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An upper tailed dependent structure to capture Wrong Way Risk

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Master Thesis

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Abstract

In the master dissertation, my objective is to analyze the impact of different copulas on Potential Future Exposure and Expected Exposure for a simple portfolio. The thesis work explains the importance of Wrong Way Risk in the finance and shows the copula approach to capture WWR.

I implemented a methodology based on the linear correlation between the exposure of the counterparty and default time, which relates to the hazard rate of the counterparty. I proposed a copula approach to evaluate Expected Exposure and Potential Future Exposure under the assumption of wrong way risk, using upper tailed copula models. Gumbel, Galambos and Husler-Reiss copulas have been implemented to analyze the copula-based exposure weights for the fixed time 3, 5 and 10 years.

The methodology has been applied to a simple portfolio with the stock of Exxon Mobile Corporation (NYSE: XOM), CDS quotes to determine survival probability. The results of the dissertation are to demonstrate the best upper-tailed copula function to capture WWR for long and short time horizons.

I believe this thesis is a good beginning of my research in Risk analysis for the future PhD dissertation.

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Chapter 1

Introduction

CVA (credit value adjustment) is a measure of the counterparty default (the difference between theoretical (risk-free) portfolio and a real portfolio). Another definition of CVA is a market value of the counterparty credit risk. For instance, there is a chance that a counterparty of the derivatives dealer will default before the maturity of the contract and the dealer adjust this value (CVA) to protect himself from losses. Furthermore, CVA is a derivative by itself, which is difficult to calculate due to the fact of its depending on the net value of the derivatives of the counterparty.

DVA is debit value adjustment, opposite to CVA, the cost to the counterparty if the dealer is default. The benefit to the counterparty is a positive amount due to the fact that the counterparty does not need to give back a loan to the dealer.

The calculation of CVA and DVA is extremely important in the financial life, under the new regulations, such as Basel III, published by Basel Committee on Banking Supervision in December 2010, the standard for CVA has been changed, and there is more and more interest in this subject for a Risk Management (in Basel II regulation the wrong-way risk should be specifically addressed by banks in the risk management practice from 2001).

The classical calculation of CVA assumes no correlation between the counterparty's probability of default and an exposure of the dealer, but when such a correlation occurs it is referred to Wrong Way Risk (WWR) (is a risk when exposure of the counterparty is negatively correlated with the credit quality of the counterparty). In contrast, Right Way Risk occurs when the correlation between the exposure and the credit quality of the counterparty is positive. It is very important that CVA framework can handle the effect of WWR and RWR.

In this thesis work I would like to concentrate on Wrong Way Risk and the copula approach to analyze the correlation between the time to default and exposures.

1.1 CVA

The explanation of CVA is given in the Introduction, but for a better understanding of its importance, the difference with VaR and intuition behind it, I would like to consider some examples and look at the important terminology concerning CVA.

Example 1.1:

Let us consider the relationship between 2 financial institutions, a bank "Standard" and a fund "Mountain". The bank gives a loan to the fund, in this case we calculate a *Default Risk* for a bank, if its counterparty defaults, it loses an amount of money the fund still own to the bank minus some collateral (an asset given to the bank by the fund to secure the loan) and, of course, the regular payments (yield) as a compensation of the given risk.

What is the Counterparty Risk? When does it occur? Generally speaking, *Counterparty Risk* is a type of credit risk, occurs when the counterparties have a derivative contract. In this case, the fund "Mountain" holds an option, written by the bank "Standard". *Market risk* is the risk of changing the price of the option, while Counterparty Risk refers to the default of the bank or its counterparty and makes no obligations of the option contract.

There are several types of exposure considered for the analysis; I would like to give a breath explanation of each of them.

Example 1.2.

Swap contract:

Vanilla interest rate swap (IRS) is a contract, when the counterparties exchange the interest rates (floating rate and fixed rate) for the given amount – a principal.

Considering the bank "Standard" is a floating-rate payer within 3-month EURIBOR rate and the fund is a fixed-rate payer, the principal is 1 million euro.

Definition 1.1. *CE – Credit Exposure* is an immediate loss if the counterparty defaults. In our example at the time of the contract it is zero, because IRS has a nominal zero.

Definition 1.2. *EE – Expected Exposure* is an average exposure for the future date and exists only for the gain counterparty, because the gain exposes from counterparty default.

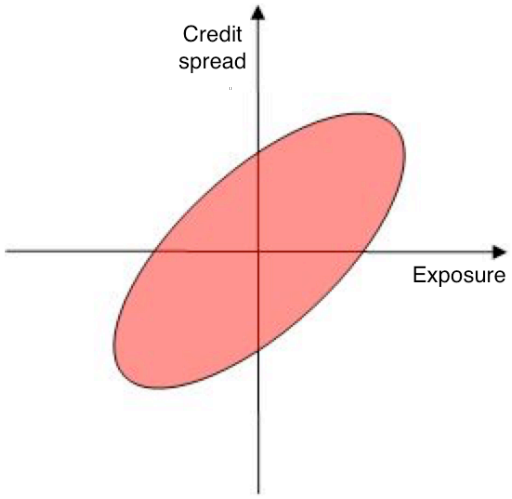
Definition 1.3. *PFE – Potential Future Exposure* is a credit exposure for the future day with the confident interval (the worst exposure is in the future). For example, PFE with 2% confident interval means that the bank will have this PFE amount in the future date with the probability no less then 98%.

Note 1.1: The definition of PFE is very close to the definition of Value-at-Risk (VaR) with several differences:

- PFE is determined for long term horizons (several years), while VaR is computed for several days;
- PFE associates with a gain but VaR predicts the worst-case loss.

1.2 Wrong Way Risk

The Introduction explains what are Wrong Way Risk and Right Way Risk; I would like to illustrate different situations, when WWR and RWR occur. Considering the credit spread and the exposure, we obtain following dependence:

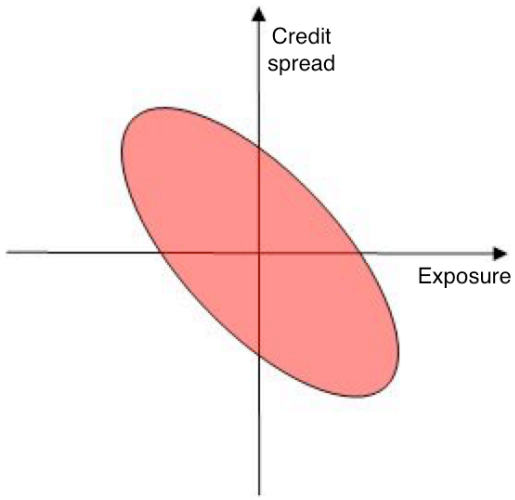


Picture 1.1. Wrong Way Risk

Example 1.3: WWR.

Let us consider 2 counterparties: a car producer and a bank. The producer enters into a forward contract with the bank to sell 1000 cars in one year for the price of 10 000 € per a car. In a year time, the price of the car decreases to 9 000 € and, meanwhile, the credit quality of the car producer became worst, while the bank obtains a gain: paying spot price 9 000 € and receiving a fixed price 10 000 € from the car producer. In this scenario the exposure of the bank increases and the credit quality of its counterparty decreases, this is exactly the situation when WWR occurs. Furthermore, the CVA on the forward contract will increase more than if those two effects were independent.

An opposite case for CVA when the exposure decreases while the credit quality of counterparty becomes worst, may be illustrated on the Figure1.2:



Picture 1.2. Right Way Risk

An example of Right Way Risk is a long forward contract on oil for an airline company to protect itself from the rising price of the fuel.

Chapter 2

The theoretical framework

There are several approaches to calculate WWR (for example, copula approach of Hull&White (2012), Pykhti&Socolov (2013) extension of WWR to systematic wrong-way risk, Rosen&Saunders (2012) correlation-based single-factor model), but all of them rely on the dependence of exposure and default, most of them represent a linear dependence of the variables. The interest of the problem has started recently, mostly due to the new regulation Basel III, where WWR represents an extreme case of exposure and default dependence used for stress testing. In the risk management copula is used to check robustness, to analysis of panic effect on the portfolio profit/loss distribution, it checks for estimation the probability distribution of losses, analyzing and pricing swap spread options.

In this thesis dissertation I will present a copula approach for WWR, comparing several copula models by illustrating dependence between Expected Exposure/Potential Future Exposure and Default Time. This approach was described in the recent paper of Boecker&Brunnbauer (2014), where the authors described different copula models (Clayton, Frank, Gauss and Gumbel) to illustrate path-consistent WWR.

In this Chapter I would like to introduce the concept of copula, describe several classes of copulas, provide examples using copulas in mathematics and finance and present 3 copula models: Gumbel, Galambos and Husler-Reiss. The choice of those particular copulas is following from the title of this work: the copula models are upper-tailed, and they are often used in quantitative finance. The pictures of copula functions and density functions of copulas will be illustrated in this Chapter. Every copula function has one parameter and to compare several models it is necessary to a chose the parameters according to the given measure. Usually this measure is a correlation coefficient such as Spearman's rank correlation coefficient or Kendall's tau (a measure of statistical dependence between random variables).

2.1 Copula model

Copula is a mathematical tool using in statistics to analyze dependence between random variables. Although correlation is used for normal distribution, copula works with skewness (in the finance the distributions are most likely to be skewed). For example, copula is applied for estimating Value-at-Risk of a portfolio by conditional copula approach, using Extreme Value Theory and copulas to evaluate Market Risk, pricing multi-asset options and credit derivatives. A paper by Chang C. Chiou and Ruey S. Tsay (2008) "A Copula-based Approach to Option Pricing and Risk Assessment" explains the importance of copula theory in Finance.

Definition 2.1. *m-dimensional copula* for a random vector $(Y_1, Y_2, \ldots, Y_m)$ with continuous marginal cumulative distribution functions $F_i(y) = P[Y_i < y]$ is a function such as:

$$C : [0,1]^m \to [0,1]$$

and

$$C(u_1, u_2, \ldots, u_m) = P[U_1 \le u_1, U_2 \le u_2, \ldots, U_m \le u_m],$$

where $(U_1, U_2, \ldots, U_m) = (F_1(Y_1), F_2(Y_2), \ldots F_m(Y_m))$ is a random vector with uniformly distributed marginals.

There are many types of copula available, depending on the choice of the function *C(u,v)*. The copula functions usually have parameters to control the strength of dependence. Some popular parametric copula models are described below.

Archimedean class of copula could be represented as

$$C(u,v) = \phi^{-1}(\phi(u) + \phi(v))$$

where $\phi(u,v)$ is continuous, strictly decreasing and convex function.

Archimedean copulas are very popular because they are easy to calculate and they allow analysing high-dimensional models using one parameter to control the strength.

Some examples of Archimedean copulas:

Gumbel copula – asymmetric copula, which shows the dependence on the positive tail;

Clayton copula – asymmetric copula, which shows the dependence on the negative tail;

Franc copula – symmetric copula.

In the book "Tail Approximation of Value-at-Risk under Multivariate Regular Variation" by Y.Sun and H.Li (2010) you could find an application of Archimedean copulas in Risk Management.

Moreover, there is a family of copulas called *Gaussian*, which does not have an explicit formula, constructing from multivariate normal distribution in R^2 for example:

$$C_R^{Gauss}(u,v) = \phi_R(\phi^{-1}(u), \phi^{-1}(v))$$

for $R \in R^{2\times 2}$ - correlation matrix, inverse cumulative distribution function for standard normal ϕ^{-1} and ϕ_R - joint cumulative distribution function for multivariate normal distribution with mean zero-vector and covariance matrix R.

In the paper "On Default Correlation: A Copula Function Approach" (2000) by David X. Li there was the first appearance of the Gaussian copula applied to CDOs, and this tool became popular for financial organizations to correlate associations between multiple securities.

Elliptical copula is another family of copulas; from the name of the family it is clear that the copula from this class has an elliptical distribution. The advantage of this class of copulas is that the model could identify different levels of correlation between the marginals. Examples of Elliptical family are Multivariate Normal distribution, t-student distribution, which are used to analyse Value-at-Risk.

I would like to obtain a model, which is precisely in the tails of the returns distribution, and this class of copulas is called *Extreme-Value*. Extreme-Value copula models use often in modelling of financial risk. The copula function has following formula:

$$C(u,v) = uv^{A[\log u/\log uv]},$$

where *A* is a function such as:

$$A:[0,1] \to [0.5;1]$$

The copula approach also gives the possibility for the analysis of extreme values in the general multivariate case. That is the reason why copulas are popular in statistical applications (high-dimensional) and also it is easy to model and estimate the distribution of random variables by estimating marginals and copulas separately.

In this paper I will consider upper-tail copula: Gumbel, Galambos and Husler-Reiss models. For each of the model coefficient of upper-tail dependence (measure for extreme co-movements in the upper tail) has to be calculated to compare results of the modelling.

Very important role in copula modelling is played by the *density function* (partial derivative of copula function with respect to each argument):

$$\phi(u,v) = \frac{\partial^2}{\partial u \partial v} C_\alpha(u,v)$$

for $(u,v) \in [0,1]^2$. I will show in the next chapter a plot of density function where the tail is visible.

2.1.1 Gumbel Copula

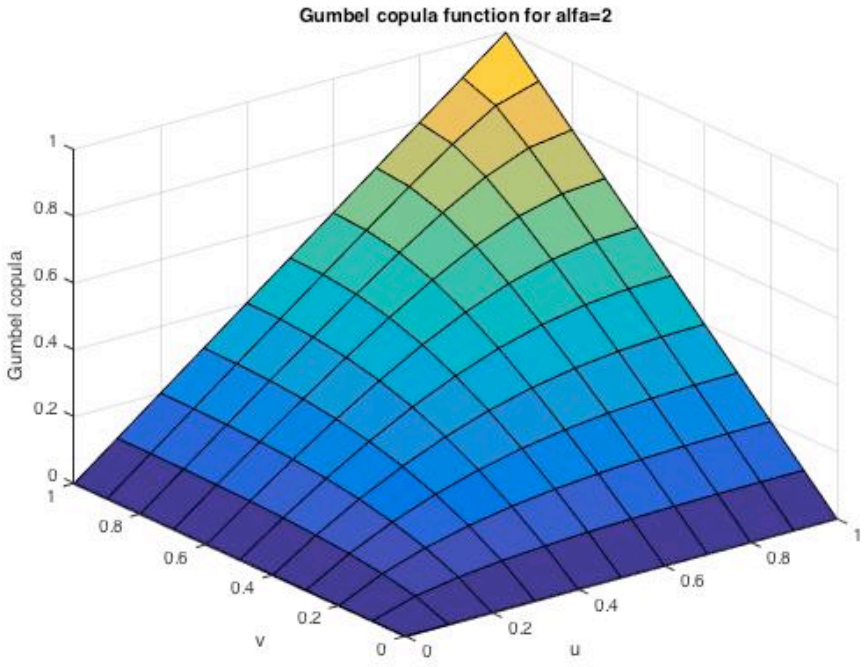
The Gumbel copula is an asymmetric copula, which shows better dependence in the positive tail than in the negative. It is an upper-tailed copula. I will consider 2-dimentional Gumbel copula model:

$$C_\alpha(u,v) = \exp\{-[(-\ln u)^\alpha + (-\ln v)^\alpha]^{1/\alpha}\} \qquad (2.1)$$

where:

$$\alpha \in [1,\infty)$$

The illustration of copula is given by *(2.1)*:



Picture 2.1. **Gumbel copula function with** $\alpha = 2$

Calculating partial derivatives of the copula function, I obtain following result for the density function:

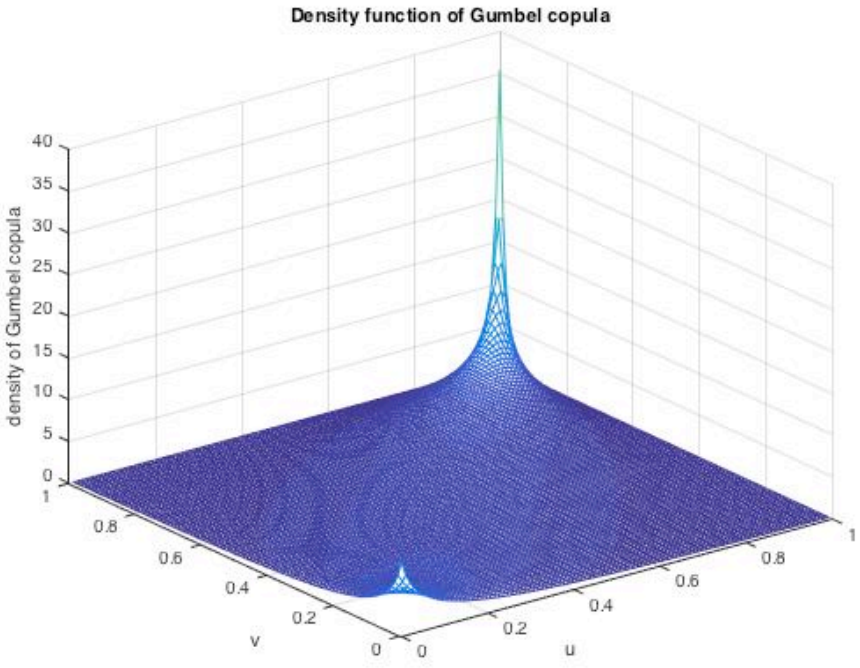
$$\frac{\partial}{\partial u} C_\alpha(u,v) = \frac{(-\ln u)^{\alpha-1}}{u}\left[(-\ln u)^\alpha + (-\ln v)^\alpha\right]^{\frac{1}{\alpha}-1} C_\alpha(u,v)$$

$$\frac{\partial^2}{\partial u \partial v} C_\alpha(u,v) = -\frac{\left[(\ln u)(\ln v)\right]^{\alpha-1}}{uv}\left(\frac{1}{\alpha}-1\right)\alpha\left[(-\ln u)^\alpha + (-\ln v)^\alpha\right]^{\frac{1}{\alpha}-2} C_\alpha(u,v) +$$

$$+\frac{(\ln u \ln v)^{\alpha-1}}{uv}\left[(-\ln u)^\alpha + (-\ln v)^\alpha\right]^{\frac{2}{\alpha}-2} C_\alpha(u,v) =$$

$$= \frac{\left[(\ln u)(\ln v)\right]^{\alpha-1}}{uv}\left[(-\ln u)^\alpha + (-\ln v)^\alpha\right]^{\frac{1}{\alpha}-2} C_\alpha(u,v)\left[(\alpha-1) + \left[(-\ln u)^\alpha + (-\ln v)^\alpha\right]^{\frac{1}{\alpha}}\right]$$

The illustration of the density function of Gumbel copula with parameters $u \in [0;1]$, $v \in [0;1]$ and $\alpha = 2$:



Picture 2.2. Gumbel density function

Picture 2.2 shows the "tails" of Gumbel copula, the tail around (0,0) is much smaller, than around (1,1). This is why this copula model belongs to the class of upper-tailed copulas.

Kendall's τ (a rank correlation to measure the dependence between random variables) for Gumbel copula is:

$$\tau(\alpha) = 1 - \frac{1}{\alpha}$$

The coefficient of tail dependence:

$$\lambda_U = 2 - 2^{1/\alpha}$$

For illustrative purposes, I have used $\alpha = 2$ and the coefficient of tail dependence $\lambda_U = 0.5858$

2.1.2 Galambos Copula

Galambos copula is another upper-tailed copula with the following function:

$$C(u,v) = uv\exp\left[\left\{(\ln u)^{-\theta} + (\ln v)^{-\theta}\right\}^{-\frac{1}{\theta}}\right], \qquad (2.2)$$

where $\theta \in [0;+\infty)$ - Galambos copula's coefficient.

The calculation of the density function:

$$\frac{\partial C}{\partial u} = v\exp\left[\left\{(\ln u)^{-\theta} + (\ln v)^{-\theta}\right\}^{-\frac{1}{\theta}}\right]\left[1 + \left\{(\ln u)^{-\theta} + (\ln v)^{-\theta}\right\}^{-\frac{1}{\theta}-1}(\ln u)^{-\theta-1}\right]$$

$$\frac{\partial^2 C}{\partial u \partial v} = \exp\left[\left\{(\ln u)^{-\theta} + (\ln v)^{-\theta}\right\}^{-\frac{1}{\theta}}\right] + \exp\left[\left\{(\ln u)^{-\theta} + (\ln v)^{-\theta}\right\}^{-\frac{1}{\theta}}\right](\ln v)^{-\theta-1} +$$

$$+\exp\left[\left\{(\ln u)^{-\theta} + (\ln v)^{-\theta}\right\}^{-\frac{1}{\theta}}\right]\left\{(\ln u)^{-\theta} + (\ln v)^{-\theta}\right\}^{-\frac{1}{\theta}-1}(\ln u)^{-\theta-1} +$$

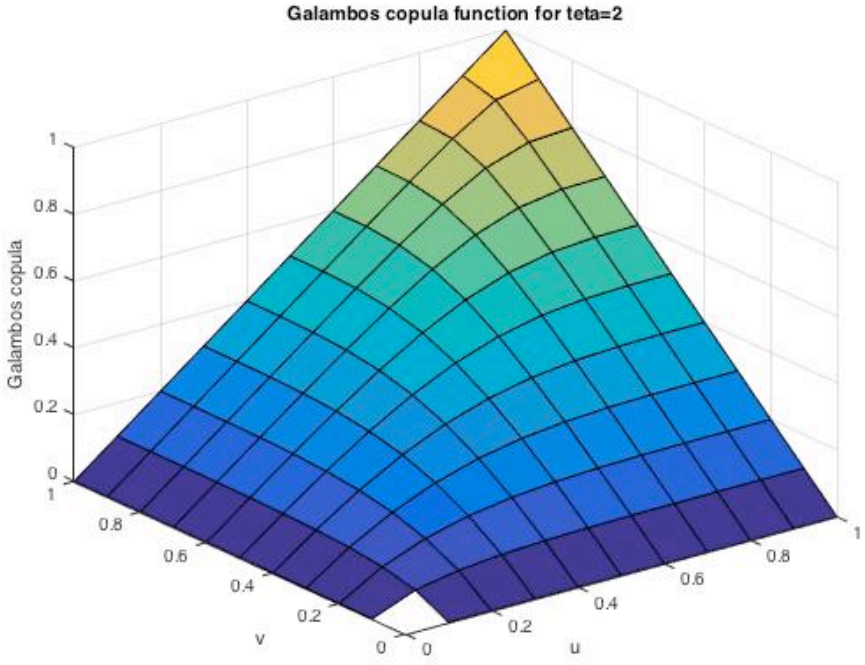
$$+\exp\left[\left\{(\ln u)^{-\theta} + (\ln v)^{-\theta}\right\}^{-\frac{1}{\theta}}\right](\ln v)^{-\theta-1}\exp\left[\left\{(\ln u)^{-\theta} + (\ln v)^{-\theta}\right\}^{-\frac{1}{\theta}}\right] +$$

$$+\exp\left[\left\{(\ln u)^{-\theta} + (\ln v)^{-\theta}\right\}^{-\frac{1}{\theta}}\right]((\ln u)^{-\theta-1}\left\{(\ln u)^{-\theta} + (\ln v)^{-\theta}\right\}^{-\frac{1}{\theta}-2}(1+\theta)(\ln v)^{-\theta-1})$$

The coefficient of upper-tail dependence is given by:

$$\lambda_U = 2 - 2^{1/\theta}$$

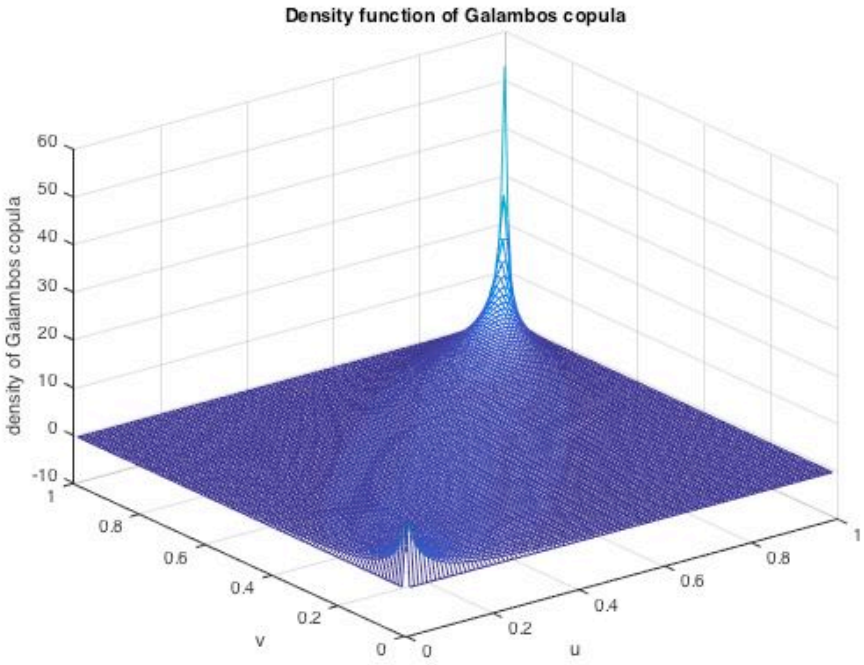
Using coefficient $\theta = 2$ I obtained the same upper-tail coefficient $\lambda_U = 0.5858$ as for Gumbel copula, it is very useful to compare all models to have the same coefficient, as it is difficult to obtain a formula for Kendall's coefficient of correlation, and even impossible for the next copula, I decide to use the same upper-tail coefficient for all models to obtain the same result.



Picture 2.3. Galambos copula function with $\theta = 2$

Picture 2.3 shows some uncertainty near (0,0), but this fact does not affect future calculations, because the chosen scale is small and I exclude this point for all the models.

The density function of Galambos copula has the same form as the density function of Gumbel copula, but the “tail” is bigger, it goes up to 55, compare with 35 for Gumbel copula.



Picture 2.4. Galambos density function

2.1.3 Husler-Reiss Copula

The Husler-Reiss copula is an upper-tailed copula, which can be represented by the following formula:

$$C(u,v) = \exp\left[\log u\psi\left\{\frac{1}{\delta}+\frac{1}{2}\delta\log(\frac{\log u}{\log v})\right\}+\log v\psi\left\{\frac{1}{\delta}+\frac{1}{2}\delta\log(\frac{\log v}{\log u})\right\}\right] \quad (2.3)$$

$\delta \in [0;+\infty)$

ψ - standard normal cumulative distribution function:

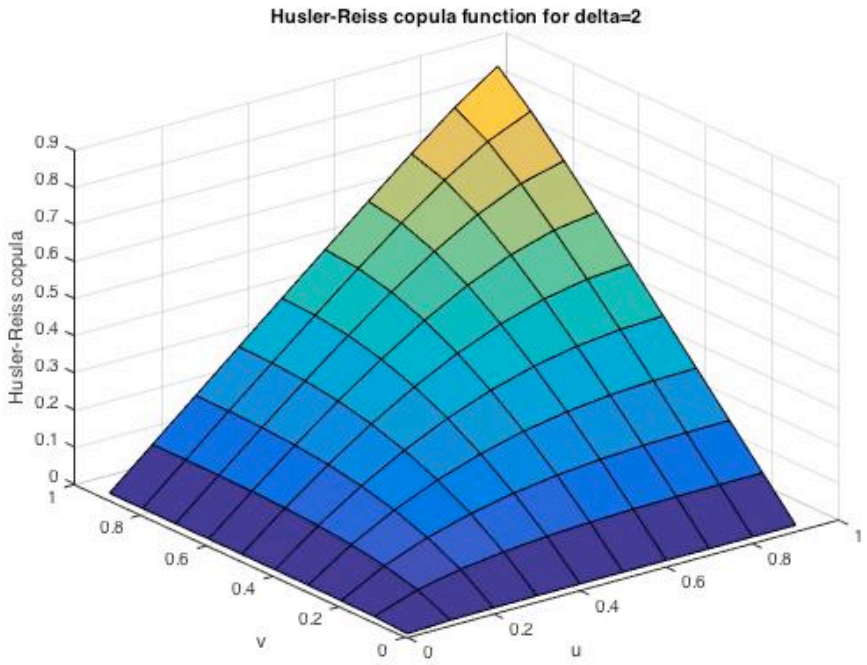
$$\psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}\,dt$$

Upper-tail dependence coefficient is:

$$\lambda_U = 2 - 2\psi(\frac{1}{\delta})$$

No formula exists for Kendall's correlation coefficient for Husler-Reiss copula model. Using upper-tail coefficient for previous models and equalise it to Husler-Reiss upper-tail dependence coefficient I obtained $\delta = 1.83 \approx 2$

The graphic representation of Husler-Reiss copula:



Picture 2.5. Husler-Reiss copula function with $\delta = 2$

According to *Figure 2.5*, Husler-Reiss copula for the given copula coefficient $\delta = 2$, the peak of the copula function is when *u*=1 and *v*=1, and it looks very much like Gumbel and Galambos copula models. Considering those 3 models, they should give similar results, but we will see later that it is not exactly right. A small notice concerning Husler-Reiss copula: the uncertainty around *u*=0 or *v*=0 and *u*=1 or *v*=1, for this reason I use the intervals: $u \in [0+\varepsilon; 1-\varepsilon]$ and $v \in [0+\varepsilon; 1-\varepsilon]$.

Chapter 3

The model

Considering formula for CVA reported in the Basel III regulation Basel Committee on Banking Supervision (2011), the model presents the evaluation of CVA as dependence between credit risk of the counterparty and the asset price movements as an extreme dependence called WWR. The copula approach for WWR has been explained in the book "CVA and Wrong Way Risk" by Hull&White, "Credit valuation adjustment and wrong way risk" by U.Cherubini and "Path-consistent Wrong-Way Risk' by Boecker&Brunnbauer.

The time to default has been calculated according to bootstrapping algorithm from J. Gregory "Counterparty Credit Risk" book using pricing formulas for CDSs to define mathematically the hazard rate of default.

3.1 CVA formula and copula approach

Assuming that the bank is default free, the formula for CVA for counterparty portfolio is:

$$CVA = E\left[LGD(\tau)\max(V(\tau),0)D(\tau)\right], \qquad (3.1)$$

where

$\tau > 0$ – random default time,
LGD(t) – loss given default of the counterparty,
D(t) – discount factor,
V(t) – portfolio value,

Definition 3.1. *Loss given default (LGD)* is the amount of money lost by the bank when the borrower is default. It is not necessary the amount of the loan, borrowed by the bank to the company, as the bank may hold collateral. One method to calculate LGD in the time of default is to compare actual total losses with the total potential exposure.

Loss Given Default of the counterparty is constant due to default free assumption. Discount factor is $D(t) = e^{-rt}$, where r is a risk-free interest rate. For simplicity of the model, let us consider $\widetilde{V}^{+}(t) = \max(D(t)V(t),0)$ with *t>0* and distribution function of the time of counterparty's default $F(t) = P(\tau \le t)$ obtaining following result:

$$CVA = LGD\int_{0}^{+\infty} E(\tilde{V}^{+}(s) \mid \tau = s)\, dF(s) \qquad (3.2)$$

By the definition of Expected Exposure (EE) for fixed time $s \in [0;+\infty)$ (see *Definition 1.2)* it is following:

$$EE_s = E\left[\left(\widetilde{V}^{+}(s)\middle|\tau = s\right)\right] \qquad (3.3)$$

From *(3.2)* and *(3.3)* we can see that CVA can be expressed as a function of expected Exposure:

$$CVA = LGD\int_{0}^{+\infty} EE_s\, dF(s) \qquad (3.4)$$

Furthermore, WWR is a risk that occurs when there is dependence between discounted portfolio value and time of default. This dependence could be represented by Expected Exposure formula.

Considering default time of the counterparty $\tau > 0$ and its distribution function $F(t) = P(\tau \le t)$ within the distribution function of discount portfolio value $G_s(\widetilde{v}) = P(\widetilde{V}(s) \le \widetilde{v}$ where $\widetilde{v} \in R$ we can apply the concept of 2-dimensional copula to receive following function:

$$C_s(G_s(\widetilde{v}), F(t)) = P(\widetilde{V}(s) \le \widetilde{v}, \tau \le t) \qquad (3.5)$$

For fixed time $s \in [0;+\infty)$

The copula model helps to analyze the dependence between exposure and default time by using their distribution functions. Furthermore, the analysis of the copula $C_s(G_s(\widetilde{v}), F(t))$ with negative dependence of default time and the exposure responds to Wrong Way Risk.

3.1.1 Expected Exposure EE

Before using formula for expected exposure, I would like to remind Sklar's Theorem for copula:

Theorem 3.1. *Sklar's Theorem*
Let *H* be a distribution function with marginal distribution *F(u)*, *G(v)* for random variables *u, v.* There exists a copula *C* such as:

$$H(u,v) = C(F(u),G(v)) \tag{3.6}$$

If *F(u), G(v)* – continuous than *C* is unique.

Using the formula (3.3) and Sklar's Theorem, expected exposure could be written as

$$EE_s = \int_{-\infty}^{+\infty} H_s(\tilde{v} \mid s)\tilde{v}^+ \, d\tilde{v}$$

and considering

$$H_s(\tilde{v} \mid s) = H_s(\tilde{v}, s) / f(s)$$

$$H_s(\tilde{v}, s) = \phi_s(G_s(\tilde{v}), F(s)) g_s(\tilde{v}) f(s),$$

where $f(s)$, $g_s(\tilde{v})$ are the density functions of default time and discounted portfolio value, respectively.

The formula for Expected Exposure I will use has the following form now:

$$EE = \int_{-\infty}^{+\infty} \phi_s\left(G_s(\tilde{v}), F(s)\right)\tilde{v}^+ dG_s(\tilde{v}) \tag{3.7}$$
$$\tilde{v}^+ = \max(\tilde{v}, 0)$$

3.1.2 Potential Future Exposure PFE

I would like to remind the reader that Potential Future Exposure (PFE) is a maximum amount of exposure, which is expected to occur on a future date with a given degree of statistical confidence.

Formula for PFE is following:

$$PFE_s = \max(\tilde{V}_s^+(s) \mid \tau = s) \tag{3.8}$$

This formula is logically evaluated from the definition of PFE, and it actually may be defined from the probability of the value of discounted portfolio, using confident interval (the result is shown in formula *(3.11).*

PFE is also called sensitivity of risk and, as I mentioned in the Introduction, it is associated with positive value rather than negative.

Also note that PFE is used for long periods (several years or decades) comparing with VaR, which is calculating for short periods (10 days). In this work I will analyze the data on the long period, using PFE not VaR.

3.2 Monte Carlo Simulations for EE and PFE

I would like to generate a set of unconditional discounted portfolio values $\{\tilde{v}_{s,1}, \tilde{v}_{s,2}, ..., \tilde{v}_{s,N}\}$ for any fixed time $s \in [0, +\infty)$. Applying formula *(3.7)*, for discrete time and using the generated set I have a formula for Expected Exposure, which is easy to calculate.

When N is big enough we could apply a Strong Law of Large Numbers:

$$\frac{1}{N}\sum_{i=1}^{N} \phi(G_s(\tilde{v}_{s,i}), F(s))\tilde{v}_{s,i}^{+} \xrightarrow{a.s.} E[\phi(G_s(\tilde{V}(s)), F(s))\tilde{V}^{+}(s)] = EE_s \qquad (3.9)$$

$N \to +\infty$.

For the Potential Future Exposure, I use a quintile to define it:

$$\frac{1}{N}\sum_{j=1}^{N} 1_{\{\tilde{v}_{s,j}^{+} > x\}} \phi(G(\tilde{v}_{s,j}), F(s)) \xrightarrow{a.s.} E[1_{\{\tilde{V}^{+}(s) > x\}} \phi(G_s(\tilde{V}(s)), F(s))] \qquad (3.10)$$

By formula *(3.8)* follows that PFE may be calculated as a quintile, I chose 95% confident interval, PFE could be derived from

$$P(\tilde{V}^{+}(s) > x \mid \tau = s) = 0.95 \qquad (3.11)$$

but it is following from formula *(3.10)*.

The formulas *(3.10)* and *(3.9)* will be used for further analysis, the results from the next Chapter were obtained precisely by those formulas. The set of unconditional discounted values will be generated using Random Walk approach; it will be also shown in the next Chapter.

Random Walk is a random process with a fixed step. In finance random walk is used to calculate the price of options: we suppose that the price of stock takes a random and unpredictable path. The probability of stock to go down is equal to the probability of the price goes up. It means using random walk it is impossible to outperform the market without assuming additional risk.

Using random walk to simulate the portfolio value, I generate a set of unconditional discounted portfolio values $\{\tilde{v}_{s,1}, \tilde{v}_{s,2}, ..., \tilde{v}_{s,N}\}$ with the starting point at the initial discounted portfolio value $\tilde{v}_s$ at the fixed time *s.* The principal is an analogue of the random walk for option pricing.

3.3 Copula weight approach

Definition 3.1. *Exposure weight* is a function such as:

$$w_s : R \to [0; +\infty)$$

For the given time *s>0,* the impact of WWR on the exposure is a rebalance of probability mass to each portfolio value $\tilde{v} \in R$:

$$\tilde{v} \to w_s(\tilde{v}) = \phi_s(G_s(\tilde{v}), F(s)) \qquad (3.12)$$

Using this approach, I can look the dependence of copula weight for the portfolio value at the fixed time; as a result the exposure weight function has bigger weight to the larger portfolio values. In the next chapter I will show the behavior of Gumbel, Galambos and Husler-Reiss copula exposure weights for the fixed time at 3, 5 and 10 years.

3.4 Hazard rate and survival probability

Default probability is probability of counterparty default in the future, the biggest credit risk – the highest probability of default. Thinking in this direction, it is logical to suppose that the default probability is inversely proportional to the rating of the counterparty. And this is one way to calculate the default rate – using the S&P or Moody's rating. Also note is that default time depends on the CDS type of contract: the buyer is paying a fixed rate (premium X_{CDS}) to protect the seller from the default of the counterparty. The default time is a random variable whose distribution is derived from CDS quotes using a bootstrapping algorithm. In fact, it estimates the default probability by backing it out (numerically) from the CDS pricing formula. *Hazard rate* is a ratio of the probability density function to the survival function. I would like to remind that survival provability is:

$$S(\tau) = 1 - D(\tau)$$

where τ -default time

$D(\tau)$ - probability of default.

Hazard rate (*h*) defines the probability of default in a short time, *dt* and it could be represented as:

$$D(\tau) = 1 - S(\tau) = 1 - e^{-h\tau}$$

The relationship between CDS premium and constant hazard rate is following:

$$h \approx \frac{X_{CDS}}{1-\beta}$$

where X_{CDS} - CDS premium (%)

β - recovery rate (%)

Assuming that CDS premium for all maturities are equal, I could consider T as a maturity time of CDS:

$$D(T) = 1 - e^{-\frac{X_T^{CDS}}{1-\beta}T} \approx -\frac{X_T^{CDS}}{1-\beta}T \qquad (3.13)$$

(3.13) is a formula to calculate default probability using CDS premium.

Chapter 4

The results

Catching WWR I have analyzed several stocks for oil and mining companies (Exxon, Total, Chevron, Billiton, Carrizo, DVN – Devon Energy Corp and CHK – Chesapeake Energy Corporation).

The stock prices were downloaded from Bloomberg, the oldest date I found was for XOM: New York (Exxon Mobil Corp) from 29/07/1980 (35 years of the stock price), which I have used in my calculation (closing price every working day). In *Appendix A* are some stock prices: 5 years every first day of the month data to illustrate the price changes (*Table A.1*). The interest rate US was downloaded from the World Bank; it is also included in *Appendix A*. The hazard rate was extrapolated from CDS.

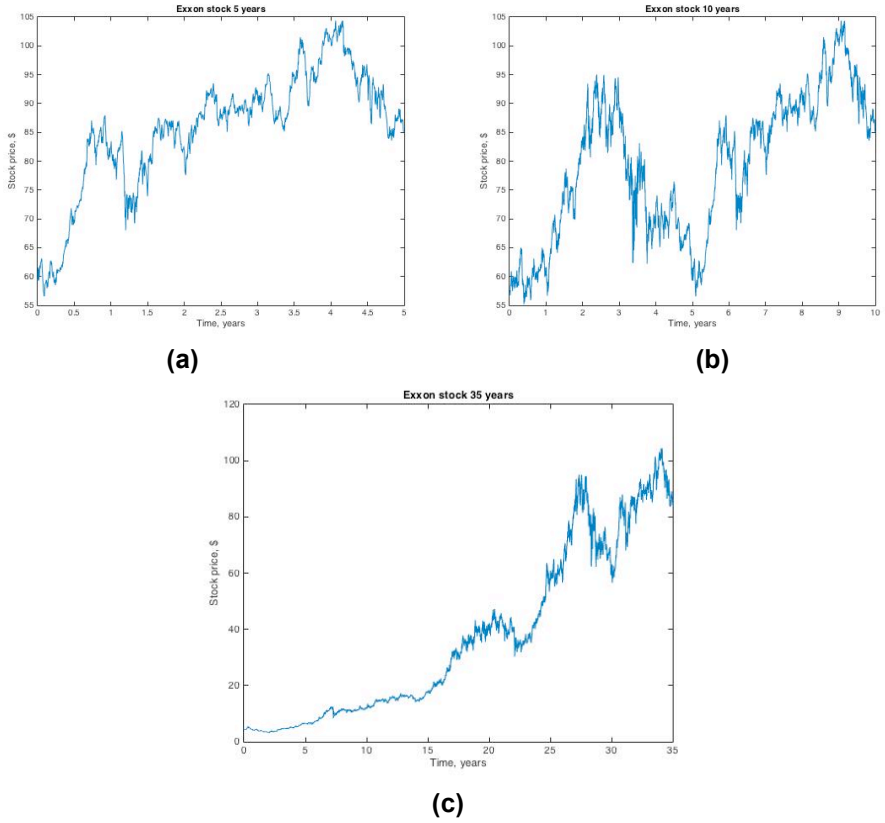
All the results are based on the theory from the previous Chapters, I used Galambos and Gumbel copulas to calculate Expected Exposure and Potential Future Exposure; Random Walk to simulate a set of unconditional discounted portfolio values; bootstrapping approach to calculate survival probability; Gumbel, Galambos and Husler-Reiss copulas to obtain exposure weights.

4.1 The data

Following the information from *Abstract*, I used is 5, 10 and 35 years stock price, but I would like to mention that the period of 35 years is a whole database, 10 years – last 10 years of the stock price and 5 years – the last 5 years of the stock.

I use the American stock of Exxon Mobil (an American multinational oil and gas corporation). I have downloaded the data from Bloomberg; the data has been available since 1980, 35 years ago from now. *Picture 4.1 (a)* shows the stock price for last 5 years (01/06/2010 – 29/05/2015), (*b)* stock price for last 10 years (01/06/1995 – 29/05/2015) and *(c)* every day closing price for 35 years (29/07/1985-29/05/2015). It is also could be found in *Appendix A*, due to the big data I will not include all 35 years. The data in *Appendix A* (*Table A.1*) is Exxon stock price and discounted stock price for the period of 5 years: monthly data (first working day of each month) from 01/06/10 to 29/05/15. *Table A2* shows interest rate of US for the period 1985-2014.

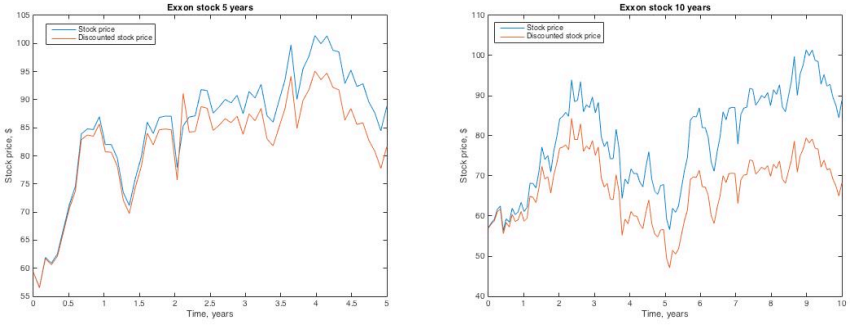
Picture 4.1 (c) shows a big movement in the price, for almost 35 years the price rose from 5.0703 to 92.8300; maximum price – 99.7500 and minimum price -3.6250.



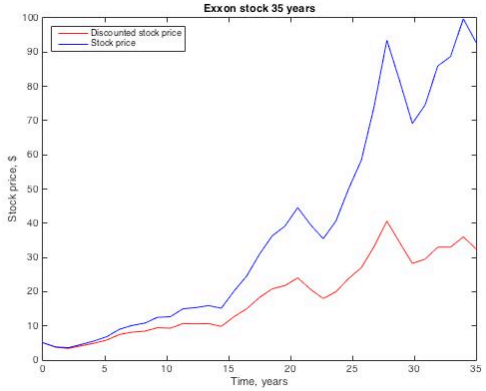
Picture 4.1. Exxon stock price (a) 5, (b) 10 and (c) 35 years

Before working with the stock it is necessary to discount its value. From the farmhouse book of Quantitative Finance we all know that “one dollar today is not equal to one dollar tomorrow!” I used the data of US interest rate from World Bank to discount the stock price as follow: for 5-years model I discounted my price forward to time t=0 (in this case it is 01/06/10), 10 years stock price was discounted to 01/06/1995 (it seems like we are at time t=0 calculating what was the price at 2015 in 1995) and 35-years data starting from 29/07/1985. The information about interest rate could be found in *Appendix A* (*Table A.2*). I used EXCEL to calculate discounted portfolio value by discount factor *D(t)* described in *Chapter 3.1*.

Picture 4.2 represents the stock price (in blue) and discounted stock price (in red). There is a significant difference between those paths, specially it is visible for the last *(c)* picture for 35 years.



(a) (b)



(c)

Picture 4.2. Exxon stock price and discounted price (a) 5, (b) 10 and (c) 35 years

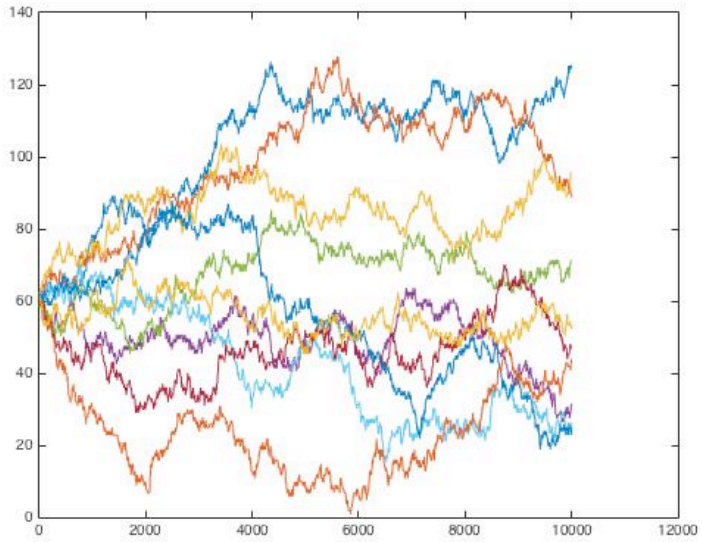
4.2 Random walk simulation

I have used random walk to simulate a set of discounted portfolio values $\{\tilde{v}_{s,1},\tilde{v}_{s,2},...,\tilde{v}_{s,N}\}$. For each discounted portfolio value $\tilde{v}_s$ the random walk algorithm simulate set of 100 possible paths (in this case a price of the asset). Although the example below shows set for N=10000, the result are not very different for simulations of N=100 but it makes easier and faster to calculate EE and PFE. I have used function 'RandDir' in Matlab to generate a random vector, it is not built-in, and the code for this function could be found in *Appendix B.*

The example in the *Picture 4.3* shows random walk for $s \in [0;10]$ with 10000 iterations. The table below has starting points for Random Walk from the *Picture 4.3.*

01/06/10	59.25		08/06/10	61.24
02/06/10	60.77		09/06/10	60.03
03/06/10	61.56		10/06/10	61.89
04/06/10	59.525		11/06/10	61.86
07/06/10	59.3		14/06/10	61.37

Table 4.1. Price of the asset 01/06/10-14/06/10



Picture 4.3. Random walk with 10000 simulations for 10 last prices of 5-years period

4.3 Default time

To derive the default probability curve I use a set of CDS for A rating from 31/03/2014. In the *Table 4.2* there are CDS from one day obtained via Bloomberg.

It is necessary to mention that CDS expires at specific date. IMM (International Monetary Market) dates are quarterly dates of each year, which some options and derivatives contracts use as a maturity day. According to CDS specialization, 20/12/2014 was a day of maturity for the CDS. 6 month CDS is actually 8.68-month contract. *Table 4.2* presents actual duration of the contracts (there were chosen 6m, 1y, 2y and 3y CDS).

I would like to obtain a formula for survival probability to use it for the period of time 35 years. In this case I use only prices at the chosen time. I applied bootstrapping algorithm from J. Gregory book "Counterparty credit risk. The new challenge for Global Financial Market" (pp. 147-155, 163) to calculate survival probabilities (1st column of 'Survival probability' in the *Table 4.2*).

Assuming a constant hazard rate, the formula for survival probability is following:

$$S(t) = e^{-\lambda t} \qquad (4.1)$$

The log-linear interpolation gives us the result $\lambda = 0.004$ and calculating Survival probability using the formula above, I obtained the 2nd column of 'Survival probability'. The results are similar but the advantage of the formula *(4.1)* is to use default probability for the longest period of the time.

Maturity	Time	Time in years	CDS quotes	Survival probability	
	0d	0	0	1	1
20/12/14	8m 17d	0.723288	0.0021	0.997499	0.9971
20/06/15	1y 2m 17d	1.221918	0.0025	0.994973	0.9951
20/06/16	2y 2m 17d	2.224658	0.0039	0.985761	0.9911
20/06/17	3y 2m 17d	3.224658	0.0053	0.972051	0.9872

Table 4.2. CDS quotes and Survival probability

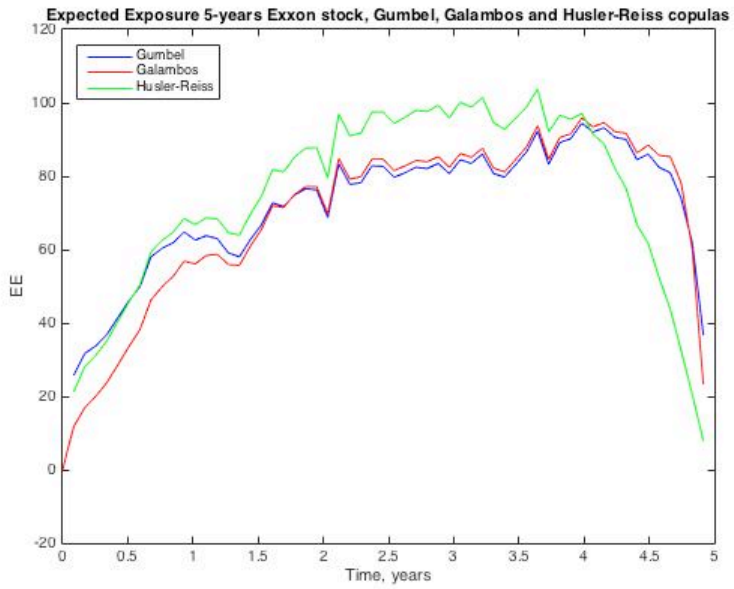
4.4 EE and PFE

The Expected Exposure as a function of Gumbel (blue), Galambos (red) and Husler-Reiss (green) copulas is shown in the *Picture 4.4*.

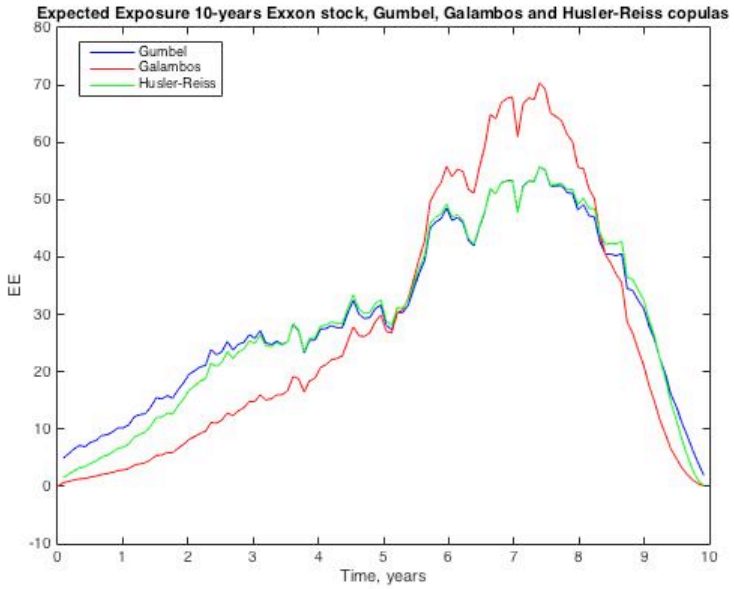
Picture 4.4 (a): Husler-Reiss model shows the greatest profile for the 4/5 first periods of time, sharply decreasing after 4-year period, while Gumbel and Galambos start to decease at the very end of the given period and have almost the same form during the period, except of the beginning, where Gumbel copula generates the largest start.

Picture 4.4 (b): The situation is completely different. Gumbel and Husler-Reiss follow the same path while Galambos copula produces a big peak at 7 years and increases much faster than other copulas.

It is seen from the pictures below that with the time the models behave differently, starting to increase sharply and have a peak at ¾ of the time.



(a)

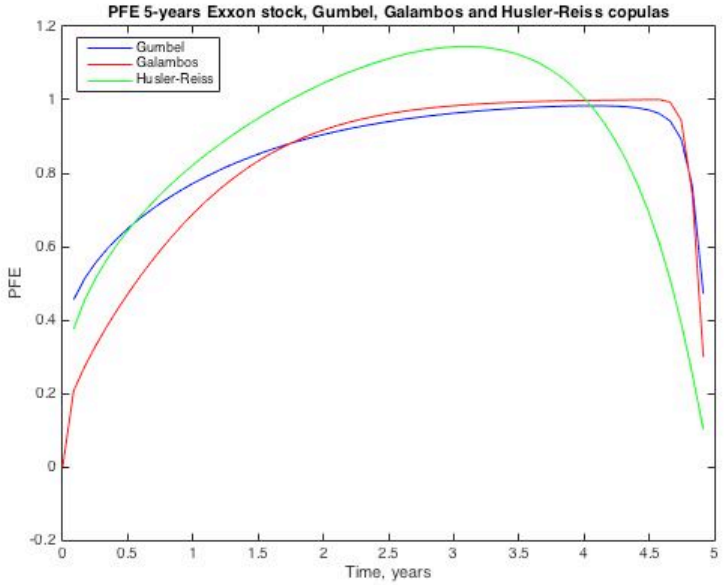


(b)

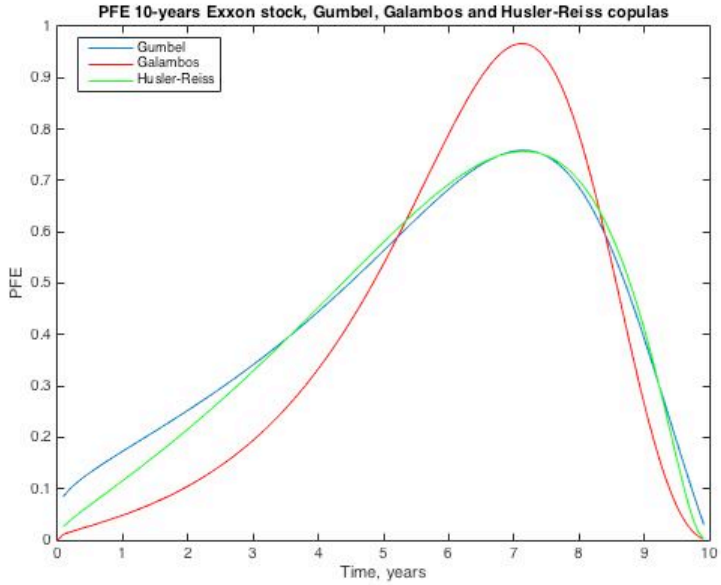
Picture 4.4. Expected Exposure as a function of Gumbel, Galambos and Husler-Reiss copulas

The very same situation can be observed for Potential Future Exposure: *Picture 4.5*. The behavior of the PFE for 5 years almost flat (except of the large increase at the beginning and big decrease at the end of the period).

Picture 4.5 (a): Husler-Reiss copula shows the different from 2 other copula profiles, while for PFE in *(b)* Galambos copula made a result of the biggest increase, and has a peak on 7 years like 2 other models, with the difference that it starts to have “tails” like normal or t-student distribution, it is not concave anymore.



(a)

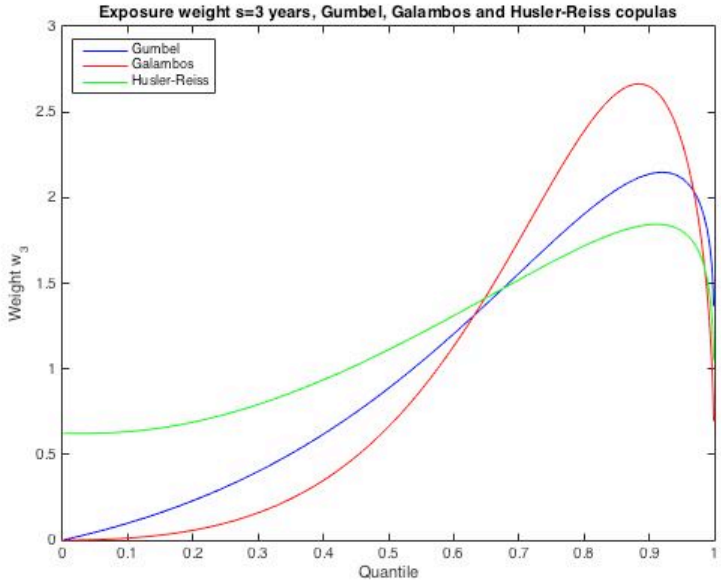


(b)

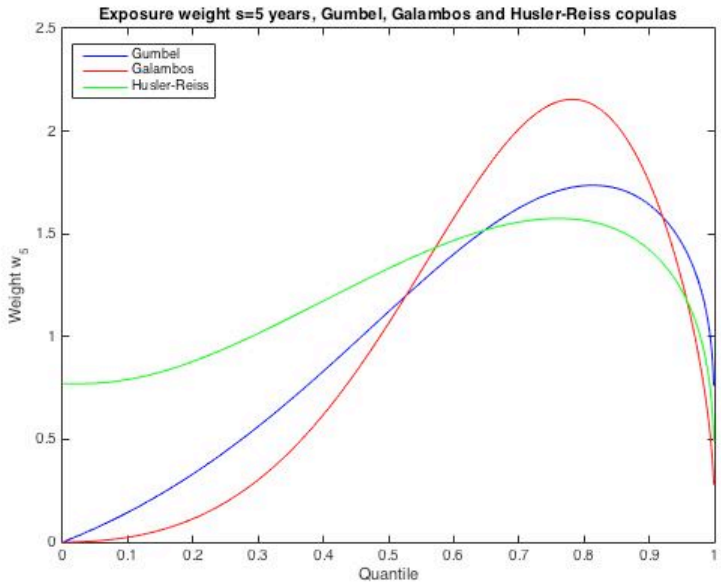
Picture 4.5. Potential Future Exposure as a function of Gumbel, Galambos and Husler-Reiss copulas

4.5 Copula-based Exposure Weights

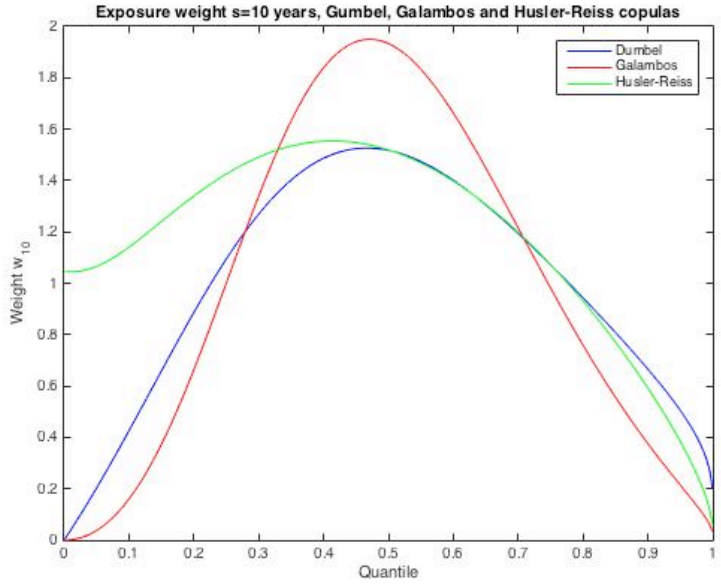
The results for copula-based exposure weights as a function of quantile of unconditional portfolio value are represented in the picture below:



(a)



(b)



(c)

Picture 4.6. Exposure weights for (a) 3, (b) 4 and (c) 10 years

The exposure weights for the fixed time 3 years has been extracted from 5-years database of stock prices, while for *s*=5 the model used 10 years of Exxon prices and for time 10 years it was required 35 years of stock.

According to the *Picture 4.6* for time 3 years *(a)* more mass is attributed to the higher quantiles, while for the time 10 years *(c)* the mass is in the middle quantile.

Moreover, Galambos copula has an extreme peak of the weight function for all horizons, while Gumbel starts to lose its position during the time.

According to Risk Management, the weight associated to the high unconditional exposure are lower than 1. For this reason the weight Husler-Reiss copula is not very useful in the analysis of exposure.

Chapter 5

Conclusion

The thesis work provides a method of the modelling of Wrong Way Risk, using general copula and weight copula approaches.

Some assumptions were made: constant hazard rate, constant Loss Given Default and parameters were chosen: copula parameters, confidence interval. Using other parameters and assumptions we could obtain different results.

Comparing different upper-tailed copula models for Expected Exposure and Potential Future Exposure, it is difficult to name the best copula model, all depends on the time frame, considering type of exposure. For example, Potential Future Exposure depends on the confidence interval, in my work I considered only 95% of confidence, the results may vary with the largest percentage.

Galambos copula yields to higher exposure values for the long period, while Husler–Reiss works better for small horizons. Gumbel copula has similar results to one of the copula, showing smaller EE and PFE.

Considering copula-based exposure weights, the greatest peak has Galambos copula, but all the models have the same movement with the time. Gumbel based exposure weights became weak with the time and even smaller than Husler-Reiss, it could be reasonable to make a conclusion that it is not a good copula model to analyse copula based exposure weights. With the time the conditional expected exposure is less affected by extreme values of unconditional exposure.

The results could be used for CVA analysis under the assumption of Wrong Way Risk. The research in WWR analysis is interesting and modern, there are more and more approaches to analyse this effect, I would like to research in this area in the future.

Appendix A

The database

The monthly stock price of Exxon Mobil Corp (XOM: New York), every first day of the month, closing price of the day, 5 years period 01/06/2010 – 01/05/2015. Discounter stock price has been calculated using real interest rate for US from World Bank. This Excel file is called `'exxon_5_years_short.xlsx'`

Stock price and discounted price, Exxon, 01/06/10 – 01/05/15

Date	Stock price	Discounted stock price
01/06/10	59.25	59.25
01/07/10	56.61	56.51572858
02/08/10	61.94	61.73387706
01/09/10	60.91	60.60621011
01/10/10	62.54	62.12445336
01/11/10	66.95	66.39440154
01/12/10	71.33	70.62025464
03/01/11	74.55	73.73444378
01/02/11	83.91	82.90909743
01/03/11	84.8	83.70473465
01/04/11	84.68	83.50274005
02/05/11	86.97	85.67518539
01/06/11	82.03	80.72796406
01/07/11	82.01	80.62761358
01/08/11	79.6	78.18001818
01/09/11	73.49	72.10687133
03/10/11	71.15	69.74113561
01/11/11	75.94	74.36188817
01/12/11	79.79	78.05378835
03/01/12	86	84.03056734
01/02/12	83.97	81.95138937
01/03/12	86.83	84.64382717

02/04/12	87.07	84.7788182
01/05/12	87.04	84.65079074
01/06/12	77.92	75.69277091
02/07/12	85.34	91.1168982
01/08/12	86.91	84.22904103
04/09/12	87.12	84.33411583
01/10/12	91.8	88.76084614
01/11/12	91.6	88.46419893
03/12/12	87.61	84.51213611
02/01/13	88.71	85.45209751
01/02/13	90.04	86.61046773
01/03/13	89.43	85.90192138
01/04/13	90.77	87.06562702
01/05/13	87.51	83.81984047
03/06/13	91.46	87.47925838
01/07/13	90.3	86.24747618
01/08/13	92.73	88.44303845
03/09/13	87.15	83.00333369
01/10/13	86	81.79209735
01/11/13	89.82	85.06999042
02/12/13	93.52	88.42681999
02/01/14	99.75	94.16046385
03/02/14	90.05	84.86245236
03/03/14	95.5	89.8486194
01/04/14	97.73	91.79353843
01/05/14	101.41	95.09138459
02/06/14	99.94	93.55691858
01/07/14	101.36	94.72821241
01/08/14	98.8	92.18194356
02/09/14	98.49	91.73968176
01/10/14	92.86	86.35151212
03/11/14	95.26	88.43578205
01/12/14	92.35	85.59147668
02/01/15	92.83	85.89307397
02/02/15	89.58	82.7479092
03/03/15	87.62	80.80261124
01/04/15	84.46	77.75877317
01/05/15	88.85	81.66424115

US interest rate 1985-2014 (%)

1985	6.526494713		2002	3.093304621
1986	6.178960114		2003	2.087708537
1987	5.619885098		2004	1.549966337
1988	5.612781691		2005	2.879449421
1989	6.723841029		2006	4.73854193
1990	6.070686754		2007	5.243852476
1991	4.982806426		2008	3.098402004
1992	3.882395302		2009	2.442585
1993	3.533042937		2010	1.999486298
1994	4.910002234		2011	1.162314282
1995	6.603302528		2012	1.425007606
1996	6.332967779		2013	1.736155482
1997	6.6071794		2014	5.20000

Appendix B

MATLAB code

B1. Defined copula functions and Density functions of copulas

```
% File gumbel.m:

function [ gumbel_copula ] = untitled( u,v,alfa )

% Gumbler copula with patameter alfa
gumbel_copula=exp(-((-log(u)).^alfa+(-
log(v)).^alfa).^(1/alfa))

end
```

```
% File galambos.m:

function [ galambos ] = untitled2( u,v,teta )

% Galambos copula with patameter teta
galambos=u.*v.*exp(((log(u)).^(-teta)+(log(v)).^(-teta)).^(-
1/teta))

end
```

```
% File huslerreiss.m:

function [ huslerreiss ] = untitled4( u,v,delta)
% Husler-Reiss copula with patameter delta
huslerreiss=exp(log(u).*normcdf(1/delta+0.5*delta*log(log(u).
/log(v)))+log(v).*normcdf(1/delta+0.5*delta*log(log(v)./log(u
))));

end
```

```
% File gumbel_density.m:

function [ gumbel_density ] = untitled( u,v,alfa )

% Density function of Gumbel copula:
gumbel_density=(exp(-((-log(u)).^alfa + (-
log(v)).^alfa).^(1/alfa)).*(-log(u)).^(alfa - 1).*(-
log(v)).^(alfa - 1).*((-log(u)).^alfa + (-
log(v)).^alfa).^(2/alfa - 2))./(u.*v) - (alfa*exp(-((-
log(u)).^alfa + (-log(v)).^alfa).^(1/alfa)).*(-log(u)).^(alfa
- 1).*(-log(v)).^(alfa - 1).*(1/alfa - 1).*((-log(u)).^alfa +
(-log(v)).^alfa).^(1/alfa - 2))./(u.*v);


end
```

```
% File galambos_density.m:

function [ galambos_density ] = untitled3( u,v,teta)

% Density function of Galambos copula:
galambos_density=exp(1./(1./log(u).^teta +
1./log(v).^teta).^(1./teta)) + exp(1./(1./log(u).^teta +
1./log(v).^teta).^(1./teta))./(log(u).^(teta +
1).*(1./log(u).^teta + 1./log(v).^teta).^(1./teta + 1)) +
exp(1./(1./log(u).^teta +
1./log(v).^teta).^(1./teta))./(log(v).^(teta +
1).*(1./log(u).^teta + 1./log(v).^teta).^(1./teta + 1)) +
exp(1./(1./log(u).^teta +
1./log(v).^teta).^(1./teta))./(log(u).^(teta +
1).*log(v).^(teta + 1).*(1./log(u).^teta +
1./log(v).^teta).^(2./teta + 2)) +
(teta.*exp(1./(1./log(u).^teta +
1./log(v).^teta).^(1./teta)).*(1./teta + 1))./(log(u).^(teta
+ 1).*log(v).^(teta + 1).*(1./log(u).^teta +
1./log(v).^teta).^(1/teta + 2))

end
```

```
% File huslerreiss_density.m:

function [ huslerreiss_density ] = untitled4( u,v,delta )

% Density function of Husler-Reiss copula:
huslerreiss_density=exp(-
log(u).*(erfc((2.^(1/2).*((delta.*log(log(u)./log(v)))/2 +
1/delta))/2)/2 - 1) -
log(v).*(erfc((2.^(1/2).*((delta.*log(log(v)./log(u)))/2 +
1/delta))/2)/2 -
1)).*((erfc((2.^(1/2).*((delta.*log(log(u)./log(v)))/2 +
1/delta))/2)/2 - 1)./u - (2.^(1/2).*delta.*exp(-
((delta.*log(log(u)./log(v)))/2 +
1/delta).^2/2))./(4*pi.^(1/2)*u) + (2.^(1/2).*delta.*exp(-
((delta.*log(log(v)./log(u)))/2 +
1/delta).^2/2).*log(v))./(4*pi.^(1/2).*u.*log(u))).*((erfc((2
.^(1/2).*((delta.*log(log(v)./log(u)))/2 + 1/delta))/2)/2 -
1)./v - (2.^(1/2).*delta.*exp(-
```

```
((delta.*log(log(v)./log(u)))/2 +
1./delta).^2/2))/(4*pi.^(1/2)*v) + (2.^(1/2).*delta.*exp(-
((delta.*log(log(u)./log(v)))/2 +
1/delta).^2/2).*log(u))./(4*pi.^(1/2).*v.*log(v))) - exp(-
log(u).*(erfc((2.^(1/2).*((delta.*log(log(u)./log(v)))/2 +
1/delta))/2)/2 - 1) –
log(v).*(erfc((2.^(1/2).*((delta.*log(log(v)./log(u)))/2 +
1/delta))/2)/2 - 1)).*((2.^(1/2).*delta.*exp(-
((delta.*log(log(v)./log(u)))/2 +
1/delta).^2/2))/(4*pi.^(1/2).*u.*v.*log(u)) +
(2.^(1/2).*delta.*exp(-((delta.*log(log(u)./log(v)))/2 +
1/delta).^2/2))./(4*pi.^(1/2).*u.*v.*log(v)) -
(2.^(1/2).*delta.^2.*exp(-((delta.*log(log(v)./log(u)))/2 +
1/delta).^2/2).*((delta.*log(log(v)./log(u)))/2 +
1/delta))./(8*pi.^(1/2).*u.*v.*log(u)) -
(2.^(1/2).*delta.^2.*exp(-((delta.*log(log(u)./log(v)))/2 +
1/delta).^2/2).*((delta.*log(log(u)./log(v)))/2 +
1/delta))./(8*pi.^(1/2).*u.*v.*log(v)))

end
```

B2. Models

B2.1. 5-years model

```
% EE and PFE for 5 years:

filename='exxon_5_years_short.xlsx';
data=xlsread(filename);
asset_price=(data(:,2))';
discount_asset_price=(data(:,4))';
data_time=data(:,1);
[d,s]=size(asset_price);

time_5=(0:5/(s-1):5);%5 years time

% Plot stock price 5 years:
figure
plot(time_5,asset_price)
hold on
plot(time_5,discount_asset_price)
title('Exxon stock 5 years')
xlabel('Time, years')
ylabel('Stock price, $')
hold off
```

```
%Calculating default probability:
lambda=0.004;
default_probab=1-exp(-lambda.*time_5);
for j=1:s;
default_distr(j)=sum(default_probab(1:j))/sum(default_probab)
;
end;

%Random walk:
N = s; % Dimensions
F = @(t,X) zeros(N,1); % Drift
G = @(t,X) eye(N); % diffusion
S = sde(F,G,'startState',discount_asset_price(1:s)'); % Start
at discounted asset price

%We now simulate 1000 steps of this process:
M=10;
simulated_set = S.simByEuler(M,'ntrials',1,'Z',@(t,X)
RandDir(N));

%density function of simulated portfolio value:
density_value=(1/M:1/M:1-1/M);

%Expected exposure:
for i=1:(s-1);
ee_gumbel(i)=sum((simulated_set(2:M,i)').*gumbel_density(dens
ity_value,default_distr(i),2))/M;
ee_galambos(i)=sum((simulated_set(2:M,i)').*galambos_density(
density_value,default_distr(i),2))/M;
ee_huslerriss(i)=sum((simulated_set(2:M,i)').*huslerreiss_den
sity(density_value,default_distr(i),2))/M;
end;

% Plot EE 5 years
figure
plot(time_5(1:end-1),ee_gumbel,'b');
hold on
plot(time_5(1:end-1),ee_galambos,'r');
hold on
plot(time_5(1:end-1),ee_huslerriss,'g')
title('Expected Exposure 5-years Exxon stock, Gumbel,
Galambos and Husler-Reiss copulas')
xlabel('Time, years')
ylabel('EE')
hold off

% Calculating Potential future exposure for Gambler copula:
for i=1:(s-1);
vv_gumbel(i)=sum(gumbel_density(density_value,default_distr(i
),2))/M;
pfe_gumbel(i)=quantile(vv_gumbel(i),0.95);
vv_galambos(i)=sum(galambos_density(density_value,default_dis
tr(i),2))/M;
pfe_galambos(i)=quantile(vv_galambos(i),0.95);
vv_huslerreiss(i)=sum(huslerreiss_density(density_value,defau
lt_distr(i),2))/M;
pfe_huslerreis(i)=quantile(vv_huslerreiss(i),0.95);
end;
```

```
% Plot PFE 5_years
figure
plot(time_5(1:end-1),pfe_gumbel,'b');
hold on
plot(time_5(1:end-1),pfe_galambos,'r');
hold on
plot(time_5(1:end-1),pfe_huslerreis,'g');
title('PFE 5-years Exxon stock, Gumbel, Galambos and Husler-
Reiss copulas')
xlabel('Time, years')
ylabel('PFE')
hold off
```

B2.2. 10-years model

```
% EE and PFE for 10 years:

filename='short_10_years.xlsx';
data_10=xlsread(filename);
asset_price_10=(data_10(:,2))';
data_time_10=data_10(:,1);
[d,c]=size(asset_price_10);
discount_asset_price_10=(data_10(:,3))';
time_10=(0:10/(c-1):10);%10 years time
% Plot stock price 10 years:
figure
plot(time_10,asset_price_10)
hold on
plot(time_10,discount_asset_price_10)
title('Exxon stock 10 years')
xlabel('Time, years')
ylabel('Stock price, $')
hold off

%Calculating default probability:
lambda=0.004;
default_probab_10=1-exp(-lambda.*time_10);
plot(time_10,default_probab_10);
for j=1:c;
default_distr_10(j)=sum(default_probab_10(1:j))/sum(default_p
robab_10);
end;

%Random walk:
N = c; % Dimensions
F = @(t,X) zeros(N,1); % Drift
G = @(t,X) eye(N); % diffusion
S = sde(F,G,'startState',(discount_asset_price_10(1:c))'); %
Start at discounted asset price
```

```
M=2;
simulated_set = S.simByEuler(M,'ntrials',1,'Z',@(t,X)
RandDir(N));

%density function of simulated portfolio value:
density_value=(1/M:1/M:1-1/M);

%Expected exposure:
for i=1:(c-1);
ee_gumbel_10(i)=sum((simulated_set(2:M,i)').*gumbel_density(d
ensity_value,default_distr_10(i),2))/M;
ee_galambos_10(i)=sum((simulated_set(2:M,i)').*galambos_densi
ty(density_value,default_distr_10(i),2))/M;
ee_huslerriss_10(i)=sum((simulated_set(2:M,i)').*huslerreiss_
density(density_value,default_distr_10(i),2))/M;
end;

% Plot EE 10 years
figure
plot(time_10(1:end-1),ee_gumbel_10,'b');
hold on
plot(time_10(1:end-1),ee_galambos_10,'r');
hold on
plot(time_10(1:end-1),ee_huslerriss_10,'g');
title('Expected Exposure 10-years Exxon stock, Gumbel,
Galambos and Husler-Reiss copulas')
xlabel('Time, years')
ylabel('EE')
hold off

% Calculating Potential future exposure for Gambler copula:
for i=1:(c-1);
vv_gumbel_10(i)=sum(gumbel_density(density_value,default_dist
r_10(i),2))/M;
pfe_gumbel_10(i)=quantile(vv_gumbel_10(i),0.95);
vv_galambos_10(i)=sum(galambos_density(density_value,default_
distr_10(i),2))/M;
pfe_galambos_10(i)=quantile(vv_galambos_10(i),0.95);
vv_huslerriss_10(i)=sum(huslerreiss_density(density_value,def
ault_distr_10(i),2))/M;
pfe_huslerriss_10(i)=quantile(vv_huslerriss_10(i),0.95);
end;

% Plot PFE 10 years
figure
plot(time_10(1:end-1),pfe_gumbel_10),'b';
hold on
plot(time_10(1:end-1),pfe_galambos_10,'r');
hold on
plot(time_10(1:end-1),pfe_huslerriss_10,'g');
title('PFE 10-years Exxon stock, Gumbel, Galambos and Husler-
Reiss copulas')
xlabel('Time, years')
ylabel('PFE')
hold off
```

```
% Function RandDir for Random Walk

function out = RandDir(N)

% Generate a random vector from the set {+/- e_1, +/-
e_2,..., +/- e_N}
% where e_i is the ith basis vector. N should be an integer.

I = round(ceil(2*N*rand));

if rem(I,2) == 1
    sgn = -1;
else
    sgn = 1;
end
out = zeros(N,1);

out(ceil(I/2)) = sgn*1;

end
```

B2.3. Copula weights

```
%Copula weights 3 years :

filename='exxon_5_years.xlsx';
data=xlsread(filename);
asset_price=(data(:,2))';
data_time=data(:,1);
[d,s]=size(asset_price);

time_5=(0:5/(s-1):5);%5 years time

%Calculating default probability:
lambda=0.004;
default_probab=1-exp(-lambda.*time_5);

for j=1:s;
default_distr(j)=sum(default_probab(1:j))/sum(default_probab)
;
end;

M=100;
%density function of simulated portfolio value:
density_value=(1/M:1/M:1-1/M);

% Calculation of exposure weights for 3 years:
weight_3_gumbel=gumbel_density(density_value,default_distr(50
6),2);
weight_3_galambos=galambos_density(density_value,default_dist
r(506),2);
weight_3_husler=huslerreiss_density(density_value,default_dis
tr(506),2);
```

```
% Plot Exposure weight s=3 years:
figure
plot(density_value(end:-1:1),weight_3_gumbel,'b');
hold on
plot(density_value(end:-1:1),weight_3_galambos,'r');
hold on
plot(density_value(end:-1:1),weight_3_husler,'g');
title('Exposure weight s=3 years, Gumbel, Galambos and
Husler-Reiss copulas')
xlabel('Quantile')
ylabel('Weight w_3')
hold off
```

```
%Copula weights 5 years :
filename='exxon_10_years.xlsx';
data_10=xlsread(filename);
asset_price_10=(data_10(:,2))';
data_time_10=data_10(:,1);
[d,c]=size(asset_price_10);
time_10=(0:10/(c-1):10);%10 years time
%Calculating default probability:
lambda=0.004;
default_probab_10=1-exp(-lambda.*time_10);
% plot(time_10,default_probab_10);
for j=1:c;
default_distr_10(j)=sum(default_probab_10(1:j))/sum(default_p
robab_10);
end;

% %Random walk:
% N = c; % Dimensions
% F = @(t,X) zeros(N,1); % Drift
% G = @(t,X) eye(N); % diffusion
% S = sde(F,G,'startState',discount_asset_price_10(1:c)); %
Start at discounted asset price

%We now simulate 10000 steps of this process:
M=1000;
% simulated_set = S.simByEuler(M,'ntrials',1,'Z',@(t,X)
RandDir(N));

%density function of simulated portfolio value:
density_value=(1/M:1/M:1-1/M);

% Calculation of exposure weights for 5 years:
weight_5_gumbel=gumbel_density(density_value,default_distr_10
(1258),2);
weight_5_galambos=galambos_density(density_value,default_dist
r_10(1258),2);
weight_5_husler=huslerreiss_density(density_value,default_dis
tr_10(1258),2);
```

```
% Plot Exposure weight s=5 years:
figure
plot(density_value(end:-1:1),weight_5_gumbel,'b');
hold on
plot(density_value(end:-1:1),weight_5_galambos,'r');
hold on
plot(density_value(end:-1:1),weight_5_husler,'g');
title('Exposure weight s=5 years, Gumbel, Galambos and
Husler-Reiss copulas')
xlabel('Quantile')
ylabel('Weight w_5')
hold off
```

```
%Copula weights 10 years :

filename='exxon.xlsx';
data_35=xlsread(filename);
asset_price_35=(data_35(:,2))';
data_time_35=data_35(:,1);
[d,k]=size(asset_price_35);

time_35=(0:35/(k-1):35);%35 years time

%Calculating default probability:
lambda=0.004;
default_probab_35=1-exp(-lambda.*time_35);

for j=1:k;
default_distr_35(j)=sum(default_probab_35(1:j))/sum(default_p
robab_35);
end;

M=100;

%density function of simulated portfolio value:
density_value=(1/M:1/M:1-1/M);

% Calculation of exposure weights for 10 years:
weight_10_gumbel=gumbel_density(density_value,default_distr_3
5(6271),2);
weight_10_galambos=galambos_density(density_value,default_dis
tr_35(6271),2);
weight_10_husler=huslerreiss_density(density_value,default_di
str_35(6271),2);

% Plot Exposure weight s=10 years:
figure
plot(density_value(end:-1:1),weight_10_gumbel,'b');
hold on
plot(density_value(end:-1:1),weight_10_galambos,'r');
hold on
plot(density_value(end:-1:1),weight_10_husler,'g');
title('Exposure weight s=10 years, Gumbel, Galambos and
Husler-Reiss copulas')
xlabel('Quantile')
ylabel('Weight w_1_0')
hold off
```

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