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Solow Model multy-country: A Network Economics Perspective

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Contents

Introduction

This thesis contributes to the study of the Solow-Swan model in the context of open economies, explicitly considering multiple interacting economies. The main model proposed in the thesis incorporates N interconnected economies, which are linked through a network system. The problem is approached from the perspective of an economic institution, which we may refer to as a Central Planner, tasked with finding a configuration of flows between the different nodes that can enhance global welfare, defined as the sum of consumption across the various economies. This Central Planner has the authority to determine the flows of income or capital between nodes, thereby orchestrating a redistribution within the entire network. The primary aim of this work is to investigate the redistributive effects on equilibrium allocations. Specifically, it seeks to examine whether the Central Planner can enhance efficiency, measured by aggregate global consumption, and/or reduce inequality among economies by guiding them towards an equilibrium that differs from the allocation under free capital mobility.

In the literature, the only available work on multiple interacting economies is Sorger (2003), who was the first to study a setting of N economies in an open framework, where the world interest rate is endogenously determined within the network system.

Before Sorger's contribution, only a few works addressed the Solow model in the context of open economies, focusing solely on the case of $N = 2$ economies. The first of these was Negishi (1965), who made an initial attempt to link two Solow economies with the aim of studying optimal fiscal policy for capital flows. His objective was to maximize net gross income, leading him to advocate for capital gains taxation in the debtor country, while the creditor country should subsidize capital gains to incentivize foreign investment in its capital.

In Manning (1975) the objective variable subject to maximization was consumption, taking into account the saving behaviour of representative households. He identified close connections between traditional golden rule of Solow-

Swan model and golden rule of two-country Solow open economy model.
The seminal work on the Solow model with two countries in an open economy context was Ruffin (1979), who formalized and consolidated earlier studies into a more streamlined model specification, even accounting for variations in the neoclassical production function. He concluded that the steady-state in an open economy is always more advantageous than that in a closed economy.
All these models assumed homogeneity in depreciation rates, an assumption strongly criticized by Sorger due to its empirical inconsistency. He also argued that the robustness of these models fails when heterogeneity in this parameter is considered, mainly because of their simplicity.
Despite the significant contribution introduced by Sorger (2003), even his model is affected by a particular problem: the model suffers from a strong form of indeterminacy of the equilibrium solution, unless it would be assumed homogeneity on saving rate or depreciation rate.
Our model is conceptually different from previous ones, as the flows between countries represent pure transfers of resources –specifically capital or income– intended for allocation to another country. This representation is better suited for studying the actions of a central planner who decides how to distribute flows within the network.
In particular, in this study, the central authority is concerned with finding the optimal configuration of flows between economies that have different characteristics in terms of model parameters, such as the saving rate, the depreciation rate, or the elasticity of capital. By optimal configuration, we mean the one that maximizes global welfare, regardless of the welfare of each individual economy.
In Chapter 1, a literature review is presented that spans from the traditional Solow-Swan model to Sorger (2003) multy-version of Solow model in open economy, covering all previously mentioned two-country Solow models. Particular attention is given to the traditional Solow model, primarily referencing toRomer (2019) and Acemoglu (2009), and the Ruffin model.
Chapter 2 provides a detailed explanation of the model introduced by Sorger (2003). Finally, the core of the thesis is presented throughout the Chapter 3, in which the structure of the model is presented together with assumptions, interpretations and computational results. Furthermore, a proposal for an objective function suitable for our framework is briefly introduced, together with an alternative version of the model in which the Central Planner reallocates income instead of capital.

Appendix A includes all additional simulation results of the model, while Appendices B and C present the MATLAB algorithms used in our analysis.

Chapter 1

Solow-Swan Model: from Closed to Open Economy

The Solow-Swan model is the father of modern macroeconomics dynamic models and a milestone of economics growth theory. It was independently developed by Robert M. Solow and Trevor W. Swan in 1956, superseding the theories stemming from the Harrod-Domar model, the so-called AK model, which is considered the precursor of their theory.
Indeed, in the introduction of Solow (1956), it is argued that his model is built on the idea of relaxing the crucial assumption that underpins the entire Harrod-Domar growth theory. That assumption was the fixed proportion of labor and capital, which ensures a balanced growth path that are fully explained by the accumulation of capital and his "coefficient" of contribution to output growth. Their model gives no room for labour variable to contribute to economics development and provides a strong Keynesian interpretation of the phenomena. "Instead Harrod and Domar talk of long run in terms of the multiplier, the accelerator, the capital coefficient. The bulk of this paper is devoted to a model of long-run growth which accepts all the Harrod-Domar assumptions except that of fixed proportions", Solow (1956). One of the strengths of the Solow model is its simplicity; nevertheless, it has managed to introduce significant contributions and shaped some of the most famous and still widely used long-term dynamic macroeconomic models. Among these, we can mention the Ramsey-Cass-Koopmans model, the Overlapping Generations model, and the multiple DSGE models, which are still widely used for policy impact evaluation.
His main implementation was to introduce the Neoclassical production function, which is characterized by decreasing marginal return to scale and constant

return to scale allowing Solow to take into account variable such as labour and technological progress. This specific innovation is important from a mathematical perspective, as it introduces concavity features to the dynamics of variables, making it possible to identify an equilibrium of the differential equation that describe capital accumulation.

This equilibrium is called steady-state, where all aggregate variables in terms of effective labor no longer grow, while aggregate variables in term of labour grow at the constant growth rate of technology.

Therefore, the Solow model presented an explanation of long-term growth dynamics, in which countries experiencing economic booms go through a phase of physical capital accumulation, during which relatively high economic growth rates are observed. However, this fast pace growth will slow down as growth rates decrease until the economy reaches the steady-state, where a country has achieved its full capital capability.

Once a country has gone through the capital accumulation phase, economic growth is driven only by technological progress.

The weakness of these conclusions lies in the fact that the model itself treats technological growth as exogenous, assigning the role of long-term driver to a variable that is not explained within the model itself.

After 9 years was published the first attempt to implement open economy in a Solow model framework in Negishi (1965).

Main variables are simply home country capital stock k, foreign country capital stock k^* and z, which denotes the capital of home country invested in the foreign country, and the capital depreciation rate α. Home and domestic production are defined as $f(k-z)$ and $f(k^*+z)$. He focused on optimal interest rate tax/subsidy policy to achieve maximum net gross income in steady-state, i.e. $f(k-z)+zf'(k^*+z)-\alpha k$. This analysis begins with the following definition of the endogenous interest rate equality condition: $(1-t)f'(k-z) = f'(k^*+z)$. He found that that optimal condition suggests a positive subsidy whenever $z > 0$; otherwise, the optimal policy is to impose a positive capital gain tax.

In Manning (1975), conclusions published by Negishi and his two-country model was extended. The main difference was the focus on find out golden rule conditions, so that consumption maximizes, instead of find out optimal condition in order to get maximum net gross income. In other words, he applied the traditional low motion of capital as saving rule: $\dot{k} = s[f(k-z)+rz] - \delta k$. Due to decreasing marginal return to scale of neoclassical production function, he found out that if the foreign economy is saving at a rate higher than golden-rule rate, the domestic economy should tax capital inflows. In the reverse case, where the foreign economy is saving at a rate less than the golden rule, the domestic economy should tax capital outflows. This conclusion could be resumed with following conditions:

$$z^* > z^{gr} > 0 \textbf{ if } s > s^*$$
$$z^* = z^{gr} = 0 \textbf{ if } s = s^*$$
$$z^* < z^{gr} < 0 \textbf{ if } s < s^*$$

The most influential contribution in the field of Solow models in open economy can be found in Ruffin (1979), in a paper published by the American Economic Review. In his work, he managed to clarify certain aspects not yet addressed in the previous literature, thanks to a relatively simple two-country model.
The national incomes of the two countries are represented by $y = f(k-z)+rz$ and $y = g(k-z)+rbz$, with the perfect capital mobility condition $f(k-z) = g'(k^*-z) = r$, which holds in every period and determines the endogenous world interest rate.
Ruffin's main conclusions indicate that the levels of national income in the steady-state of an open economy are always higher than those in a closed economy. Additionally, he reported the long-term effects of international trade on interest rates and wages: in the exporter country, perfect capital mobility lowers the long-run wage, while increasing the long-run interest rate. For the importing country, the reverse occurs.
The first attempt of implementing a network of N generic Solow economies is presented in Sorger (2003), where the idea of two-country was dropped, analyzing a more complex system in which the interest rate is determined endogenously among multiple countries. Sorger criticized in particular the simplicity of two-country version model with only K,K^* and Z which aren't enough to compute equilibrium when we adopt heterogeneity on depreciation rates. Furthermore, Sorger highlighted that in Ruffin model it's not allowed for both countries to reduce their home capital stock by the same quantity to be invested in the other country. Precisely, main aggregate variables are not affected.
Main result that emerges in Sorger multy-country Solow model is that the model suffers from a strong form of indeterminacy. Nevertheless, multiplicity of equilibria is proved to be bounded within a small rage of interest world rate. Indeterminacy problem ceases to exist whenever homogeneity on saving rate

or depreciation rate is imposed.

1.1 Solow-Swan Model

1.1.1 Assumptions

Solow-Swan modeled long run growth in a closed economy framework, where commodities are represented by a single good, which can be directly interpreted as national income.
The economy is populated by a large number of agents, all equal to each other, who choose a constant fraction $s \in (0,1)$ of output to invest in future production in every period t, while the remaining fraction will be consumed.
This setup allows us to refer to "representative agent", since the results of the model can be interpreted as if they result from a single agent.
The three main variables in the model are capital stock $K(t)$, labour $L(t)$ and technology $A(t)$. The latter enters in the production as a multiplier of labour, defining the "effective labour" quantity $A(t)L(t)$.
Solow's fundamental assumptions concern the mathematical properties of the production function and the dynamics of these three key variables. The production function takes the following implicit functional form:

$$Y(t) = F(K(t), A(t)K(t)) \tag{1.1}$$

The way in which productivity of labour was employed is called "Labour-augmenting" or "Harrod-neutral". This choice is due to mathematical convenience as it represents the only way to achieve a balanced growth path in the long run, as proved by Uzawa (1961). We will focus on it later.

Assumption 1.1: production function is assumed twice differentiable in $K(t)$ and $L(T)$ and satisfies the two following features:

$$F_K(K, AL) = \frac{\partial F(K, AL)}{\partial K} > 0 \tag{1.2}$$

$$F_L(K, AL) = \frac{\partial F(K, AL)}{\partial K} > 0 \tag{1.3}$$

$$F_{KK}(K, AL) = \frac{\partial F(K, AL)}{\partial K} < 0 \tag{1.4}$$

$$F_{LL}(K, AL) = \frac{\partial F(K, AL)}{\partial K} < 0 \tag{1.5}$$

and it is assumed also to exhibit constant return to scale. It means that multiplying capital and effective labour by a constant λ, output also must increase by the same proportion.

$$\lambda F(K(t), A(t)K(t)) = F(\lambda K(t), \lambda A(t)K(t)) \tag{1.6}$$

This property is also called homogeneity of degree one and is very useful to ensure perfect competition in firms' maximisation problem, thereby guaranteeing the zero-profit condition.

Assumption 1.2: production function must satisfy the so-called INADA conditions:

$$\lim_{K \to 0} F_K(K, AL) = \lim_{L \to 0} F_L(K, AL) = \infty \tag{1.7}$$

$$\lim_{K \to \infty} F_K(K, AL) = \lim_{L \to \infty} F_L(K, AL) = 0 \tag{1.8}$$

Thus, the marginal productivity of production factors exhibits strict concavity, ensuring the internal existence of a solution. These two assumptions about the production function represent the main deviations from the Harrod-Domar growth model, replacing the linearity of output with respect to capital and incorporating diminishing marginal returns on production factors. Consequently, Solow dropped the extreme assumption of a fixed capital-labor proportion, allowing economic growth to converge towards a steady-state.

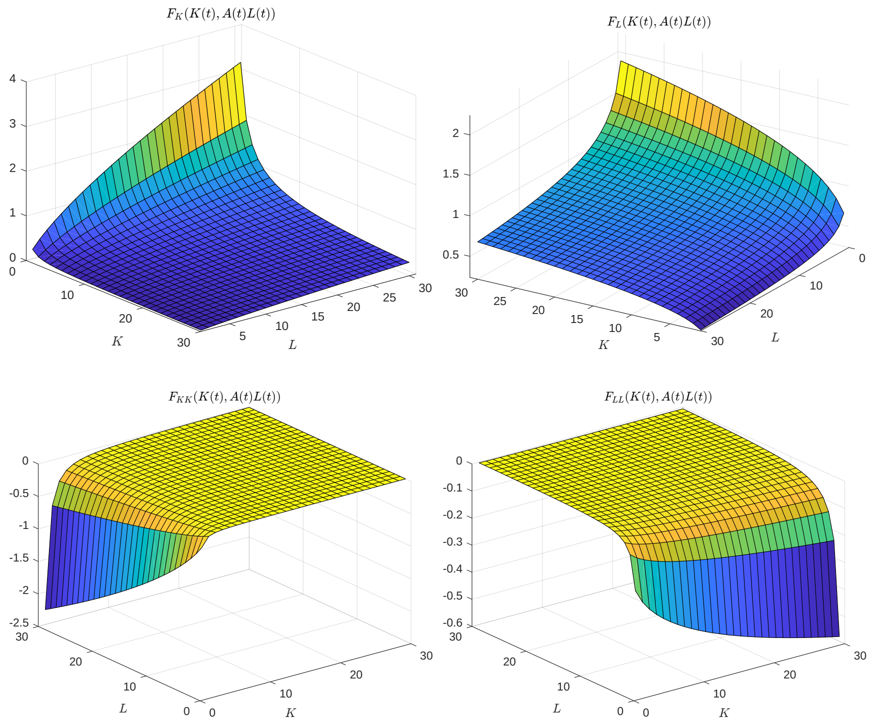


Figure 1.1: Example of well-behaved production function, own elaboration.

1.1.2 Firm's Problem

Now we will discuss the firm's profit maximization problem. Let's assume the presence of a representative firm operating in a perfectly competitive market. The firm faces an aggregate production function and chooses optimal $K(t)$ and $L(t)$ given competitive factor prices $w(t)$, $r(t)$ and for a given $A(t)$. The price of the unique good is normalized to one, and we know from *Assumption 1* and *Assumption 2* that $F(K(t), A(t)L(t))$ is strictly concave.

$$\begin{aligned} \max_{K(t),L(t)} \quad & \pi(t) = Y(t) - (r(t) + \delta)K(t) + w(t)L(t) \\ \text{s.t.} \quad & Y(t) = F(K(t), A(t)L(t)) \end{aligned} \tag{1.9}$$

Due to the perfectly competitive market, factor prices equalize their marginal productivity:

$$r(t) = F_K(K(t), A(t)L(t)) - \delta \tag{1.10}$$

$$w(t) = F_L(K(t), A(t)L(t)) \tag{1.11}$$

Here, we must recall the CRS (Constant Returns to Scale) property. By applying Euler's Theorem to a linearly homogeneous function, we can derive the firm's zero-profit condition:

$$Y(t) = (r(t) + \delta)K(t) + w(t)L(t) \tag{1.12}$$

All profits of the firm are paid out to the production factors, and since all capital and labor supply comes from the representative household, it is unnecessary to specify who owns the firms.

1.1.3 Quantitative Dynamics

As we previously stated, representative households choose a constant share of income to invest in future production $s \in (0, 1)$. Thus, investment in each period is defined by the share of output saved, determined by the saving rate s, while consumption $C(t)$ is the residual part of the output that is not saved by households.

$$I(t) = sY(t) \tag{1.13}$$

$$C(t) = (1 - s)Y(t) \tag{1.14}$$

Capital at time t is equal to past accumulated $K(t)$ plus the new investment of the period $I(t)$, excluding the depreciated part $\delta K(t)$.
Then, we know by national accounting that in a closed economy output must be either consumed or saved. Therefore, the following accounting identity must hold to ensure the general equilibrium of the model:

$$Y(t) = I(t) + C(t) \tag{1.15}$$

Hence, we could identify equation that describes the evolution of capital in time:

$$\dot{K}(t) = I(t) - \delta K(t) = sY(t) - \delta K(t) \tag{1.16}$$

This differential equation is called the fundamental law motion of capital of Solow model.
Now we introduce population and technological growth. i.e. $L(t)$ and $A(t)$. Initial level of technology $A(0)$ and population $L(0)$ is taken as given and they grow to a constant growth rate n and g, showing an exponential growth trend.

Thus, population and technological take the following dynamics:

$$L(t) = L(0)e^{nt} \tag{1.17}$$

$$A(t) = A(0)e^{gt} \tag{1.18}$$

Exploiting mathematical properties of constant returns to scale (CRS), we can express quantities in terms of effective labor by dividing the production function by $A(t)L(t)$. This transformation will be helpful for representing equilibrium along the long-run growth path.

$$y(t) = F\left(\frac{K(t)}{A(t)L(t)}, 1\right) = f(k(t)) \tag{1.19}$$

Now, we shall express capital low of motion in effective-labour term ($\dot{k}$). First of all, by growth rate of capital per effective-labour $\dot{k}$ and by logarithm properties we can compute:

$$\frac{\dot{k(t)}}{k(t)} = \frac{\dot{K(t)}}{K(t)} - \frac{\dot{L(t)}}{L(t)} - \frac{\dot{A(t)}}{A(t)} = \frac{\dot{K(t)}}{K(t)} - n - g \tag{1.20}$$

Now, replacing $\dot{K(t)}$ with his law of motion $\dot{K(t)} = sY(t) - \delta K(t)$, dividing and multiplying by $A(t)L(t)$ and rearranging, we would obtain $k(t)$ growth rate

$$\frac{\dot{k(t)}}{k(t)} = \frac{sf(K(t))}{k(t)} - (\delta + n + g) \tag{1.21}$$

and the following law of motion of effective capital-labour ratio takes the form:

$$\dot{k}(t) = sf(k(t)) - (\delta + n + g)k(t) \tag{1.22}$$

Capital law of motion in effective-labour term states that the absolute change in capital per effective-labour depends by two different terms. The first one is the saved share of output per effective-labour, which represents the investment per effective-labour. The second term $(\delta + n + g)k(t)$ is called break-even-investment line and it could be interpreted as the amount of investment that must be done in order to keep $k(t)$ at the existing level.

1.1.4 Equilibrium: Balanced Growth Path

Solow model is a neoclassical macroeconomic dynamic model with non optimizing agents, indeed Solow kept basic Keynesian features that comes by Harrod-Domar model, especially when we talk about agent who's saving a constant fraction of income. As a neoclassical framework model, we should define what are model equilibrium and market clearing conditions in steady-state.

The equilibrium model is characterized by $k(t)$ zero growth, while aggregate variables in absolute terms must feature equal growth rates, i.e. $\frac{\dot{K(t)}}{K(t)} = \frac{\dot{Y(t)}}{Y(t)} = \frac{\dot{C(t)}}{C(t)}$.

In this specification, once the steady-state is reached, the model achieves sustained growth forever. This can be easily noted by looking at the following equation: $\frac{\dot{\frac{Y(t)}{L(t)}}}{\frac{Y(t)}{L(t)}} = \frac{\dot{K(t)}}{K(t)} + n + g - \frac{\dot{K(t)}}{K(t)} - n = g$, which indicates that, thanks to constant returns to scale (CRS), $Y(t)$ grows at the rate of $\frac{\dot{K(t)}}{K(t)} + n + g$.

Definition 1.1: for an exogenous initial capital endowment $K(0)$, the equilibrium path of Solow model in continuous time with population and technological growth at constant rate n and g is a sequence of $\{k(t), Y(t), r(t), w(t)\}$, such that:

-$k(t)$ evolves following dynamic described in 1.22.

-$Y(t)$ is given by 1.15.

- $r(t)$ and $w(t)$ is defined in 1.25 and 1.26.

The path of steadily growing equilibrium is referred to as the "Balanced Growth Path" due to the behavior of variables in the steady-state and the presence of labor-augmenting technological progress. As previously mentioned, we observe a constant and sustained growth of output per labor and capital per labor, which aligns well with the empirical data presented by Kaldor (1961).

-k^*, y^* and c^* are constant.

-$\frac{Y}{K}$ and $\frac{Y}{L}$ grow at g.

-K^*, Y^* and C^* grow at $n + g$.

In conclusion, Solow model implies that, regardless the initial conditions, economy will always converge towards BGP, in which long run growth is entirely driven by technological progress. Exogenous technological growth represents the main weakness of Solow model, since it's just the main element that determine long run growth. In other words, sustained long run growth is completely explained by something that is not explained within the model.

1.1.5 Equilibrium Example

The most famous and adopted production function in economics literature is the Cobb-Douglas function. It's perfectly suitable for the Solow model, as it satisfies both *Assumption 1* and *Assumption 2*.

$$F(K(t), A(t)K(t)) = K(t)^{\alpha}(A(t)L(t))^{1-\alpha} \tag{1.23}$$

where $\alpha \in (0, 1)$, in order to impose CRS assumption.
Dividing output by $A(t)L(t)$, we could work with model in effective labour unit terms. Then we get:

$$y(t) = k^{\alpha} \tag{1.24}$$

While from FOCs of firm's maximization problem $r(t)$ and $w(t)$ becomes:

$$r(t) = f'(k(t)) \tag{1.25}$$

$$w(t) = f'(k(t)) - k(t)f'(k(t)) \tag{1.26}$$

Definition 1.2: the equilibrium solution of Solow model in continuous time coincide with the steady-state solution of the ODE that represent capital in effective-labour dynamic: $f(k) = k^*$.

In other words, steady-state equilibrium represents the solution in which $\dot{k} = 0$. This occurs when $\dot{Y(t)} = \dot{K(t)} = \dot{C(t)} = 0$, implying that output growth rate is null. Then we should set

$$\dot{k} = 0 \tag{1.27}$$

so that

$$sk(t)^{\alpha} = (\delta + n + g)k(t) \tag{1.28}$$

Hence, isolating $k(t)$ we would get k_{ss} steady-state level.

$$k^* = \left(\frac{s}{\delta + n + g}\right)^{\frac{1}{1-\alpha}} \tag{1.29}$$

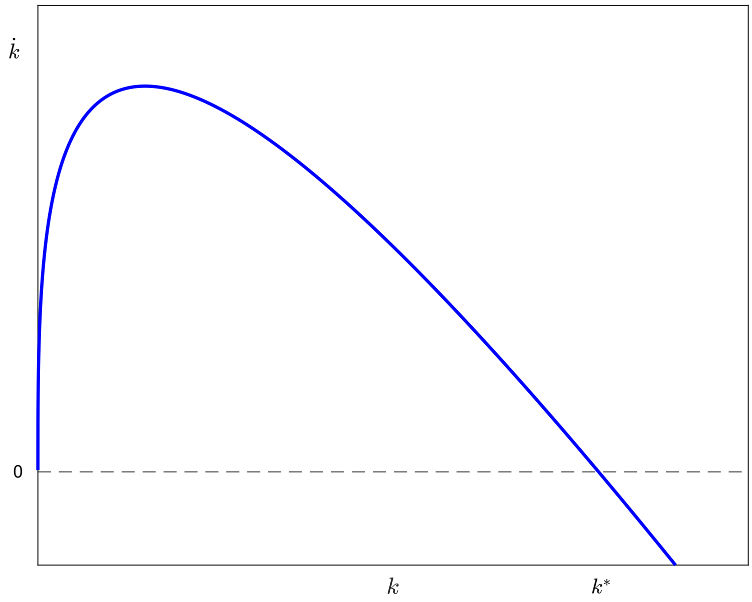


Figure 1.2: Steady-state convergence of $k(t)$, own elaboration

From k^*, we can also derive the steady-state levels of y^* and c^*.

$$y^* = f(k^*) = \left(\frac{s}{\delta + n + g}\right)^{\frac{\alpha}{1-\alpha}} \tag{1.30}$$

$$c^* = (1-s)f(k^*) = (1-s)\left(\frac{s}{\delta + n + g}\right)^{\frac{\alpha}{1-\alpha}} \tag{1.31}$$

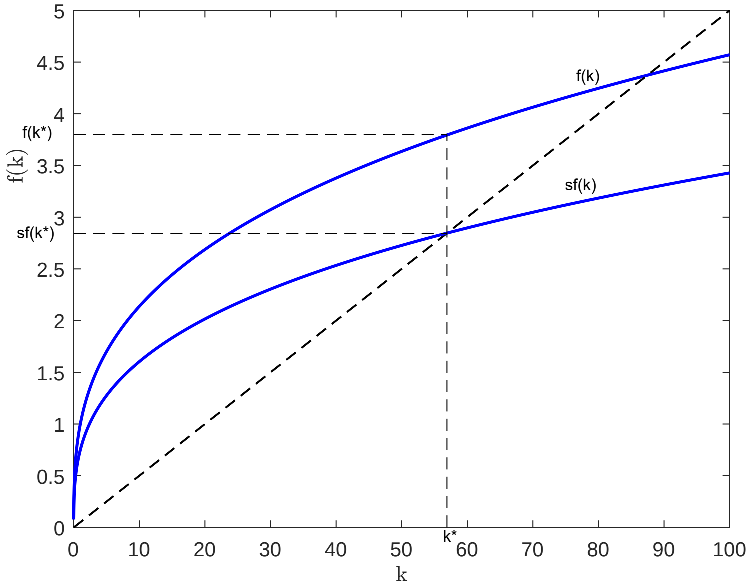


Figure 1.3: Steady-state level of k^* and y^*, own elaboration

Transition dynamic shows that in the left side of k^* the capital per effective-

labour accumulates faster than $(\delta + n + g)k(t)$. Instead, in the right side we note that capital investment per effective labour is not enough to counteract depreciation of $k(t)$ and the growth of effective-labour units (i.e. n+g).
Due to strictly concavity of $y(t)$ and $sy(t)$ and the linearity of $(\delta+g+n)$ with respect to $k(t)$, the unique internal equilibrium solution of the capital law of motion is achieved when investment per effective-labour equals beak-even investment line $sy(t) = (\delta + g + n)k(t)$.
Steady-state equilibrium is guaranteed is guaranteed by every production function that are strictly concave and satisfies INADA conditions. Those functions are called well-behaved production function, since its behaviours allow for achieving steady-state of capital's ODE.
This is straightforward to demonstrate for the Cobb-Douglas production function, since $f'(k(t)) = \alpha k(t)^{\alpha-1}$ goes to zero when k goes to infinity and goes to infinity when k goes to zero. Finally, $f''(k(t)) = -(1-\alpha)\alpha k(t)^{\alpha-2}$ has negative sign.

1.2 Ruffin Two Economies Model

In Raffin model we have two Solow economies, the home economy and the foreign economy, which interact with each other through international investment flows.
The assumptions remain the same as those of the traditional Solow model. That is, each economy produce one identical good produced in perfect-competition market. The production functions satisfy INADA conditions and exhibit constant returns to scale (CRS) with full employment.
The home and foreign capital stocks are denoted as k and k^*, respectively, while L and L^* represent the population which, both growing at the same constant rate n.
Z denotes home country capital unit owned abroad, so that domestic capital per-capita for home country is determined by $k - z$.
The production function of home country is $f(k - z)$, while the production function for the foreign country is $g(k^* + bz)$, where b is defined as $\frac{L}{L^*}$.
Free capital mobility requires that the following equation holds:

$$f'(k - z) = g'(bk^* + z) = r \tag{1.32}$$

Equation 1.32, together with the conditions for well-behaved production functions, these assumptions imply a unique solution for the (k,k^*) pair in the world economy. National income must account for the international balance of net positions, where the home country receives rz, while the foreign country pays rbz.

$$y = f(k - z) + rz \tag{1.33}$$

$$y^* = g(k^* + bz) - rbz \tag{1.34}$$

Now we can express the capital laws of motion in per-capita terms for both countries:

$$\dot{k} = sy - nk \tag{1.35}$$

$$\dot{k^*} = sy^* - nk^* \tag{1.36}$$

Finally, by imposing steady-state condition $\dot{k} = \dot{k^*} = 0$, together with equations 1.32, 1.33 and 1.34,we derive the unique steady-state solution of the Ruffin model.

Now, let's examine some behavioral properties of the model. Considering the effect of capital accumulation on z, diminishing returns to scale imply that an increase on k must be accompanied by an increase in z. The opposite holds for the foreign country, so that $\frac{\partial z}{\partial k} > 0$ and $\frac{\partial z}{\partial k^*} < 0$.

The effects of capital on income per capita are particularly interesting. Assuming $z > 0$ (i.e. the home country is the world creditor), we obtain $\frac{\partial y}{\partial k^*} < 0$ and $\frac{\partial y^*}{\partial k} > 0$. This result indicates that as global capital increases, depressing the global interest rate, the creditor country experiences a loss in terms of interest payments, while the debtor country benefits from a lower cost of interest on foreign debt.

1.2.1 Steady-State

Along the steady-state equilibrium in the Ruffin model, it is proved that "..solutions for per capita incomes and the capital-labor ratios with perfect capital mobility exceed the steady-state solutions with prohibited capital movements for both countries", Ruffin (1979). Looking at Figure 3.6, it is clear that that autarky equilibrium curve is upward sloping, since each incentive for capital to migrate must be eliminated by increasing both capitals.

By implicit function rule, Ruffin proved that $\Delta k = 0$ is upward sloping above autarky equilibrium curve and downward sloping below it. For $\Delta k^* = 0$ is exactly the opposite.

Steady-state in open economy is achieved when both capital growth is null $\Delta k = \Delta k^* = 0$.
Since the two curves has opposite slope, uniqueness of solution is ensured.

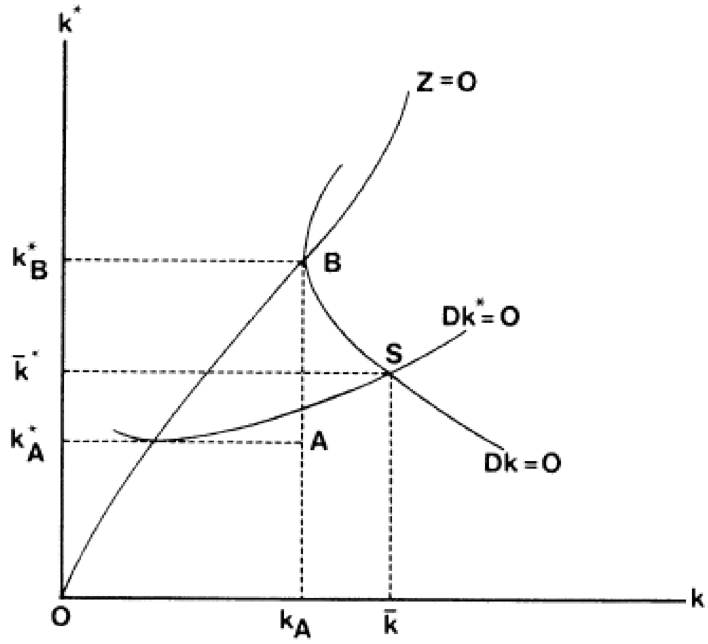


Figure 1.4: Open economy steady-state transition, Ruffin (1979)

Looking at the intersection of the equations reveals that the open economy equilibrium (S) is located in the northeast relative to the closed economy starting point (B).
Finally, Ruffin emphasized long run effect of open economy on wages and interest rate. As illustrated by the evolution of capital in the home country (see Figure 3.6), it is clear that the interest rate increases while the wage rate also rises. Conversely, in the foreign country, where capital follows a decreasing trajectory towards the new equilibrium, there will be a decline in long-run wages accompanied by an increase in the interest rate.

Chapter 2

Multy-Country Version of the Solow Model

In the following chapter, we are going even further into the analysis of free capital mobility in a Solow model framework.
Sorger contribution is particularly interesting for our purposes since it represents the first attempt of introducing a network system composed by N generic Solow economies.
Sorger's main idea was to expand the simple Solow models to two economies in an open economy framework, as developed by Negishi (1965), Manning (1975), Ruffin (1979), and Deardorff (1994), since the extreme simplification of these models don't allow to consider economies with heterogeneity in depreciation rates and saving rates. Hence, he introduced a more complex model with new variables that allow to take into account more than just two economies also different from each other.

2.1 Model Structure

The model features N Solow-economy with effective labour force that grows in each country at the same rate γ, i.e. $L_i(t) = L_{i0}e^{\gamma t}$.
Output and capital are mobile across boarders, then part of capital of country i could be owned by other countries. K_{ji} denotes capital stock located in country i that's owned by country j, so that we denote total capital stock of country i as follow:

$$K_i(t) = \sum_{j=1}^{N} K_{ji}(t) \quad (2.1)$$

where $K_{ji}(t) > 0$ for each i,j and t.
In the current specification, each country save a constant fraction of GNP, denoted as $Y_i(t)$ and represents GDP plus net investment flow:

$$Y_i(t) = F_i(K_i(t), L_i(t)) - r_i(t)\sum_{l\neq i} K_{li}(t) + r_l(t)\sum_{l\neq i} K_{il}(t) \quad (2.2)$$

where $r_i(t) = \frac{\partial}{\partial K} F_i(K_i(t), L_i(t))$, since firms operate in a perfectly competitive goods market.
Each country saves a constant fraction of GNP $s_i Y_i(t)$, which is also equal to gross investment. Furthermore, we know that residents of country i could invest abroad, then gross saving must equal the sum of all investments of resident of country i in every country j: $s_i Y_i(t) = \sum_{j=1}^{N} I_{ij}(t)$.
Sorger introduced a new variable $a_{ij} = I_{ij}/[s_i Y_i(t)]$, which is interpreted as the share of country i gross saving invested on country j, then it must hold the following condition:

$$\sum_{j=1}^{N} a_{ij}(t) = 1 \quad (2.3)$$

Given all gross saving shares and investment function, the law motion of capital of country i owned by residents of a generic country j is defined as

$$\dot{K_{ij}} = a_{ij}(t)s_i[F_i(K_i(t), L_i(t)) - r_i(t)\sum_{l\neq i} K_{li}(t) + r_l(t)\sum_{l\neq i} K_{il}(t)] - \delta_j K_{ij}(t) \quad (2.4)$$

Combining all model equations presented so far, we could express capital law of motion in intensive form.

$$\dot{k_{ij}} = a_{ij}(t)s_i[f_i(\sum_{l=1}^{N} K_{il}(t)) - f_i'(\sum_{l=1}^{N} K_{il}(t))\sum_{l\neq i} k_{il}(t) + \sum_{l\neq i} f_i'(\sum_{m=1}^{N} k_{ml}(t))k_{il}(t)] - (\delta_j + \gamma)k_{ij}(t) \quad (2.5)$$

Then, we are going to talk about open economy features of the model. Since Sorger presented a network system of multy Solow-Swan economies, an arbitrage condition that ensures perfect capital mobility is needed:

$$f_i'(\sum_{l=1}^{N} k_{il}(t)) - \delta_i = \rho(t) \quad (2.6)$$

The latter condition states that exist a world interest rate $\rho(t)$ of equilibrium such that equalize net rental rate $r_i(t)$ of each country.

2.2 Steady-State: Problem Structure

Sorger collected all functions k_{ij} and a_{ij} into two matrix $\mathbf{K}$ and $\mathbf{A}$

Definition 3.1: the triple ($\mathbf{K}$, $\mathbf{A}$, ρ) forms an equilibrium if satisfies the following system of equations:

$$\begin{cases} \dot{k}_{ji} = a_{ij} s_i y_i(\mathbf{K}(t)) - (\delta_i + \gamma) k_{ji}(t) \\ f_i' \left(\sum_{l=1}^{N} k_{il}(t) \right) - \delta_i = \rho(t) \\ k_{ij}(t) \geq 0 \\ \sum_{j=1}^{N} a_{ij}(t) = 1 \end{cases} \tag{2.7}$$

Reformulating stationary equilibrium conditions as a function of world interest rate, we would get a simpler system of equations, because it doesn't involve investment shares a_{ij}.
Denoting $\hat{\delta}$ as the smallest δ_i, i.e. $\hat{\delta}$ =$(\delta_i | i = 1, 2, ..., N)$, given $\rho \geq -\hat{\delta}$, we would get a unique capital stock $k_i^*(\rho)$ such that $f_i'(k_i^*(\rho)) = \rho + \delta_i$. Then, it's defined the parameter $\beta_{ij} = \frac{\gamma + \delta_i(s_i)}{s_i}$.

Definition 3.2: the number ρ is the world interest rate in a stationary equilibrium if and only if the following is true: $\rho > -\hat{\delta}$, and there exists a constant matrix $\mathbf{K} \geq 0$ (of capital stock) such that:

$$\begin{cases} \sum_{j=1}^{N} k_{ij} = k_j^*(\rho) \\ \sum_{j=1}^{N} (\beta_{ij} - \rho) k_{ij} = w_i(k_i^*(\rho)) \end{cases} \tag{2.8}$$

It's straightforward to note that the system suffers from indeterminacy of solution, since the conditions involve $2N$ equations in $N^2 + 1$ variables, N^2 capital stocks k_{ij} and ρ. Only in the trivial case $N = 1$, the number of unknowns coincide with the number of equations, whereas $N \geq 2$ we would face a continuum of world interest rates that are supported by stationary equilibria.

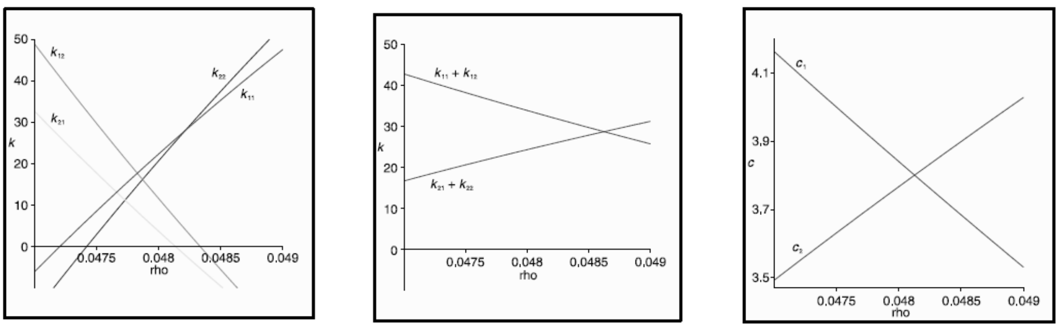
However, it's proved by Sorger that the multiplicity of interest rates coinciding with steady-state equilibrium is bounded within a small range of ρ, unless homogeneity on depreciation rates or saving rates is assumed.

2.3 Steady-State: Example

In this section we will present the example of a steady-state continuum equilibria exposed in Sorger (2003) and the analysis of the effect of world interest rate on main aggregate variables.

Example: $N = 2$ countries where $f_1(k) = f_2(k) = k^{\frac{1}{2}}$, $\delta_1 = 1/25$, $\delta_2 = 1/25$, $s_1 = 1/3$, $s_2 = 1/4$ and $\gamma = 1/100$.

Given that both k_1^* and k_2^* are decreasing with respect to ρ, because of neoclassical production function properties, it's not true when we consider capital stocks. Indeed, capital stock owned by country 1, i.e. that country with higher saving rate and lower depreciation of capital, is decreasing with respect to world interest rate, while for country 2 is the opposite.
It's worth to note that domestic capitals are increasing with respect to ρ, whereas cross-country owned capitals are decreasing. Then, when an interest rate reduction occurs, countries close interest gaps in order to exploit incentives of investing abroad.
In terms of consumption, the country with larger saving and accumulation attitude is better-off in equilibrium with smaller interest rate, since it would exploit the higher marginal return of country with smaller investing capability, while country 2 prefers higher interest rates. In other words, the reason lies again on decreasing marginal returns property.



(a) Capital stock as function of ρ

(b) Owned capital stock as function of ρ

(c) Consumption rates as function of ρ

Figure 2.1: Aggregates behaviour in steady-state, Sorger (2003).

Chapter 3

Network Interactions of Multiple Solow Models

After reviewing the main works in the literature that aim to analyze the effects of interactions among multiple Solow economies, we will present our contribution. The model that we propose applies the Solow model in a network framework, where each node represents a Solow-economy connected to each other through bilateral flows. The analysis aims to explore the possibilities of a Central Planner, which is represented by a central node that determines the flows of capital (or income) between nodes, in allocating resources more efficiently and equally compared to the allocations resulting from free capital mobility.

3.1 Network Framework of the Model

We have N economies, each regulating its relations with other economies independently. Net flow between two nodes (i and j) is represented by γ_{ij}, with the reverse direction net flow satisfying the condition $\gamma_{ij} = -\gamma_{ji}$.
In this network framework, the system consists of $\frac{N(N-1)}{2}$ bilateral net flows γ_{ij}, and the total net flow for country i is given by $\sum_{j=1}^{N} \gamma_{ij} = \gamma_i$. Furthermore, international trade accounting requires that the sum of all countries' net flows must equal zero: $\sum_{i=1}^{N} \gamma_i = 0$.

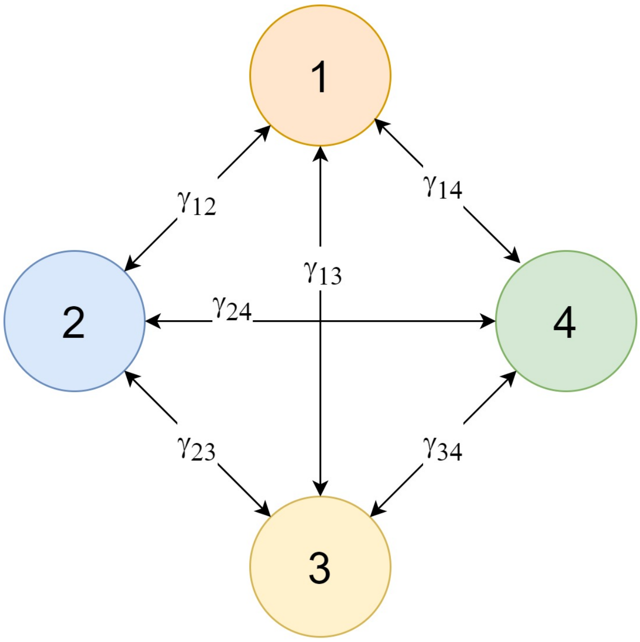


Figure 3.1: Example of 4 connected economies that exchange γ_{ij} flows.

To reformulate the problem within a general network framework, the Central Planner is empowered to determine flows that connect various nodes. This Central Planner can be identified as a central node that authoritatively set transfers, with the objective of managing a trade-off between maximum aggregate consumption and equality in the steady-state levels across the entire system of interconnected economies. The trade-off could be analytically expressed as an optimization problem of an objective function that take into account aggregate consumption and inequality together, that should be maximized with respect to the set of feasible γ_{ij}.
The network framework could be summarized as in the following graph.

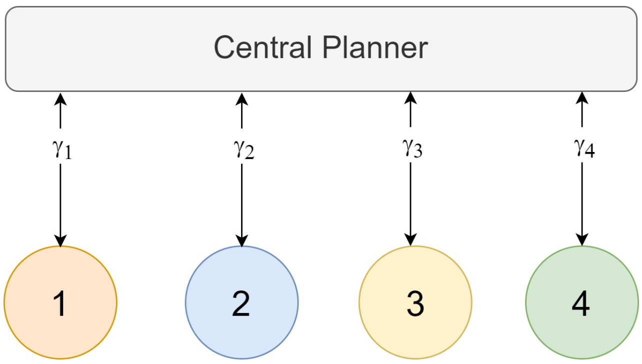


Figure 3.2: 4 connected economies through a Central Planner.

3.1.1 The Model: Microfoundation

After introducing network characteristics, we will derive the model analytically. Thus, in the context where capital can flow among the N economies, we should have the following capital law of motion for a generic country i:

$$\dot{k}_i = s_i f_i(k_i) + \gamma_i(\sum_{j \neq i} k_j) - \delta_i k_i \tag{3.1}$$

Each country produces the same unique good with a neoclassical production function in effective-labour term. To simplify, we assume a constant level $L(t)$ for each country and we exclude technological growth, with $A(t)$ normalized to 1 in every period.

We know that R_i represents the rental price for a unit of capital and that capital depreciates over time at a rate δ_i, so that the rate of return after depreciation is given by $R_i - \delta_i$. However, in this economy capital can flow between two nodes at a rate γ_i, this share represents the incoming/outgoing capital, which is determined by a Central Planner.

Since we know that the increase in k_i also affects the depreciated capital with rate δ_i, we also know that it affects net imports of capital with rate γ. This implies that the net interest rate of investing in capital k_i is:

$$\rho_i = f_i'(k_i) + \gamma - \delta_i. \tag{3.2}$$

Equation 3.2 assumes that the owners of capital k_i is the household of country i. In other words, there is no crossed capital ownership between the two countries. The parameter γ acts reinforcing or countering depreciation, according to its sign. (See Barro and Sala-i-Martin (1995), page 32).

After imposing neoclassical production function $F(K, L) = K_i^{\alpha} L_i^{(1-\alpha)}$, representative firm's problem in effective labour terms take the following form:

$$\max_{K_i, L_i} \pi_i = [k_i^{\alpha_i} + \gamma_i \sum_{j=1}^{n} k_j - (\rho_i + \delta_i)k_i - w_i]L_i \tag{3.3}$$

Let's compute FOCs:

$$\frac{d\pi_i}{dK_i} = \alpha_i k_i^{\alpha_i - 1} + \gamma_i - (\rho_i + \delta_i) = 0 \tag{3.4}$$

$$\frac{d\pi_i}{dL_i} = (1 - \alpha_i(1 - \gamma_i))k_i^{\alpha_i} + \delta_i k_i - w_i = 0 \tag{3.5}$$

Final structure of the model with N generic economies is as follows::

$$\dot{k}_i = s_i k_i^{\alpha_i} + \gamma_i k_T - \delta_i k_i \tag{3.6}$$

with

$$k_T = \sum_{j=1}^{N} k_j \tag{3.7}$$

and payment to factors

$$\rho_i = \alpha_i k_i^{\alpha_i - 1} + \gamma_i - \delta_i \tag{3.8}$$

$$w_i = k_i^{\alpha_i} - (\rho_i - \gamma_i + \delta_i) k_i \tag{3.9}$$

Where γ_i represents a fraction of total capital of network system that are determined by the Central Planner and since the latter affects interest rates, arbitrary condition couldn't be ensured in every achievable steady-state. In order to keep the model numerically computable, we should consider a new variable ϵ which denotes the bias between interest rates.
Moreover, in every macroeconomic model, it is necessary to define accounting identities. In our framework, the new amount of resources allocated as investments in country i consists of a fraction of domestic savings, denoted by $s_i y_i$ (that we denote as i_i), where y_i represents the domestic production, given by $k_i^{\alpha_i}$. Additionally, a portion is redistributed from country j, represented by $\gamma(k_1 + k_2)$, which we denote as z_i.
We should define the gross national income as x_i:

$$x_i = y_i + z_i = k_i^{\alpha_i} + \gamma_i \sum_{j=1}^{N} k_j \tag{3.10}$$

This quantity is subject to maximization through the firm's profit maximization problem. Since domestic production must be allocated for both consumption and domestic investment, the following relationship must hold:

$$y_i = c_i + i_i \tag{3.11}$$

We can summarize the accountability equations of total saving $\tilde{s}_i$ and gross national income x_i as:

$$\tilde{s}_i = i_i + z_i = s_i k_i^{\alpha_i} + \gamma_i \sum_{j=1}^{N} k_j \tag{3.12}$$

$$x_i = y_i + z_i = c_i + i_i + z_i = k_i^{\alpha_i} + \gamma_i \sum_{j=1}^{N} k_j \tag{3.13}$$

Finally, since perfect competition imposes a zero-profit condition, to ensure general equilibrium, it must hold that the total gross national income is distributed among the production factors:

$$k_i^{\alpha_i} + \gamma_i \sum_{j=1}^{N} k_j = (\rho_i + \delta_i) k_i + w_i \tag{3.14}$$

3.1.2 The Model: Numerical Treatment

For analytical purposes, we derive a compact formulation of the system of differential equations, considering that γ parameters are stored into an inverse asymmetric matrix as:

$$\begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1N} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{N1} & \gamma_{N2} & \cdots & \gamma_{NN} \end{bmatrix}$$

In order to represent an inverse asymmetric matrix, following conditions must be respected:

- $\gamma_{ji} = -\gamma_{ij}$
- $\gamma_{ii} = 0$

so that $\boldsymbol{\gamma}$ matrix becomes

$$\begin{bmatrix} 0 & \gamma_{12} & \cdots & \gamma_{1N} \\ -\gamma_{12} & 0 & \cdots & \gamma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma_{1N} & -\gamma_{2N} & \cdots & 0 \end{bmatrix}$$

Since we adopted a network system that includes a central node, we could simplify γ_{ij} matrix into a vector representing only the net flows. Specifically, we express it as $\sum_{j=1}^{N} \gamma_{ij} = \gamma_i$, eliminating the combinations of bilateral flows that complicate the analysis from a Central Planner's perspective. This leads to:

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_N \end{bmatrix}$$

A compact formulation of capital law of motion that exploit gamma vector could be represented as:

$$\dot{\mathbf{k}} = \mathbf{s} \odot \mathbf{k}^{\boldsymbol{\alpha}} - \boldsymbol{\delta} \odot \mathbf{k} + \boldsymbol{\gamma} \sum_{j=1}^{n} k_j \tag{3.15}$$

While compact formulation of interest rates should be

$$\boldsymbol{\rho} = \boldsymbol{\alpha} \odot \mathbf{k}^{\boldsymbol{\alpha - 1}} + \boldsymbol{\gamma} - \boldsymbol{\delta} + \boldsymbol{\epsilon} \tag{3.16}$$

Finally, numerical solution of the model is obtained by imposing $\dot{\mathbf{k}} = 0$ in the following system.
Furthermore, since arbitrage condition can't hold due to the imposition of capital redistribution by Central Planner, it should be considered an additional variable ϵ_i that denotes interest rate biases of countries with respect to endogenous world interest rate. To clarify the role of ϵ_i, in a network considering only two nodes, it would be interpreted as the spread between country 2 interest rate with respect to country 1. Moreover, for mathematical purposes this new variable is simply a degree of freedom of equation system, since it is crucial for breaking arbitrage condition (denoted by the third equation of the system). In other words, it allows interest rates to take on any possible value.
For computational analysis, this ϵ is useful to keep truck of "distance" between a generic equilibrium allocation and the corresponding free capital mobility allocation, that is achieved when $\epsilon_1 = \epsilon_2 = ... = \epsilon_n = 0$.

$$\begin{cases} \dot{\mathbf{k}} = \mathbf{s} \odot \mathbf{k}^{\alpha} - \boldsymbol{\delta} \odot \mathbf{k} + \boldsymbol{\gamma} \sum_{j=1}^{n} k_j \\ \boldsymbol{\rho} = \boldsymbol{\alpha} \odot \mathbf{k}^{\alpha - 1} + \boldsymbol{\gamma} - \boldsymbol{\delta} + \boldsymbol{\epsilon} \\ \rho_i = \rho_j \quad \forall i, j \in [1:N] \end{cases} \tag{3.17}$$

It is also worth noting that solving the system without the second and third equations would yield the same results. Then, in the Central Planner specification of the model, the endogenous variables are two: the vector of capitals and the vector of interest biases, although the latter is simply a degree of freedom.

Definition 3.1: given the $\boldsymbol{\gamma}$ vector, the couple $(\mathbf{k},\ \boldsymbol{\rho})$ form an equilibrium if 3.15 and 3.16 hold together with $\boldsymbol{\rho} = c \cdot \mathbf{1}$ and $\dot{\mathbf{k}} = 0$.

3.1.3 Model With N=2

When we consider the model with two countries, the numerical problem simplifies to the following system of equations:

$$\begin{cases} \dot{k}_1 = s_1 k_1^{\alpha_1} + \gamma(k_1 + k_2) - \delta_1 k_1 \\ \dot{k}_2 = s_2 k_2^{\alpha_2} - \gamma(k_1 + k_2) - \delta_2 k_2 \\ \alpha_1 k_1^{\alpha_1 - 1} + \gamma - \delta_1 = \alpha_2 k_2^{\alpha_2 - 1} + \gamma - \delta_2 + \epsilon \end{cases} \tag{3.18}$$

Following Sorger (2003), the model could be expressed as the evolution of the different components of capital of each node:

$$\begin{cases} \dot{k}_{11} = s_1 y_1 - \delta_1 k_{11} \\ \dot{k}_{21} = \gamma(k_1 + k_2) - \delta_1 k_{21} \\ \dot{k}_{22} = s_2 y_2 - \delta_2 k_{22} \\ \dot{k}_{12} = -\gamma(k_1 + k_2) - \delta_2 k_{12} \\ \alpha_1 k_1^{\alpha_1 - 1} + \gamma - \delta_1 = \alpha_2 k_2^{\alpha_2 - 1} + \gamma - \delta_2 + \epsilon \end{cases} \tag{3.19}$$

This is the model specification subject to our computational analysis, where we simulated the steady-state levels of network system as a function of the γ. The solution for equilibria is determined by imposing the steady-state condition for both countries: $\dot{k}_1 = \dot{k}_2 = 0$.
It is important to specify that, up to now, we have not succeeded in obtaining an analytical solution to the mathematical problem. We were only able to derive the trivial solution, namely the solution under autarky condition ($\gamma = 0$).

3.2 Existence and Uniqueness of Solution

A strength of our model compared to Sorger (2003) lies in the determinacy of equations system, even in the presence of heterogeneity in the parameters α_i, s_i, and δ_i. We know that the system of equations defined in Sorger's model suffers from indeterminacy of the solution due to an excess of variables com-

pared to the number of equations.
In our case, we have a system of $N+2$ equations with N unknowns, where the two additional equations, compared to the capital laws of motion, are merely additional conditions that do not introduce any new endogenous variables to the system (ϵ represents a degree of freedom as explained in previous sections). In addition, based on the computational results of our model, there exists a certain narrow range of γ for which the model's solution exists and is unique. In the following section, we attempt to draw some conclusions regarding this constraint.
Let us focus on the two-economy model, and then generalize the proof to N economies. First of all, we delete interest rate equation since it's not determinant on dynamic of the system and we impose steady-state conditions ($\dot{k_1} = \dot{k_2} = 0$). Rearranging we get the following two equations

$$\begin{cases} s_1 k_1^{\alpha_1} = -\gamma(k_1 + k_2) + \delta_1 k_1 \\ s_2 k_2^{\alpha_2} = +\gamma(k_1 + k_2) + \delta_2 k_2 \end{cases} \tag{3.20}$$

- LHSs: since production functions respect *Assumption 1.1* and *Assumption 1.2*, it's straightforward to note that LHS terms (the so-called saving functions) are strictly concave: when $k_1 \to 0$ and $k_2 \to 0$, we know that $s_1 k_1^{\alpha_1} \to \infty$ and $s_2 k_2^{\alpha_2} \to \infty$. While $k_1 \to \infty$ and $k_2 \to \infty$, we have that $s_1 k_1^{\alpha_1} \to 0$ and $s_2 k_2^{\alpha_2} \to 0$.
- RHSs: both equation terms are linear with respect to capitals. But proof it's not straightforward as in traditional Solow-Swan model, where the RHS (the so called break-even investment line) is linearly dependent just on k_i and it's known that it starts from the origin, so that LHS=RHS is ensured (see 1.1.5).

In our case, the two capital laws of motion are interdependent, as are their corresponding break-even investment lines (because of $\pm\gamma(k_1 + k_2)$ terms).
Hence, we should focus on the two RHSs behaviour: rearranging country 1 equation, we would get $(\delta_1 - \gamma)k_1 - \gamma k_2$. Since RHS couldn't be negative, a necessary but not sufficient condition on gamma must be: $\gamma < \delta_1$. Specularly, by country 2 we obtain $\gamma > -\delta_2$. Finally, combining the two conditions, we get:

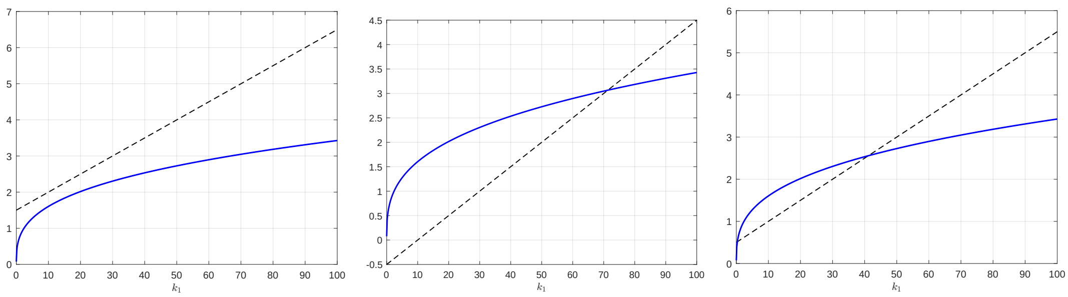
$$-\delta_2 < \gamma < \delta_1 \tag{3.21}$$

The economic interpretation of the following condition establishes that the capital in-flows for net importer countries must not be excessively large, avoiding

that it would completely offset capital depreciation. This would cause the capital of the importer country to explode and that of the exporter to become negative.
To generalize this necessary condition in an N generic framework, we need to adopt specific notation to differentiate between importer and exporter countries. That is, δ_j denotes depreciation rate of exporter countries, while δ_l denotes depreciation rate of importer ones.

$$\max(-\delta_j) < \gamma_i < \min(\delta_l) \tag{3.22}$$

Assuming the necessary but insufficient condition 3.21 is satisfied, we can analyze three different equilibrium cases for country 1 based on its equations graph.
a) Break-even investment line lies above origin and above saving function along all domain (i.e. $\mathbb{R}^+$). In the current case the model equilibrium isn't achieved.
b) Break-even investment line starts below the origin and crosses saving function once, implying existence and uniqueness of a stable equilibrium.
At the same time, since γ enters specularly into country 2 equation, it shouldn't be too large, otherwise it falls in case a).
c) Break-even investment line starts above the origin and crosses saving function in two different points. The first intersection indicates an unstable equilibrium, whereas the second intersection represents a stable equilibrium.



(a) Case a) example (b) Case b) example (c) Case c) example

Figure 3.3: Dynamics behaviour, own elaboration

Definition 3.2: If the values of γ_i in the $\boldsymbol{\gamma}$ vector are sufficiently small, the equilibrium defined in *Definition 3.1* exists and is unique.

3.3 Investigation Strategy

In order to investigate the utility of the Central Planner's intervention, we will perform a series of computational simulations of multiple steady-state levels as function of $\boldsymbol{\gamma}$ vector.

For simplicity, we will assume that the network is composed of two economies ($N = 2$) and then investigate the Central Planner's incentive to shift the equilibrium away from the one determined by the free market (when $\boldsymbol{\epsilon}$=$\mathbf{0}$).

Starting from the simplest case, in which both economies are identical ($\alpha_1 = \alpha_2$ and $s_1 = s_2$) and satisfy the golden rule condition ($\alpha_i = s_i$), before testing different and less conventional parameter combinations. To be more precise, the order of the cases to be analyzed has been arranged according to the following criteria explained in the graph.

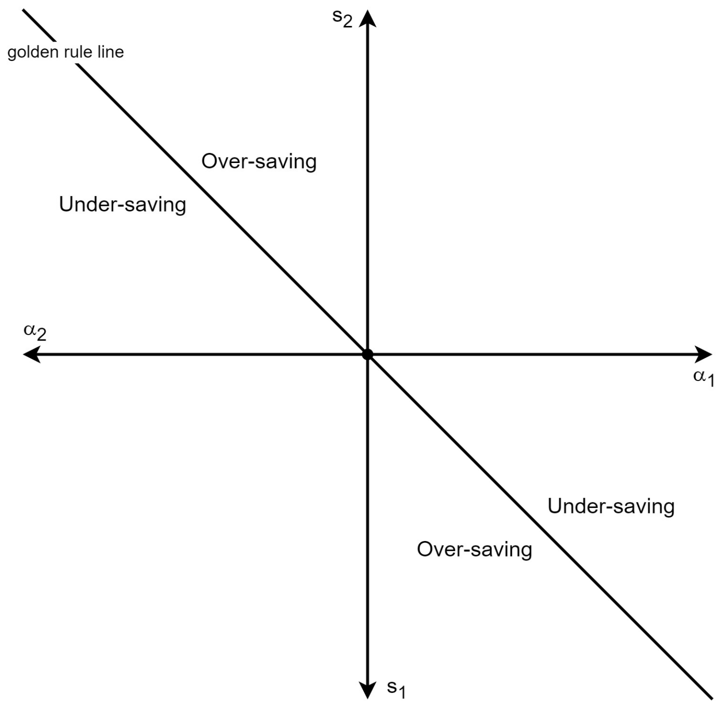


Figure 3.4: Economic behaviour (α, s)

In the fourth quadrant, we find all possible combinations of the pairs (α_1,s_1), while in the second quadrant we find the same for country 2 (α_2,s_2). All the combinations where $\alpha_i = s_i$ lie along the 45-degree line, that is where the golden rule for country i is satisfied.

As it is depicted within the graph, we will test different combinations of homogeneity and heterogeneity on parameters between two countries. We also take into account the role of the golden rule in our specific setting.

Here are the possible cases:
-Identical economies with golden rule achieved.
-Different economies with golden rule achieved.
-Identical economies without golden rule achieved.
-Both countries in oversaving (or both countries in undersaving).
-One country in over-saving and the other in under-saving (the reverse is analogous).

3.4 Assumptions and Interpretations

The main restrictive assumptions we face in our model are those found even in the traditional Solow model. Beyond the presence of a single commodity, we refer especially to the constant saving rate over time assumption. Although we are not dealing with a representative household that maximizes utility, when we refer to network models that consider multiple regions, we can rely on empirical evidence related to the historical series of saving rates in advanced countries. Indeed, we can observe that in the long run, differences in saving rates tend to be stable over time, beyond cyclical components. There appears to be a sort of structural differences in the consumption-saving habits of residents from different countries that remain stable in time.

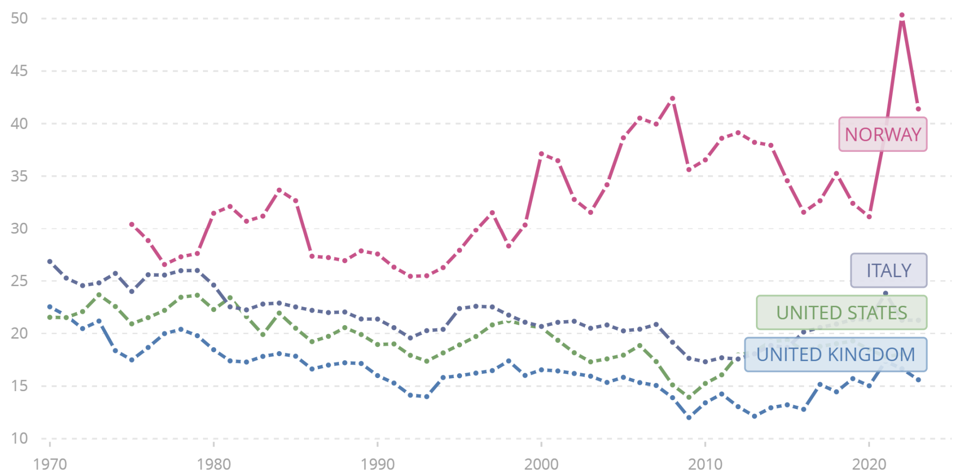


Figure 3.5: Gross savings over GNI, World Bank Data

As specified in the previous subsection, the analysis goes also through cases

where perfect rationality doesn't hold, that is in cases where the golden rule is not respected (overaccumulation and underaccumulation). In these cases, we are asserting that households are not rational enough to determine the saving rate that maximizes aggregate consumption. Supporting this claim is literature in the field of behavioral finance. A significant portion of the literature recognizes that investors tend to be affected by overconfidence, as evidenced by the high volume of financial transactions carried out even under conditions of high risk and low net return. This behavior, revealed in price patterns, has often raised doubts about rational expectations theories. (See Lovallo and Kahneman (2003), Malmendier and Tate (2005) and Daniel and Hirshleifer (2015)).
Additionally, debate in the field of Overlapping Generations Models (OLG) can be mentioned, which has paid particular attention to long-term overaccumulation phenomena. However, there are also some countries whose low capital stock per worker can be attributed to conditions of under-accumulation (see Acemoglu (2009) pag. 354).
Regarding the interpretation of the network framework adopted in our case, we can relate it to a federal system, where each individual node represents a singular state, while the central node (Central Planner) is represented by the Central Government, which determines the flows of fiscal transfers between regions.

3.5 Computational Results

In this subsection we will present numerical simulations of the model, adopting specification described in the previous subsection (i.e. with $N = 2$). We are going through just most significant cases among those highlighted in 4.2, namely identical economies, oversaving vs undersaving economies, sharing the same α and oversaving (high α) vs undersaving (low α) economies. These three cases are respectively depicted as the yellow, green and purple stars in 3.6.

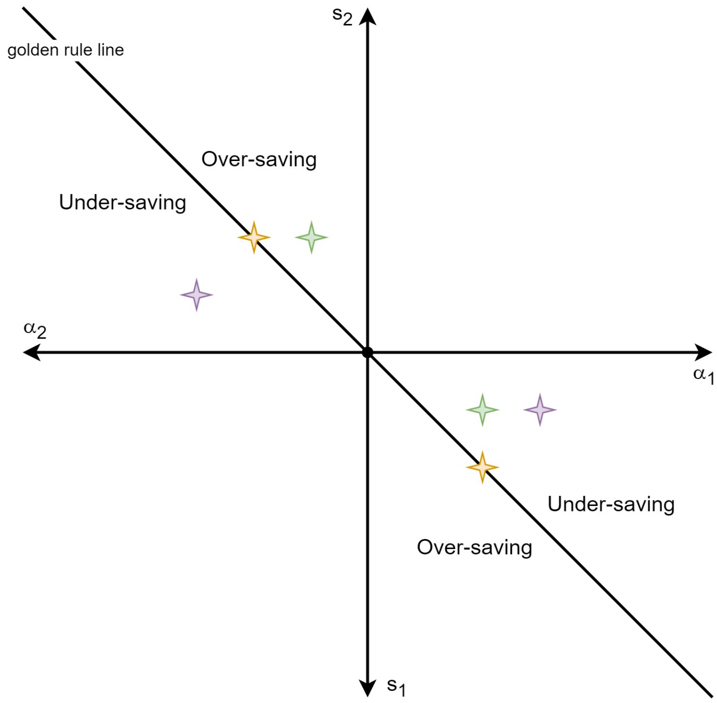


Figure 3.6: Economic behaviour (α, s)

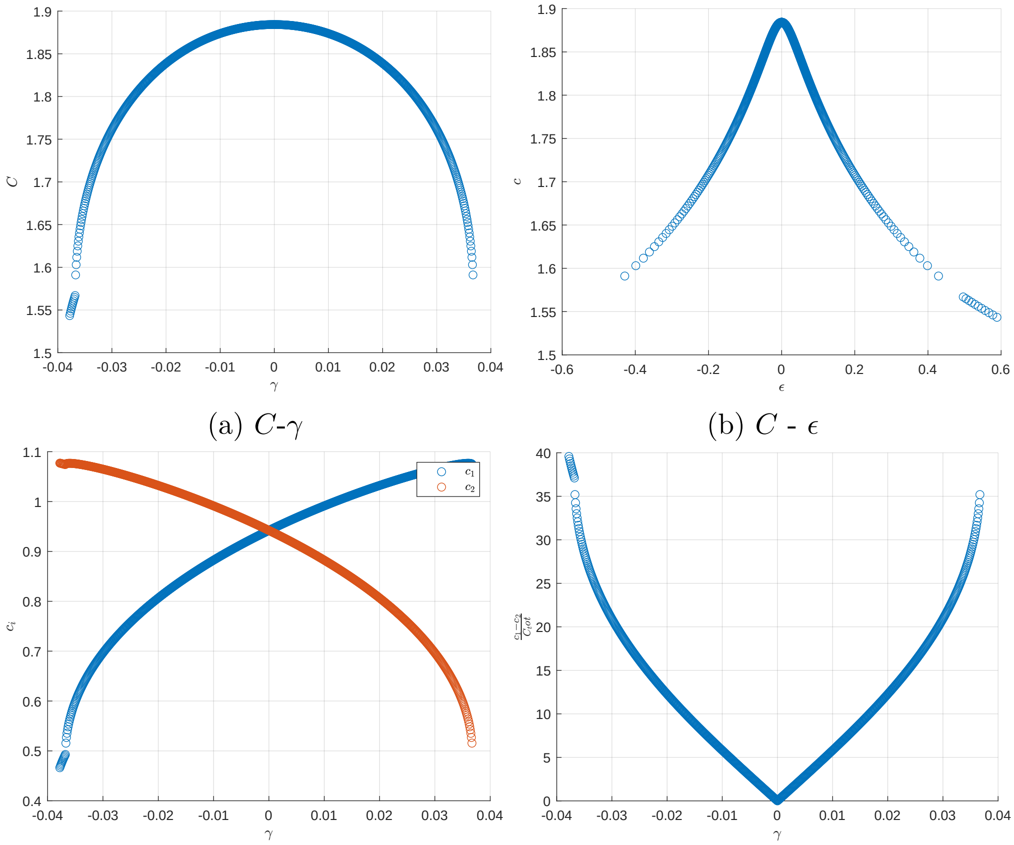
While remaining cases will be exposed within the appendix.
For each model computation, we will present aggregate consumption as a function of γ and ϵ within the first row of plots, in the second one, instead, we will present single economies consumption and a measure of relative inequality with respect to γ. As a measure of welfare dispersion, we adopted the absolute difference of single consumption divided by total consumption, i.e. $\frac{c_1 - c_2}{C}$. In which c_i is the single consumption of country i, while C denotes aggregate consumption of the whole network system.
Within graphical representations are involved two equilibrium benchmark cases, which are useful to compare with what come out by other equilibrium that deviate by those equilibrium. Specifically, we are talking about autarky equilibrium, in which $\gamma = 0$, and when allocation are determined by free capital mobility, that is when the equilibrium is characterized by interest rates equality $\epsilon(\gamma) = 0$. It's worth to note that it is possible for the two previous cases to overlap, especially in most symmetric cases (i.e. identical economies).
Concerning numerical experiment, we rely on the software MATLAB. Algorithm adopted employees a classical numerical solver for system of equations (See Appendix B and Appendix C).
Numerical computation of equilibrium could be managed assuming the interest

rate parity, that together with steady-state equation expressed as ratio gives:

$$\frac{k_1}{k_2} = \frac{\left(\frac{\delta_1}{s_1} - \frac{\gamma}{s_1}(1+\frac{k_2}{k_1})\right)^{\frac{1}{\alpha_1 - 1}}}{\left(\frac{\delta_2}{s_2} + \frac{\gamma}{s_2}(1+\frac{k_1}{k_2})\right)^{\frac{1}{\alpha_2 - 1}}} \tag{3.23}$$

$$\alpha_1 k_1^{\alpha_1 - 1} + \frac{\gamma}{s_1} - \delta_1 = \alpha_1 k_2^{\alpha_2 - 1} - \frac{\gamma}{s_2} - \delta_2 + \epsilon \tag{3.24}$$

3.5.1 Identical Economies



(a) C-γ (b) C - ϵ

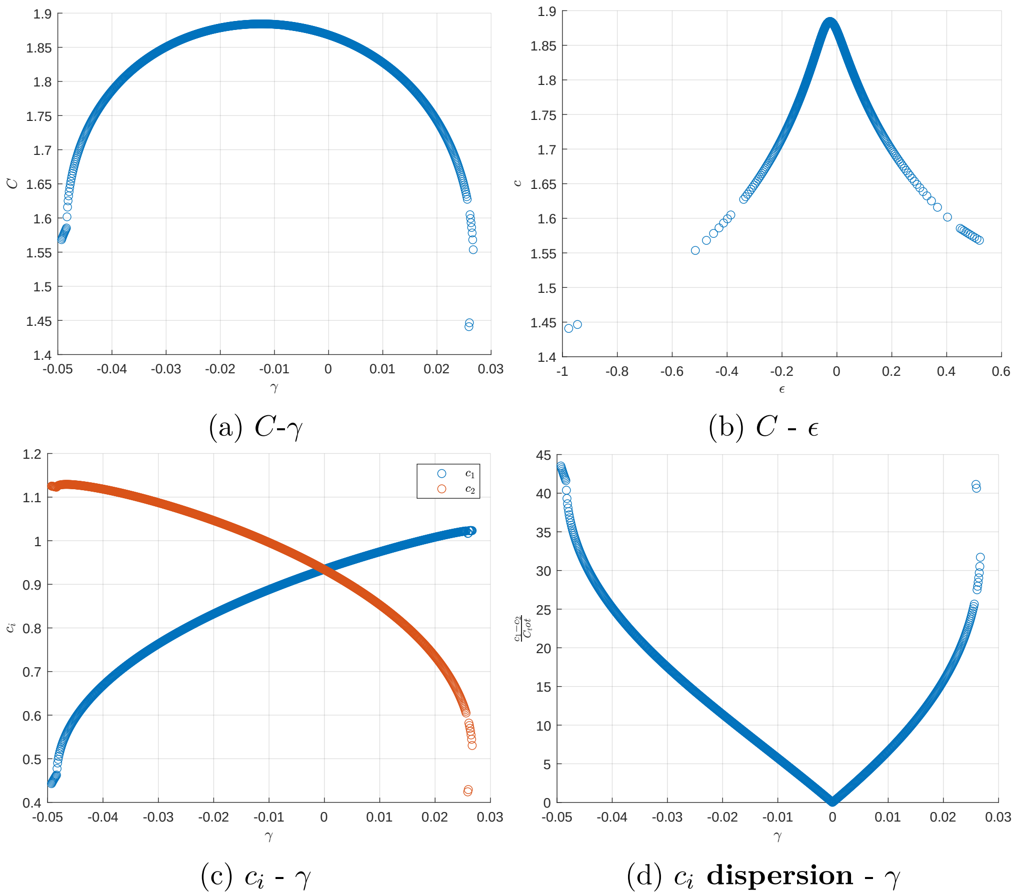
(c) c_i - γ (d) c_i **dispersion** - γ

Figure 3.7: Complete homogeneity with golden rule achieved

Computations presented in Figure 3.7 are the results from completely symmetric network, since both economies have the same capital share, the same saving rate and golden rule is achieved by both countries. The parameters are as follows: $\alpha_1 = \alpha_2 = 0.30$, $s_1 = s_2 = 0.30$ and $\delta_1 = \delta_2 = 0.15$.
This case represents the simplest one and we adopt it as starting point before testing more complex parameter combinations.
Indeed, the two nodes behave as two unconnected traditional Solow economies: the Pareto efficient allocation is reached autonomously by free market and it

leads to maximum welfare and minimum inequality without Central Planner redistribution, i.e. $\gamma = \epsilon = 0$ (allocation in question also correspond to autarky equilibrium). Finally, every Central Planner intervention damages economy in terms of both efficiency and inequality.

3.5.2 Oversaving vs Undersaving Economy, Sharing the Same α



(a) C-γ (b) C - ϵ

(c) c_i - γ (d) c_i **dispersion** - γ

Figure 3.8: Oversaving vs undersaving, same α

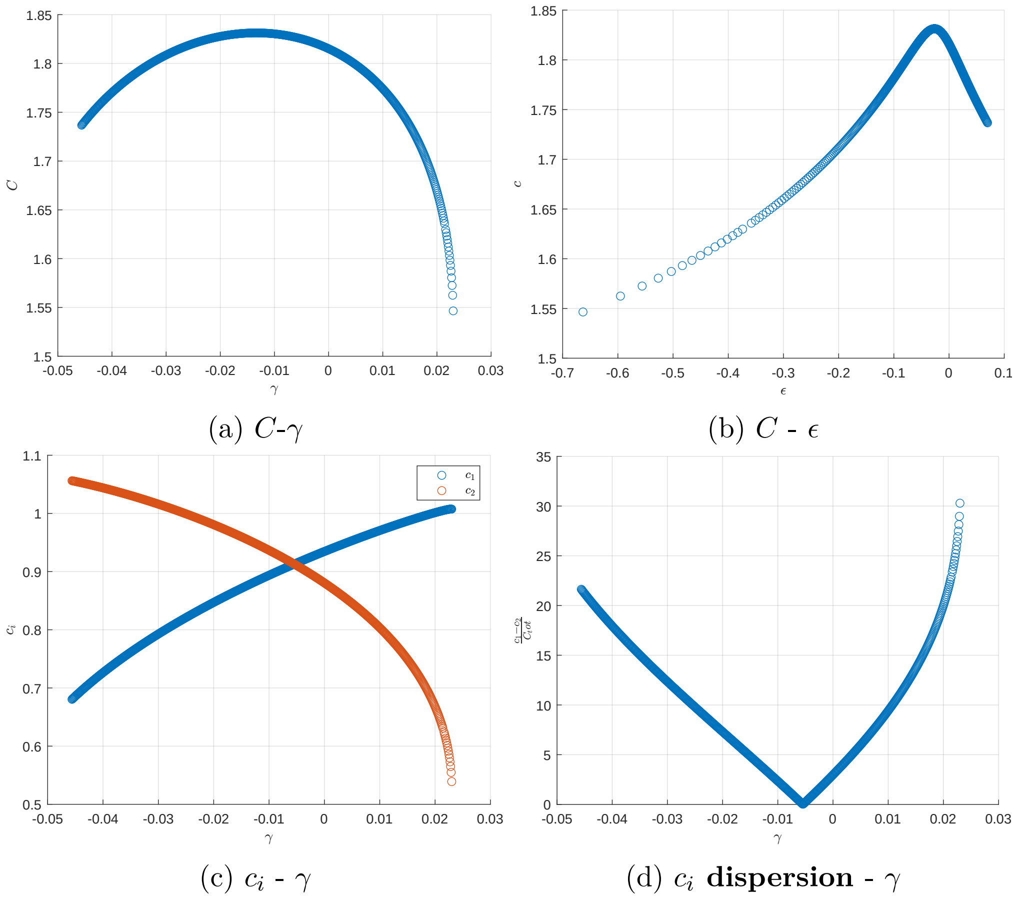
.

In the current case we are facing two economies with same capital share $\alpha_1 = \alpha_2 = 0.30$, where both countries are failing to achieve golden rule condition. Specifically, country 1 is oversaving ($s_1 = 0.35$), while country 2 is undersaving ($s_2 = 0.25$). Depreciation rates are $\delta_1 = \delta_2 = 0.15$.
The following simulation reveals that in the $C - \gamma$ plot, optimal welfare is achieved when the oversaver country transfers resources to the undersaver

one. This implies a $\gamma < 0$ and it can also be observed by looking at aggregate consumption as a function of the interest rate gap ϵ, where the global maximum differs from the equilibrium determined by free capital mobility. In other words, country 2 must maintain a higher interest rate than country 1, which engages in oversaving.

Moreover, when Central Planner reallocates resources in favor of the undersaving economy, it generates inequality, indicating that the Central Planner faces an efficiency-inequality trade-off.

3.5.3 Oversaving (High α) vs Undersaving (Low α) Economy



(a) C-γ (b) C - ϵ (c) c_i - γ (d) c_i **dispersion** - γ

Figure 3.9: Oversaving vs undersaving, different α.

Now we are going to present the most interesting case.

The two economies feature full heterogeneity on parameters, where country 1 is oversaving and country 2 is undersaving. Unlike the previous case, the countries have also different α_i. Specifically, country 1 is characterized by a

higher capital share compared to country 2: $\alpha_1 = 0.30$, $\alpha_2 = 0.25$, $s_1 = 0.35$, $s_2 = 0.20$ and $\delta_1 = \delta_2 = 0.15$.
From the computational results, it emerges that it is optimal for the Central Planner to transfer resources from the oversaving country to the undersaving one (i.e., from country 1 to country 2). Moreover, shifting resources to country 2 up to a certain level leads to a decrease in inequality. This is straightforward to note looking at subplot (a) and subplot (d), where γ of absolute equality falls within the range defined by the γ of free market and the γ of maximum aggregate consumption equilibrium.

Definition 3.3: whenever we are facing a case such that $s_2 < \alpha_2 < \alpha_1 < s_1$ (or $s_1 < \alpha_1 < \alpha_2 < s_2$), the Central Planner it's able to enhance global welfare and reduce inequality at the same time.

3.6 Sharing Income Version

In this section we present a variant of the model where the central node of the network, instead of redistributing capital, he controls flows of income. Assumptions and network framework still exactly the same of original version of the model presented so far.
In this section we will go through Microfoundation, equilibrium solution conditions and computational results (the same three cases), as we done for the main version of the model. Since most concepts are shared with the capital-sharing version, we will only present the differences relative to the previous model.

3.6.1 Microfoundation

Now consider the setting where income can flow between the N economies:

$$\dot{k}_i = s_i \left[f_i(k_i) + \gamma_i \sum_{j=1}^{N} f_j(k_j) \right] - \delta_i k_i \tag{3.25}$$

Here, γ_i denotes the share of global GDP that flows into or out of country i. Moreover, GNP is subject to maximization on firm's problem, i.e. $y_i = f_i(k_i) + \gamma_i \sum_{j=1}^{N} f_j(k_j)$.
After imposing neoclassical production function $F(K, L) = K_i^{\alpha} L_i^{(1-\alpha)}$, repre-

sentative firm's problem in effective labour terms takes the following form:

$$\max_{K_i,L_i} \pi_i = [k_i^{\alpha_i} + \gamma_i \sum_{j=1}^{n} k_j^{\alpha_j} - (\rho_i + \delta_i)k_i - w_i]L_i \tag{3.26}$$

Let's compute FOCs:

$$\frac{d\pi_i}{dK_i} = \alpha_i k_i^{\alpha_i - 1}(1 + \gamma_i) - (\rho_i + \delta_i) = 0 \tag{3.27}$$

$$\frac{d\pi_i}{dL_i} = (1 - k_i)(1 + \gamma_i)\alpha_i k_i^{\alpha_i - 1} - w_i = 0 \tag{3.28}$$

Final structure of the model with N generic economies in sharing income version model is:

$$\dot{k}_i = s_i \left[k_i^{\alpha_i} + \gamma_i y_T\right] - \delta_i k_i \tag{3.29}$$

with

$$y_T = \sum_{j=1}^{N} k_j^{\alpha_j} \tag{3.30}$$

and payment to factors

$$\rho_i = \alpha_i k_i^{\alpha_i - 1}(1 + \gamma_i) - \delta_i \tag{3.31}$$

$$w_i = (1 - k_i)(1 + \gamma_i)\alpha_i k_i^{\alpha_i - 1} \tag{3.32}$$

Similarly to capital sharing version, γ_i represent a fraction of total GDP of network system that are determined by the Central Planner and implications on free market conditions still the same.
Nevertheless, while γ_i still positively affect the interest rate even in this version, it no longer acts as a counter to capital depreciation. Instead, it now boosts the marginal return on production linearly in the interest rate equation 3.31
Now, we will also interpret this second version from the perspective of accountability and the clearing conditions.
Since the resources subject to redistribution are no longer capital, which was previously destined to increase capital stock immediately, but rather income, the accounting logic must differ. In this version, the inflow of resources, z_i, must be divided between consumption and savings. Therefore, the gross national income for country i is defined as:

$$x_i = y_i + z_i = k_i^{\alpha_i} + \gamma_i \sum_{j=1}^{N} k_j \tag{3.33}$$

and it must be equal to the sum of consumption and investment, i.e.

$$i_i = s_i[k_i^{\alpha_i} + \gamma_i \sum_{j=1}^{N} k_j] \tag{3.34}$$

$$c_i = (1 - s_i)[k_i^{\alpha_i} + \gamma_i \sum_{j=1}^{N} k_j] \tag{3.35}$$

where $c_i + i_i = x_i$.
Finally, zero-profit condition imposed by perfect competition leads to:

$$x_i = k_i^{\alpha_i} + \gamma_i \sum_{j=1}^{N} k_j = (\rho_i + \delta_i)k_i + w_i \tag{3.36}$$

3.6.2 Numerical Treatment

Now let's find out a compact formulation of differential equations system as done in the capital sharing model. Given the shares vector $\boldsymbol{\gamma}$, a new adapted compact formulation of capital law of motion that exploit gamma vector and a compact formulation of the new interest rate, sharing income equilibium conditions comes out as:

$$\begin{cases} \dot{\mathbf{k}} = \mathbf{s} \odot \left[\mathbf{k}^{\boldsymbol{\alpha}} + \boldsymbol{\gamma} \sum_{j=1}^{n} k_j^{\alpha_j} \right] - \boldsymbol{\delta} \odot \mathbf{k} \\ \boldsymbol{\rho} = \boldsymbol{\alpha} \odot \mathbf{k}^{\boldsymbol{\alpha}-1} \odot (\mathbf{1} + \boldsymbol{\gamma}) - \boldsymbol{\delta} + \boldsymbol{\epsilon} \\ \rho_i = \rho_j \quad \forall i, j \in [1:N] \end{cases} \tag{3.37}$$

Definition 3.4: given the $\boldsymbol{\gamma}$ vector, the couple ($\mathbf{k}$, $\boldsymbol{\rho}$) form an equilibrium if $\dot{\mathbf{k}} = 0$ holds.

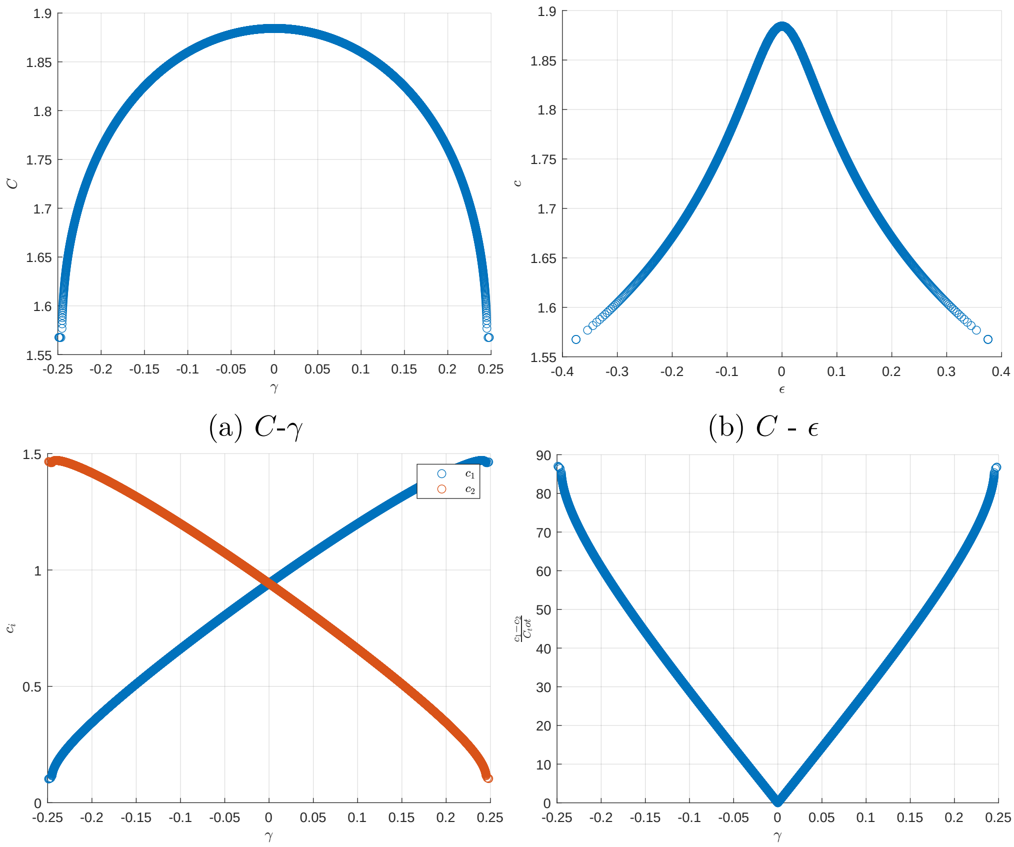
3.6.3 Model with N=2

When we face the model with just two countries, the numerical problem simplifies to the following equations system:

$$\begin{cases} \dot{k}_1 = s_1 \left[k_1^{\alpha_1} + \gamma(k_1^{\alpha_1} + k_2^{\alpha_2})\right] - \delta_1 k_1 \\ \dot{k}_2 = s_2 \left[k_2^{\alpha_2} - \gamma(k_1^{\alpha_1} + k_2^{\alpha_2})\right] - \delta_2 k_2 \\ \alpha_1 k_1^{\alpha_1 - 1}(1 + \gamma) - \delta_1 = \alpha_2 k_2^{\alpha_2 - 1}(1 - \gamma) - \delta_2 + \epsilon \end{cases} \tag{3.38}$$

As with the capital-sharing model, we will present the numerical solution for the $N = 2$ specification just introduced, beyond the fact that, for this variant of the model, we were unable to get an analytical solution.

3.6.4 Identical Economies



(a) C-γ

(b) C - ϵ

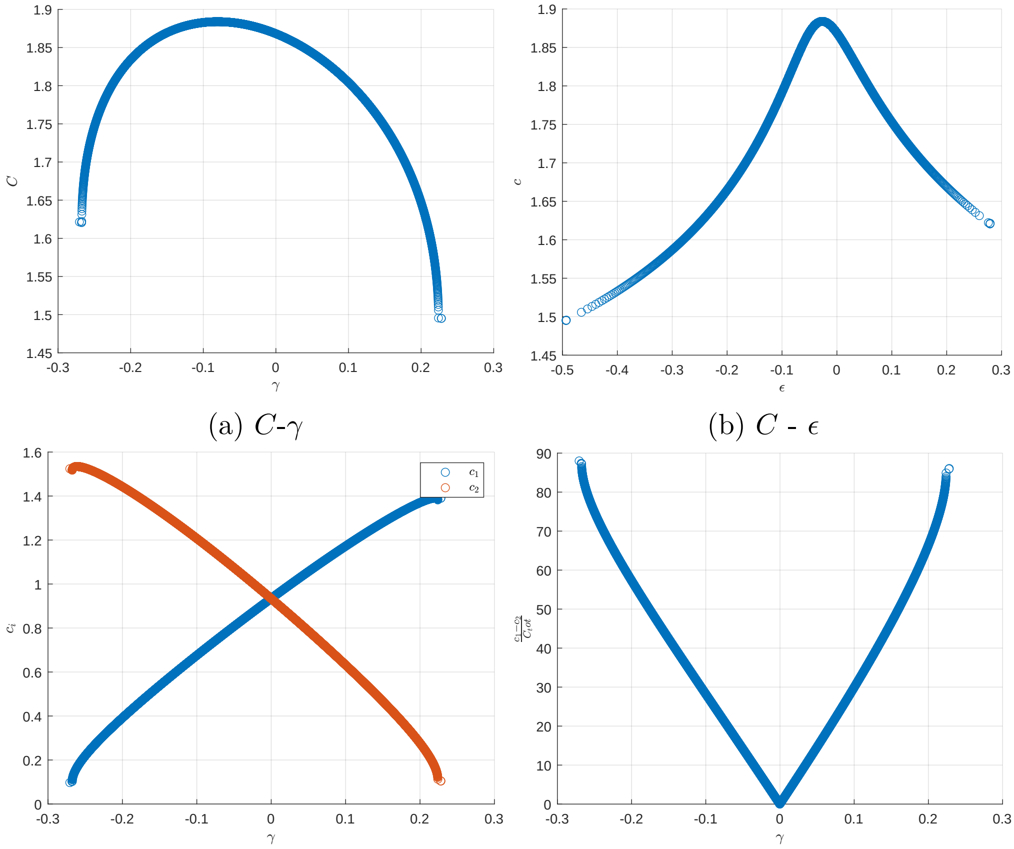
(c) c_i - γ

(d) c_i **dispersion** - γ

Figure 3.10: Complete homogeneity with golden rule achieved

Simulating equilibria with two identical economies, i.e. $\alpha_1 = \alpha_2 = s_1 = s_2 = 0.30$ and $\delta_1 = \delta_2 = 0.15$, the optimal allocation, both in terms of global consumption and inequality, coincides with autarky allocation. Nothing different from sharing capital model.

3.6.5 Oversaving vs Undersaving economy, Sharing the Same α



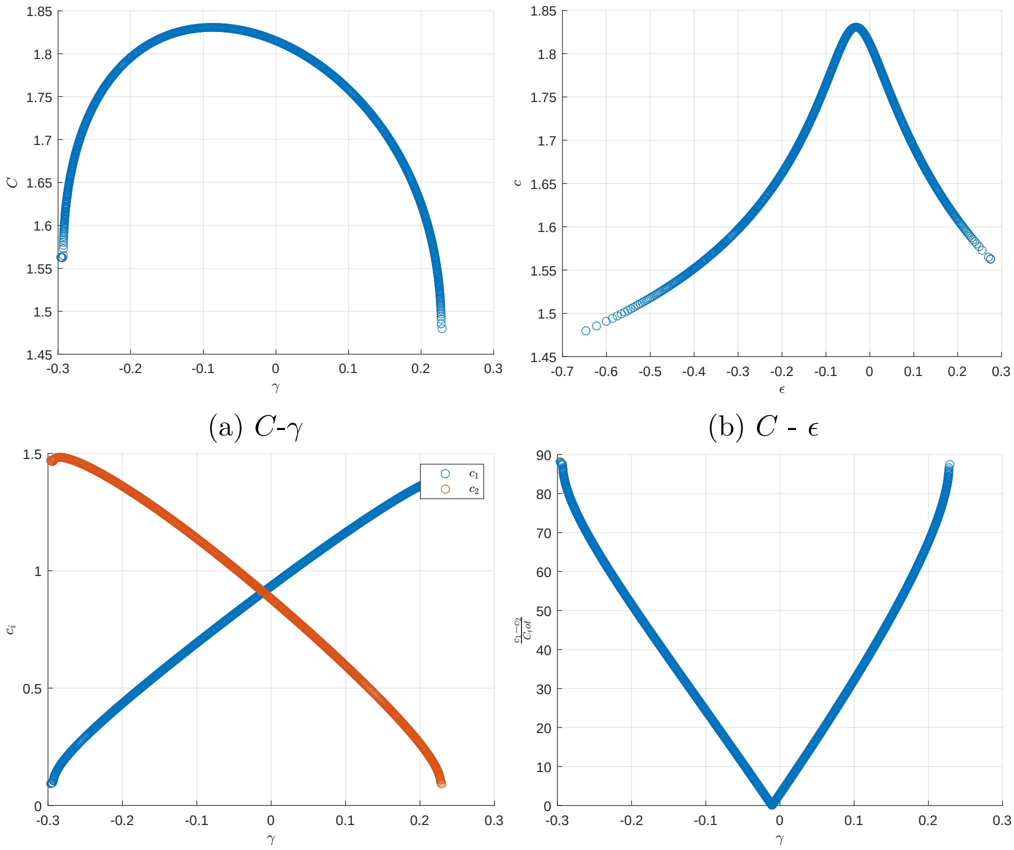
(a) C-γ (b) C - ϵ

(c) c_i - γ (d) c_i **dispersion** - γ

Figure 3.11: Oversaving vs undersaving, same factor share

Even when heterogeneity on parameters is introduced, we observe that the results do not differ from those in the capital-sharing model. Exactly as in 3.5.2, we are facing two economies with same capital share $\alpha_1 = \alpha_2 = 0.30$, where both countries are failing to achieve golden rule condition. Precisely, country 1 is oversaving ($s_1 = 0.35$) and country 2 is undersaving ($s_2 = 0.25$), implying that the Central Planner faces an efficiency-inequality trade-off as in capital sharing version.

3.6.6 Oversaving (high α) vs undersaving (low α) economy



(a) C-γ

(b) C - ϵ

(c) c_i - γ

(d) c_i **dispersion** - γ

Figure 3.12: Oversaving vs undersaving, different α

The case in which the two economies feature full heterogeneity on parameters, in which country 1 is oversaving and country 2 is undersaving, confirms that sharing income model behaves like the capital share model, i.e. shifting resources from the oversaving country to the undersaving one, the Central Planner could simultaneously achieve an increase in global consumption and a reduction in inequality.

3.7 Welfare Analysis

Since the problem of analysis is framed in terms of a Central Planner that redistributes flows of resources and in most cases we saw that he would face efficiency-inequality trade-off, it may be useful to introduce an objective function related to global welfare, which would be penalized by dispersion of global

consumption among nodes.
A famous metric that considers both aggregate consumption and inequality can be represented by the social welfare function proposed in Atkinson (1970):

$$W(c_i) = \sum_{i=1}^{n} u_i(c_i) \tag{3.39}$$

where is adopted a CRRA (Constant Relative Risk Aversion) utility function:

$$u_i(c_i) = (c_1^{1-\eta})^{\frac{1}{1-\eta}} \tag{3.40}$$

In this function, η represents the coefficient of inequality aversion: the higher the value of η, the greater the penalty that function $W(c_i)$ incurs as the difference between the c_i increases.

$$W(c_i) = \sum_{i=1}^{n} (c_i^{1-\eta})^{\frac{1}{1-\eta}} \tag{3.41}$$

This type of welfare function is particularly useful in our case, as it allows us to narrow the analysis within the domain of γ between γ_C (the γ that maximizes the sum of consumption) and γ_0 (the γ that ensures perfect equality in consumption). This enables us to maximize the welfare function and obtain a socially optimal γ_w^*.
Analyzing the two extreme cases and the intermediate one, we would obtain:

-$\eta = 0$: in this case, the welfare function becomes a simple sum of consumption:

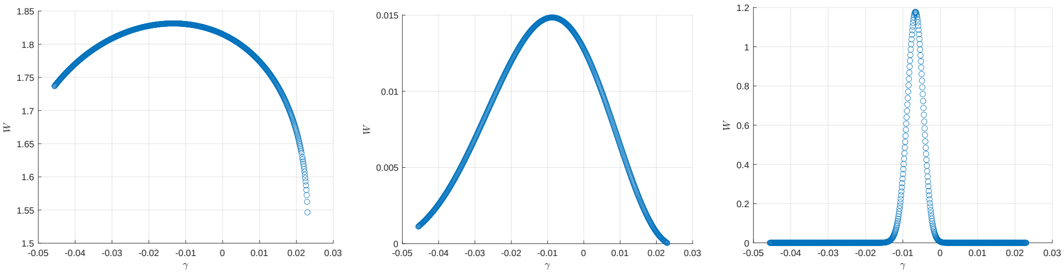
$$W(c_i) = \sum_{i=1}^{n} c_i \tag{3.42}$$

and in this scenario the social optimum γ overlaps the γ maximizer of global consumption, i.e. $\gamma_w^* = \gamma_C$.

-$\eta = +\infty$: even the slightest inequality in consumption leads to a severe penalty in the welfare function. It's easy to see that the γ maximizing welfare tends towards the γ of perfect equality: $\gamma_w^* \to \gamma_0$.

- $\eta \in [0, +\infty)$: in the third and intermediate case, we set inequality aversion parameter such that social optimum doesn't lead to one of the two boundary cases, giving balanced weight to both inequality and aggregate consumption.

Hence, we would get $\gamma_w^* \in [\gamma_0, \gamma_C)$ or $\gamma_w \in (\gamma_C^*, \gamma_0]$.



(a) $\eta = 0$ (b) $\eta = 5$ (c) $\eta = 20$

Figure 3.13: Employing welfare objective function to case 3.5.3.

Conclusion

This master's thesis contributes to the field of Solow-Swan models within an open economy framework, as well as to the field of network economics models. The literature on open economy with the Solow model is not particularly extensive, likely due to the model's excessive simplicity and its limitations in studying typical international economic issues. However, this same simplicity makes it a suitable model to be studied within a structure of direct interactions, such as a network.

As we mentioned in the introduction and in Chapter 1, the only contribution in which a Solow model in open economy with a multi-country network has been adopted is found in Sorger (2003). Differently by our model, the network system introduced by Sorger involves a series of N countries whose residents can freely invest in the capital of other countries and obtain the marginal return on the capital in which they have invested. In our network framework, we adopt a centralized model in which the central node redistributes capital among the various nodes, akin to a central government or a Central Planner.

Given our framework focused on the Central Planner's perspective, we experimented with various combinations of model parameters to identify when the central node would have the opportunity to intervene. More specifically, I am referring to cases where the redistribution of capital (or income) results in a more efficient equilibrium allocation in terms of global consumption and/or a more equal distribution.

The main results presented throughout this thesis can be summarized in two points:

1) Differently from the model introduced by Sorger (2003), the system of equations that defines our model does not suffer from indeterminacy of solution, even in the case of complete heterogeneity of parameters. Furthermore, within a certain range of γ, the model seems to exhibit existence and uniqueness of the solution. We note that this statement is confirmed by the computational simulations performed and reported in Section 3.5. Analytically, we were able

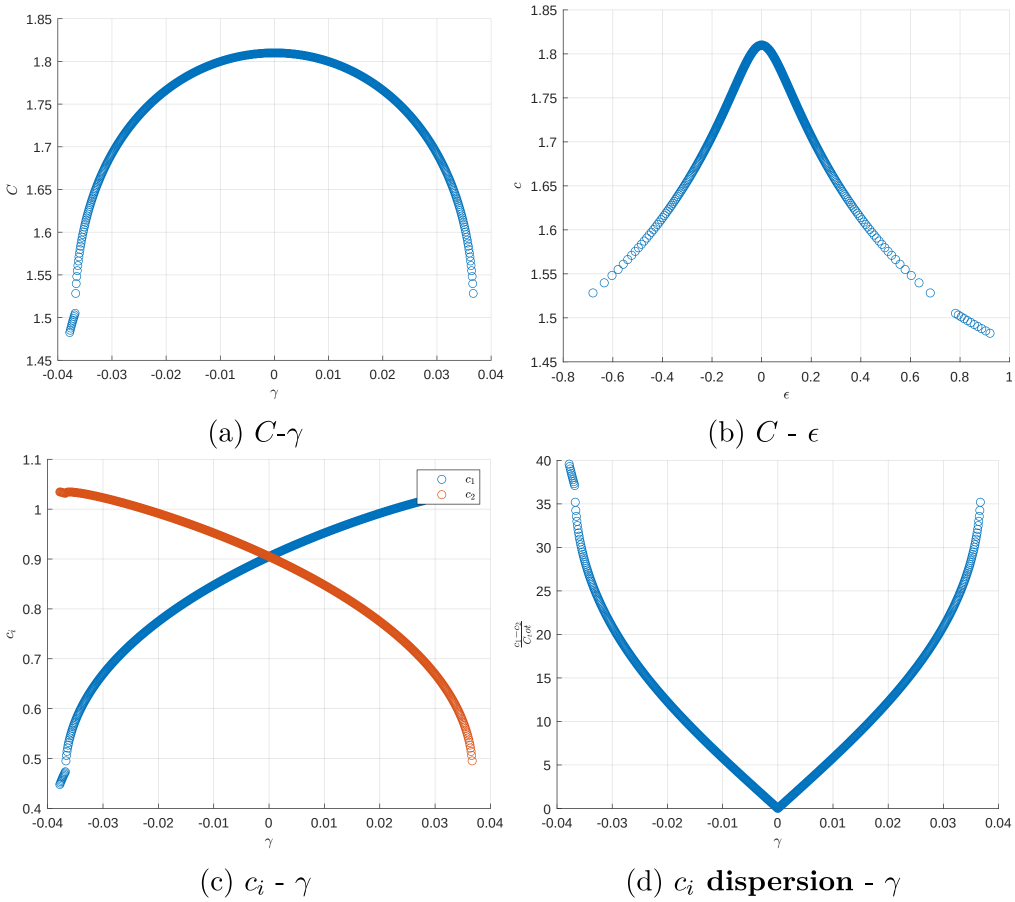
to derive only a necessary but not sufficient condition, as discovering a sufficient or necessary and sufficient condition for existence and uniqueness still a challenging task until an analytical solution of the model is obtained.
2) Whenever Solow economies don't achieve golden rule condition ($s_i \neq \alpha_i$) and countries show differences in technology and behavior (α_i, s_i), the intervention of a Central Planner can be beneficial since he can enhance global welfare, reduce inequality, or achieve both objectives.
The cases that were subject to computational analysis and respectively results could be summarized as follow:
- Identical economies without golden rule achieved: as in the case of two identical economies that achieve the golden rule, any redistribution by a Central Planner will be harmful to both efficiency and equality. The only difference compared to case 3.5.1 is that C^* will reach a smaller amount since the golden rule is not achieved.
-Different economies with golden rule achieved: in this case, the most efficient allocation still coincides with that of the free market, which is easy to understand from the fact that both countries respect the golden rule. Differently from the previous case, in a laissez faire equilibrium, there is inequality in consumption, as the country with a lower capital accumulation capacity would be richer if it received resources from the wealthier country. In this case, a trade-off between efficiency and equity occurs.
-Overaccumulation vs overaccumulation or underaccumulation vs underaccumulation (heterogeneity on parameters): since both countries are facing the same kind of dynamic inefficiency, it would be efficient for both to save more or save less at the same time. This implies that the most efficient allocation is very close to the free market allocation (because the bilateral capital flows compensate each other, leading a γ close to zero).
-Oversaving vs undersaving economy, sharing the same α: in order to achieve maximum efficient allocation, Central Planner should shift capital from oversaver to undersaver country. However, both countries share the same capital factor share, which leads to increasing inequality, negatively impacting the consumption of the oversaving country.
-Oversaving (high α) vs undersaving (low α) economy: this represents the most interesting case, where the Central Planner can increase efficiency and reduce inequality at the same time. This occurs by shifting capital from the oversaving country to the undersaving country. Furthermore, since the countries have different α, it is easy to observe that the free market equilibrium leads to an

initially unequal allocation: this is why shifting resources to the undersaving (and initially poorer) country not only improves efficiency, as both countries offset the dynamic inefficiency (oversaving and undersaving), but also provides more resources to the country with a lower marginal productivity of capital (assuming other factors are the same).

Appendix A

Additional Results

A.1 Identical Economies without Golden Rule Achieved.



(a) C-γ

(b) C - ϵ

(c) c_i - γ

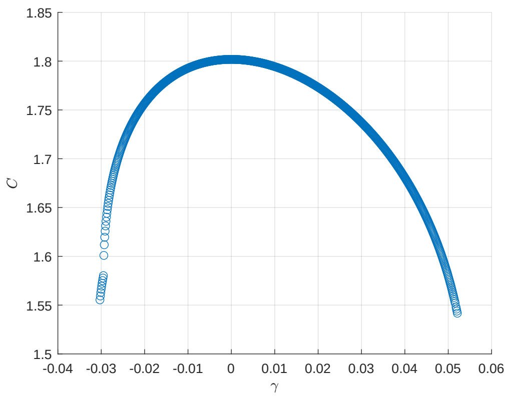
(d) c_i **dispersion** - γ

Figure A.1: $s_1 = s_2 = 0.20$ and $\alpha_1 = \alpha_2 = 0.30$

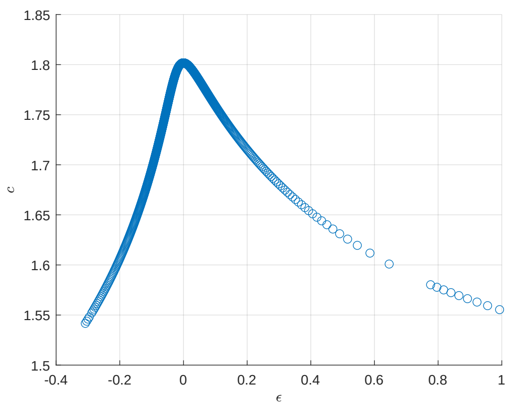
The results suggest that in the case of identical countries, even when the golden rules are not respected, the most efficient and equal allocation corresponds to that of both the free market and autarky. ($\gamma = \epsilon = 0$).

Definition A.1.1: whenever all countries are identical in terms of parameters, the most efficient allocation corresponds to both the free market and autarky allocation ($\gamma = \epsilon = 0$), regardless of the golden rule.

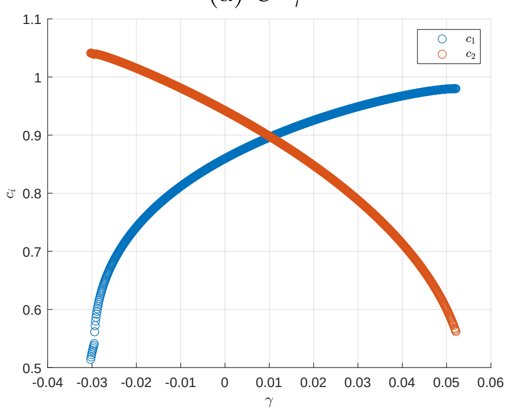
A.2 Different Economies with Golden Rule Achieved.



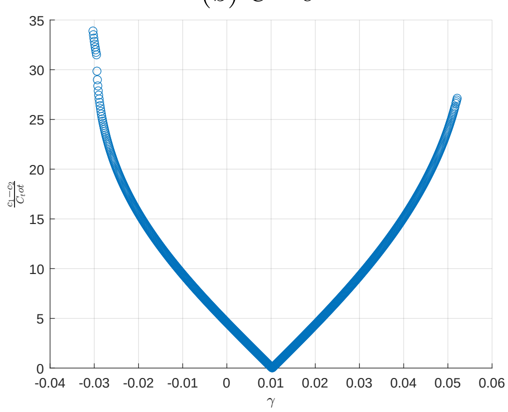
(a) C-γ



(b) C - ϵ



(c) c_i - γ



(d) c_i **dispersion** - γ

Figure A.2: $\alpha_1 = s_1 = 0.20$ and $\alpha_2 = s_2 = 0.30$

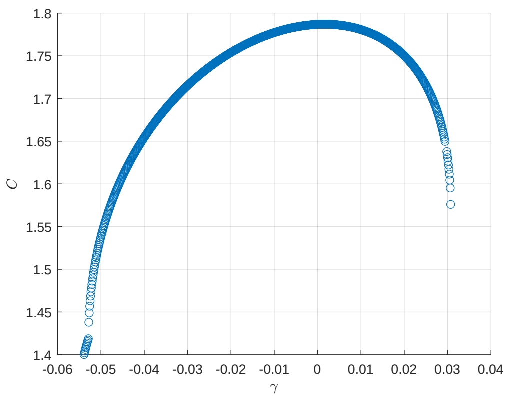
In a golden rule regime respected by both countries, but with heterogeneity in parameters, it can be observed that the maximum global consumption is achieved in a free market regime (and at the same time in autarky), such as in the case of Identical Economies 3.5.1.
Nevertheless, because of heterogeneity on behaviour parameters, the most equal allocation doesn't coincide anymore with most efficient allocation.

It is worth noting that whenever the most efficient allocation perfectly coincides with the autarky allocation, it means that is optimal for both economies to operate as two distinct traditional Solow-Swan economies.

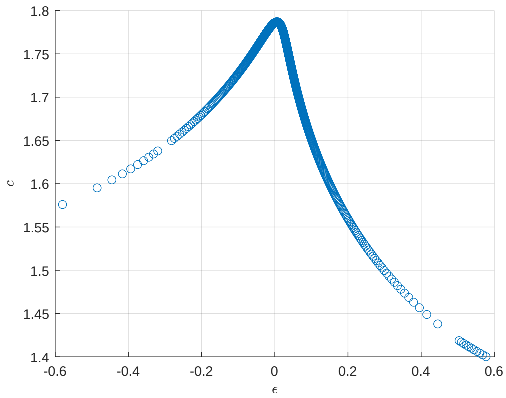
Definition A.1.2: whenever all countries achieve the golden rule condition, the most efficient allocation corresponds to both the free market and autarky allocation ($\gamma = \epsilon = 0$), regardless of differences in country parameters.

Definition A.1.3: Central Planner could achieve C^* through γ only when all countries have differences and parameters and at the same time do not respect the golden rule.

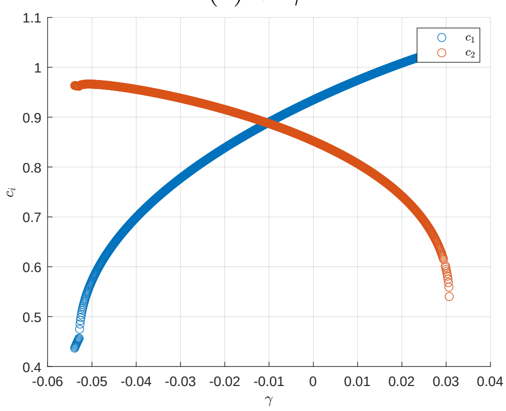
A.3 Overaccumulation vs Overaccumulation (Heterogeneity on Parameters)



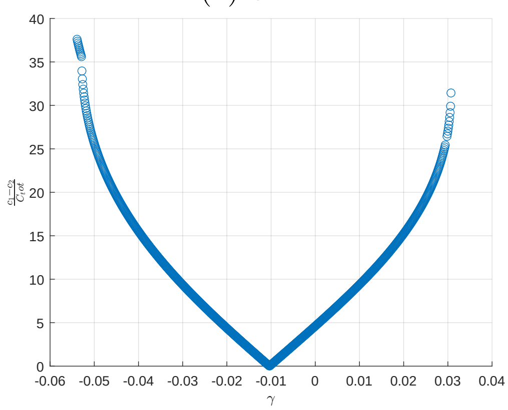
(a) C-γ



(b) C - ϵ



(c) c_i - γ



(d) c_i **dispersion** - γ

Figure A.3: $\alpha_1 = s_1 = 0.20$ and $\alpha_2 = s_2 = 0.30$

The equilibrium allocation with maximum global consumption corresponds to very small γ. This is due to the fact that both countries are in overaccumulation, so both would prefer to accumulate less. However, since country two has a smaller capital accumulation capability compared to country one, the allocation of perfect equality would require a transfer from the country with greater accumulation capacity to the country with less, i.e. $\gamma < 0$.
The case of underaccumulation vs underaccumulation will not be presented, as it is exactly specular with respect to current case.

Appendix B

MATLAB Code: Sharing Capital Model

```
%% CASO N=2 ECONOMIA
clear all
step=0.1;

%% Definizione intervallo variazione parametri
% saving rates
% s1l= 0.1:0.001:0.4;
s1l= 0.35;
s2l= 0.20;
% capital shares
a2l=0.25; %0.2:0.001:0.4;
a1l=0.30;
% depreciation rate
d1l=0.15; %[0.1:0.001:0.2];
d2l=0.15;
gamma_nl = -0.1:0.0001:0.1;
% gamma_nl = 0;

eta=20;
C_mat = zeros(length(s1l),length(s2l));

%% preallocazione NAN
k2k1=nan(length(s1l),length(s2l),length(a1l),length(a2l),length(d1l),
    length(d2l),length(gamma_nl));
```

```
gamma_n=nan(length(s1l),length(s2l),length(a1l),length(a2l),length(d1l),length(d2l),length(gamma_nl));
err=nan(length(s1l),length(s2l),length(a1l),length(a2l),length(d1l),length(d2l),length(gamma_nl));
ff=nan(length(s1l),length(s2l),length(a1l),length(a2l),length(d1l),length(d2l),length(gamma_nl));
% preallocazione tabella
tabella=array2table(zeros(0,21));
% dichiaro columns tabella
tabella.Properties.VariableNames=["k2k1","eps","gamma","e","f","s1","s2","a1","a2","d1","d2","k1","k2","r1","r2","c","c1","c2","cdiff","w_diff","w"];
% contatore
count=0;
% numero di combinazioni
NN=numel(s1l)*numel(s2l)*numel(a1l)*numel(a2l)*numel(d1l)*numel(d2l)*numel(gamma_nl);
%% LOOP
for i=1:numel(s1l)
    for j=1:numel(s2l)
        for h=1:numel(a1l)
            for l=1:numel(a2l)
                for m=1:numel(d1l)
                    for n=1:numel(d2l)
                        for p=1:numel(gamma_nl)
                            s1=s1l(i);
                            s2=s2l(j);
                            a1=a1l(h);
                            a2=a2l(l);
                            d1=d1l(m);
                            d2=d2l(n);
                            gamma = gamma_nl(p);
                            % GUESS del problema numerico (K2/K1 e gamma)
                            guess=[(s2/d2)^(1/(1-a2))/(s1/d1)^(1/(1-a1)) 0];
                            ii=0;
                            try
                                % SOLVER del sistema non lineare
```

```
        [x,f,e]=fsolve(@(xx) semi_ratio_analitic(xx,a1,a2,s1,s2,d1,d2,gamma),guess,optimset('Display','off','TolFun',10^-10,'MaxIter',1000));
    catch
        x(1)=nan;
        x(2)=nan;
        e=-100;
        f=nan;
    end
    % SE ABBIAMO UNA SOLUZIONE...
    if e>0
        k2k1(i,j,h,l,m,n)=x(1);
        eps(i,j,h,l,m,n)=x(2);
    else
        k2k1(i,j,h,l,m,n)=nan;
        eps(i,j,h,l,m,n)=nan;
    end
    % OTTENGO CAPITALI E INTEREST RATES
    k1=(1/s1*(d1-gamma*(1+k2k1(i,j,h,l,m,n))))^(1/(a1-1));
    k2=(1/s2*(d2+gamma*(1+1/k2k1(i,j,h,l,m,n))))^(1/(a2-1));
    r1=a1*k1^(a1-1)+gamma-d1;
    r2=a2*k2^(a2-1)-gamma-d2;
    c1=(1-s1)*(k1^a1);
    c2=(1-s2)*(k2^a2);
    cdiff = c1 - c2;
    % CONSUMO AGGREGATO
    c=c1+c2;
    w_diff = abs(cdiff)/c*100;
    w= (c1^(1-eta) + c2^(1-eta))^(1/1-eta);

    if imag(c) ~= 0
        C_mat(i,p)=nan;
    else
        C_mat(i,p)=c;
    end

    % REGISTRO CASO COME RIGA nella tabella
```

```
                        tabella=[tabella;{x(1),x(2),gamma,e,mean(f),
                            s1,s2,a1,a2,d1,d2,k1,k2,r1,r2,c,c1,c2,
                            cdiff,w_diff,w}];
                        % CONTO E DISPLAY
                        count=count+1;
                        disp(count/NN*100)
                    end
                end
            end
        end
    end
end
end
% SALVATAGGIO
save results_e_uguale k2k1 gamma s1l s2l a1l a2l d1l d2l tabella
```

```
function [f,x]=semi_ratio_analitic(x,a1,a2,s1,s2,d1,d2, gamma)

% ROOTS DEL SISTEMA
k2k1=x(1);
ep=x(2);
% SISTEMA DI EQUAZIONI tale che F=0
f(1)=1/k2k1-(((1/s1*(d1-gamma*(1+k2k1)))^(1/(a1-1)))/((1/s2*(d2+gamma
    *(1+1/k2k1)))^(1/(a2-1))));
f(2)=a1/s1*(d1-gamma*(1+k2k1))+gamma-d1-(a2/s2*(d2+gamma*(1+1/k2k1))-
    gamma-d2)-ep;
end
```

Appendix C

MATLAB Code: Sharing Income Model

```
%% CASO N=2 ECONOMIA
clear all
step=0.1;

%% Definizione intervallo variazione parametri
% saving rates
s1l= 0.35;
s2l= 0.20;
% capital shares
a2l=0.25;
a1l=0.30;
% depreciation rate
d1l=0.15;
d2l=0.15;
gamma_nl = -0.30:0.0001:0.30;

eta=20;
C_mat = zeros(length(s1l),length(s2l));

%% preallocazione NAN
k2k1=nan(length(s1l),length(s2l),length(a1l),length(a2l),length(d1l),
    length(d2l),length(gamma_nl));
gamma_n=nan(length(s1l),length(s2l),length(a1l),length(a2l),length(
    d1l),length(d2l),length(gamma_nl));
```

```
err=nan(length(s1l),length(s2l),length(a1l),length(a2l),length(d1l),length(d2l),length(gamma_nl));
ff=nan(length(s1l),length(s2l),length(a1l),length(a2l),length(d1l),length(d2l),length(gamma_nl));
% preallocazione tabella
tabella=array2table(zeros(0,20));
% dichiaro columns tabella
tabella.Properties.VariableNames=["k1","k2","eps","gamma","e","f","s1","s2","a1","a2","d1","d2","r1","r2","c","c1","c2","cdiff","w_diff","w"];
% contatore
count=0;
% numero di combinazioni
NN=numel(s1l)*numel(s2l)*numel(a1l)*numel(a2l)*numel(d1l)*numel(d2l)*numel(gamma_nl);
%% LOOP
for i=1:numel(s1l)
    for j=1:numel(s2l)
        for h=1:numel(a1l)
            for l=1:numel(a2l)
                for m=1:numel(d1l)
                    for n=1:numel(d2l)
                        for p=1:numel(gamma_nl)
                            s1=s1l(i);
                            s2=s2l(j);
                            a1=a1l(h);
                            a2=a2l(l);
                            d1=d1l(m);
                            d2=d2l(n);
                            gamma = gamma_nl(p);
                            % GUESS del problema numerico (K1, K2 e epsilon)
                            guess=[(s1/d1)^(1/(1-a1)) (s2/d2)^(1/(1-a2)) 0];
                            ii=0;
                            try
                                % SOLVER del sistema non lineare
                                [x,f,e]=fsolve(@(xx) semi_ratio_analitic(xx,a1,a2,s1,s2,d1,d2,gamma),guess,optimset('Display','off','TolFun'
```

```
       ,10^-10,'MaxIter',1000));
catch
    x(1)=nan;
    x(2)=nan;
    e=-100;
    f=nan;
end

if e>0
    k1(i,j,h,l,m,n)=x(1);
    k2(i,j,h,l,m,n)=x(2);
    eps(i,j,h,l,m,n)=x(3);
else
    k1(i,j,h,l,m,n)=nan;
    k2(i,j,h,l,m,n)=nan;
    eps(i,j,h,l,m,n)=nan;
end
% OTTENGO r e c
r1=a1*k1^(a1-1)*(1+gamma)-d1;
r2=a2*k2^(a2-1)*(1-gamma)-d2;
c1=(1-s1)*(k1^a1+gamma*(k1^a1+k2^a2));
c2=(1-s2)*(k2^a2-gamma*(k1^a1+k2^a2));
cdiff = c1 - c2;
% CONSUMO AGGREGATO
c=c1+c2;
w_diff = abs(cdiff)/c*100;
w= (c1^(1-eta) + c2^(1-eta))^(1/1-eta);

if imag(c) ~= 0
    C_mat(i,p)=nan;
else
    C_mat(i,p)=c;
end

% REGISTRO CASO COME RIGA nella tabella
tabella=[tabella;{k1,k2,eps,gamma,e,mean(f),
    s1,s2,a1,a2,d1,d2,r1,r2,c,c1,c2,cdiff,
    w_diff,w}];
% CONTO E DISPLAY
count=count+1;
```

```
                              disp(count/NN*100)
                          end
                      end
                  end
              end
          end
      end
  end
% SALVATAGGIO
save results_e_uguale k2k1 gamma s1l s2l a1l a2l d1l d2l tabella
```

```
function [f,x]=semi_ratio_analitic(x,a1,a2,s1,s2,d1,d2, gamma)

% ROOTS DEL SISTEMA
k1=x(1);
k2=x(2);
ep=x(3);
% SISTEMA DI EQUAZIONI tale che F=0
f(1)=s1*(k1^a1 + gamma*(k1^a1+k2^a2))-d1*k1;
f(2)=s2*(k2^a2 - gamma*(k1^a1+k2^a2))-d2*k2;
f(3)=a1*k1^(a1-1)*(1+gamma)-d1 -a2*k2^(a2-1)*(1-gamma)+d2-ep;
end
```

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