08/05/2015 11/05/2015 12/05/2015 13/05/2015
14/05/2015 15/05/2015 18/05/2015 19/05/2015 20/05/2015
21/05/2015 22/05/2015 25/05/2015 26/05/2015 27/05/2015
28/05/2015 29/05/2015 01/06/2015 02/06/2015 03/06/2015
04/06/2015 05/06/2015 08/06/2015];
axis([0 T-S 0.00 1]);
plot(r,'k-','linewidth',1)
plot(k,'g-','linewidth',1)
plot(o,'m-','linewidth',1)
plot(p,'b-','linewidth',1)
plot(q,'y-','linewidth',1)
plot(CBI,'r-','linewidth',3)
fronter=ones(1,length(r))*0.5;
plot(fronter,'k:')
legend('Bubble Indicator based on MSE','Bubble Indicator based on MAPE','Bubble Indicator based on AIC','Bubble Indicator based on BIC',
'Bubble Indicator based on Likelihood','Consolidated Bubble Indicator')
hold off
```

```
% Comparison of the CBI and the CAC 40

figure(7)
hold on
plot(10000*CBI,'Color',[.5 .5 .5],'linewidth',1)
fronter=ones(1,length(r))*0.5*10000;
plot(fronter,'k:')
plot(exp(y(S:T)),'r-','linewidth',2)
hold off

%-----------------------------------------------------------------------------------------------------------------------------------
% Assessment of the White Noise hypothesis in the residuals of the Random Walk model
%-----------------------------------------------------------------------------------------------------------------------------------

% Restauration of the original dataset on the year 2015
y=[4252.29          4111.36     4083.50     4112.73     4260.19     4179.07     4228.24     4290.28     4223.24     4323.20
                    4379.62     4394.93     4446.02     4484.82     4552.80     4640.69     4675.13     4624.21     4610.94
                    4631.43     4604.25     4627.67     4677.90     4696.30     4703.30     4691.03     4651.08     4695.65
                    4679.38     4726.20     4759.36     4751.95     4753.99     4799.03     4833.28     4830.90     4862.30
                    4886.44     4882.22     4910.62     4951.48     4917.32     4869.25     4917.35     4963.51     4964.35
                    4937.20     4881.95     4997.75     4987.33     5010.46     5061.16     5028.93     5033.42     5037.18
                    5087.49     5054.52     5088.28     5020.99     5006.35     5034.06     5083.52     5033.64     5062.22
                    5074.14     5151.19     5136.86     5208.95     5240.46     5254.12     5218.06     5254.35     5224.49
                    5143.26     5187.59     5192.64     5211.09     5178.91     5201.45     5268.91     5173.38     5039.39
                    5046.49     5081.97     4974.07     4981.59     4967.22     5090.39     5027.87     4974.65     4961.86
                    5029.31     4993.82     5012.31     5117.30     5133.30     5146.70     5142.89     5117.17     5083.54
                    5182.53     5137.83     5007.89     5025.30     5004.46     5034.17     4987.13     4920.74     4857.66];
y=log(y);

T=length(y);
res_RW=y(2:T)-y(1:T-1);
res_RW=res_RW';

% COMPUTATION OF THE AUTOCORRELATION FUNCTION OF THE RESIDUALS
% Computation of the empirical autocovariances
K=20;
gama_emp_res=autocovec_emp(res_RW,K);

% Computation of the empirical autocorrelations
var_res=mean(res_RW.*res_RW)-mean(res_RW)^2; % Empirical variance of the residuals
ACF_res=gama_emp_res/var_res;

% Plot of the ACF

% Definition of the lower/upper bounds

% Level of significance (using the quantile alpha of the SN distribution):
alpha=0.99; % Level of significance=(1-alpha)*2
quantile=norminv(alpha);

% ACF bounds for looking at correlation:
ACF_lconf=-(quantile/sqrt(length(residuals2)))*ones(K);
ACF_upconf=(quantile/sqrt(length(residuals2)))*ones(K);

figure(8)
hold on
bar(ACF_res, 'b')
plot(ACF_lconf, 'r')
plot(ACF_upconf, 'r')
title('Autocorrelogram of the residuals of the Random Walk Model')
legend('ACF','Confidence interval at 99%')
hold off

% Computation of the coefficient of determination of the Random Walk Model
R2_RW=(corr(y(1:T-1),y(2:T))^2)

% Computation of the mean of the residuals

Mean_Res_RW=mean(res_RW)

% Plot of two histograms of the residuals:
half=floor((T-1)/2);
```

```
Res_RW1=res_RW(1:half-1);
edgesRW1=linspace(min(Res_RW1),max(Res_RW1),10);
histRes_RW1=histc(Res_RW1,edgesRW1);

Res_RW2=res_RW(half:T-1);
edgesRW2=linspace(min(Res_RW2),max(Res_RW2),10);
histRes_RW2=histc(Res_RW2,edgesRW2);

figure(9)

hold on
subplot(2,1,1)
bar(histRes_RW1, "edgecolor", "r")
title("Histogram of the residuals of the random walk model, first part of 2015")
subplot(2,1,2)
bar(histRes_RW2,'b')
title("Histogram of the residuals of the random walk model, second part of 2015")
hold off

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                                                          %
% Assessment of the link between the Bubble and the Quantitative Easing                        %
%                                                                          %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% IMPLEMENTATION OF THE DATA

% Objective variable:
% Consolidated Bubble Indicator (computed using a bandwidth of 75 trading days, daily data)
% For the period from 17/04/2014 to 08/06/2014 (i.e. 289 observations)

Z=[0.3828           0.3834      0.39689 0.3965  0.3972  0.39651 0.39642 0.38353 0.38672 0.38685
                    0.38977     0.39417 0.40016 0.41776 0.42319 0.42536 0.41461 0.41797 0.41754
                    0.41867     0.40577 0.40901 0.41581 0.42523 0.43835 0.44639 0.44222 0.4455
                    0.4443      0.44498 0.44511 0.44378 0.4456  0.44364 0.43074 0.44471 0.44604
                    0.44716     0.45243 0.45574 0.46479 0.47015 0.45862 0.45288 0.45819 0.46532
                    0.46339     0.46159 0.4587  0.45209 0.45049 0.44137 0.44863 0.44754 0.45172
                    0.43295     0.42908 0.42462 0.41352 0.41242 0.41411 0.42017 0.42099 0.42282
                    0.43944     0.44596 0.44894 0.44203 0.44975 0.44879 0.44372 0.44289 0.4464
                    0.4377      0.43605 0.43507 0.4337  0.43341 0.45062 0.44964 0.45064 0.44984
                    0.4467      0.44528 0.44043 0.4309  0.42393 0.41205 0.40629 0.40212 0.39307
                    0.38501     0.38023 0.37657 0.37309 0.36971 0.36604 0.36126 0.35444 0.34796
                    0.34386     0.33976 0.33702 0.33436 0.33263 0.31857 0.33049 0.32909 0.31791
                    0.32827     0.33381 0.31886 0.31992 0.31904 0.31883 0.32123 0.32766 0.33885
                    0.34144     0.35047 0.35505 0.35238 0.35145 0.3445  0.33444 0.32864 0.33089
                    0.30631     0.32928 0.34282 0.35774 0.35876 0.35776 0.35842 0.35955 0.35981
                    0.35974     0.3589  0.35998 0.35931 0.36106 0.36345 0.36346 0.36111 0.35971
                    0.35861     0.35811 0.35393 0.34993 0.3468  0.34523 0.34125 0.34539 0.35328
                    0.35379     0.35357 0.35338 0.35272 0.35196 0.34984 0.34827 0.3471  0.34713
                    0.34802     0.34478 0.34713 0.34746 0.34669 0.34984 0.35181 0.33767 0.3518
                    0.35522     0.34319 0.35975 0.36055 0.35911 0.35721 0.35498 0.35352 0.36381
                    0.36545     0.36557 0.37085 0.36912 0.36342 0.36991 0.3735  0.37554 0.38219
                    0.38809     0.39244 0.38903 0.39581 0.40028 0.39837 0.39755 0.3933  0.38754
                    0.39679     0.39399 0.38483 0.38694 0.38175 0.37782 0.35919 0.35809 0.35595
                    0.35462     0.35319 0.36588 0.36608 0.3638  0.36085 0.35634 0.35936 0.34326
                    0.34216     0.33841 0.33765 0.34866 0.34841 0.34948 0.3515  0.35239 0.35292
                    0.35451     0.35568 0.35747 0.36056 0.36435 0.36854 0.37384 0.37981 0.38596
                    0.39657     0.40514 0.40796 0.41992 0.4328  0.43601 0.44275 0.44963 0.44293
                    0.43987     0.43615 0.43548 0.42988 0.43063 0.43097 0.4338  0.43859 0.44749
                    0.44961     0.45442 0.45566 0.44764 0.45702 0.46736 0.45581 0.4654  0.46945
                    0.46392     0.47375 0.46226 0.45025 0.44397 0.45063 0.44988 0.44174 0.4345
                    0.44335     0.44532 0.44755 0.45035 0.44462 0.438   0.43227 0.43046 0.43073
                    0.4293      0.4273  0.4339  0.43677 0.43275 0.43258 0.43609 0.44095 0.43589];
Z=Z';

% Explanatory variable:
% Quantitative easing variable (Yield spot rate, spread between the 10-year and 3-month maturity)
```

```
Q=[1.612463     1.634729    1.61781   1.59247   1.572343  1.594816  1.629407  1.595577  1.573901  1.553856
            1.54301     1.585085  1.55938   1.562728  1.555721  1.520003  1.470599  1.432438  1.444192
            1.471548    1.475678  1.503178  1.493511  1.486783  1.498871  1.460103  1.430892  1.440125
            1.469478    1.464496  1.526936  1.571419  1.569603  1.412705  1.456352  1.516735  1.52289
            1.525893    1.530036  1.491713  1.524682  1.520224  1.438856  1.483071  1.468142  1.470699
            1.408887    1.386301  1.395784  1.389747  1.374307  1.394545  1.428617  1.372689  1.375467
            1.340949    1.340532  1.309486  1.322594  1.321481  1.305113  1.302632  1.297179  1.254942
            1.260874    1.283207  1.245981  1.298047  1.275362  1.252401  1.231261  1.264729  1.30294
            1.287121    1.238595  1.277678  1.236322  1.202444  1.171102  1.185259  1.191649  1.166156
            1.128459    1.12168   1.142834  1.128483  1.115344  1.112446  1.101657  1.088028  1.074488
            1.022062    1.01016   1.034721  1.03747   1.086414  1.094884  1.130592  1.105079  1.098243
            1.177903    1.191061  1.18748   1.234788  1.203186  1.203539  1.210062  1.248072  1.199311
            1.171001    1.152794  1.136847  1.120479  1.118733  1.099113  1.086825  1.042142  1.063861
            1.080125    1.063328  1.059984  1.041026  1.035515  1.035425  1.04623   0.995241  0.917447
            0.952177    1.02829   1.007751  1.029814  1.031253  1.040437  1.040367  1.027544  1.034257
            1.034426    0.995795  0.986901  0.986919  0.952025  0.965234  0.969705  0.968947  0.958687
            0.960695    0.936738  0.937473  0.928667  0.937974  0.929321  0.972517  0.938034  0.907568
            0.912391    0.884451  0.867618  0.829238  0.817639  0.822453  0.868462  0.880291  0.871756
            0.889599    0.8394    0.81842   0.809098  0.791288  0.761665  0.764123  0.706962  0.712878
            0.753478    0.742315  0.73621   0.730619  0.731747  0.685694  0.670915  0.672072  0.634161
            0.613805    0.551026  0.588626  0.624306  0.637822  0.58217   0.57949   0.519546  0.527843
            0.518103    0.51742   0.531588  0.721405  0.77567   0.573381  0.558386  0.581429  0.586786
            0.555561    0.539154  0.520111  0.549372  0.59637   0.607614  0.599874  0.590807  0.655356
            0.648659    0.57752   0.560232  0.588589  0.609215  0.628764  0.611086  0.586888  0.614863
            0.621523    0.589848  0.543053  0.576812  0.567808  0.6174    0.598729  0.584981  0.611724
            0.550907    0.463359  0.445931  0.474843  0.501344  0.501062  0.508126  0.46127   0.459021
            0.421496    0.479174  0.511974  0.518446  0.506071  0.501061  0.489726  0.478711  0.489113
            0.508472    0.503483  0.495305  0.497398  0.508527  0.486541  0.471467  0.433845  0.436899
            0.421738    0.379561  0.415078  0.451737  0.506763  0.5013    0.525426  0.521796  0.619899
            0.69865     0.783131  0.846781  0.930203  0.959355  0.910234  0.93845   1.057013  1.010806
            1.066254    1.001768  1.018614  0.969064  0.970267  1.001141  0.983869  0.985153  0.946323
            0.9083      0.901585  0.850868  0.858524  1.024952  1.163387  1.195665  1.225564  1.228924];
Q=Q';

n=length(Z); %=length(Q) !

%-----------------------------------------------------------------------------------------------------------------------
% Computation of linear regressions
%-----------------------------------------------------------------------------------------------------------------------

%cQ=cos(Q);
QQ=[ones([n,1]),Q];
%cQQ=[ones([n,1]),cQ];

Beta1=ols(Z,QQ)
%Beta2=ols(Z,cQQ)

%[Beta1]=polyfit(Q,Z,1)
%[Beta2]=polyfit(cQ,Z,1)

Res_1=Z-(Beta1(1)+Q*Beta1(2));
%Res_2=Z-(Beta2(1)+cQ*Beta2(2));

figure(10)
hold on
plot(Q,Z,'k*','linewidth',1)
plot(Q,(Beta1(1)+Q*Beta1(2)),'r-','linewidth',2)
%plot(Q,(Beta2(1)+cQ*Beta2(2)),'g-','linewidth',1)
legend('CBI (with a width of 75)','Linear estimation')
hold off

% Computation of the coefficient of determination for the simple regression
R2_4=(corr(Q,Z))^2

% Computation of the coefficient of determination for the cosinus regression
%R2_5=(corr(cQ,Z))^2

%-----------------------------------------------------------------------------------------------------------------------
% Non-Parametric Kernel Regression
%-----------------------------------------------------------------------------------------------------------------------

% NADARAYA-WATSON ESTIMATION
```

```
% Definition of the Silverman's h:

varZ=mean(Z.*Z)-mean(Z)^2; % Empirical variance
sig_emp=sqrt(varZ);
Sh=1.06*sig_emp*n^(-0.2)

% Definition of the Nadaraya-Watson estimator:

function phi=reg_ker(arg,X,Y,h)
d=length(arg);
phi=zeros(size(arg));
for k=1:d
  N=sum(Y.*dnorm((arg(k)-X)/h));
  D=sum(dnorm((arg(k)-X)/h));
  phi(k)=N/D;
endfor
endfunction

NW_est1=reg_ker(Q,Q,Z,Sh);
NW_est2=reg_ker(Q,Q,Z,Sh+0.05);
NW_est3=reg_ker(Q,Q,Z,Sh+0.10);

% Plot of the Nadaraya-Watson non-parametric estimation

figure(11)
hold on
plot(Q,Z,'*','Color',[.5 .5 .5])
plot(Q,NW_est1,'b+','linewidth',1)
plot(Q,NW_est2,'m+','linewidth',1)
plot(Q,NW_est3,'r+','linewidth',1)
%plot(Q,0.6+cos(Q+2.24)/4,'k+')
title('Nadaraya Watson Non-Parametric Estimation')
legend('Data','Nadaraya-Watson Estimator, for h=Sh','Nadaraya-Watson Estimator, for h=Sh+0.05','Nadaraya-Watson Estimator, for h=Sh+0.10')
hold off
```