Università Ca' Foscari Venezia

DIPARTIMENTO DI ECONOMIA

CORSO DI LAUREA IN ECONOMIA E FINANZA

CURRICULUM FINANCE

PROVA FINALE

A Model of Pollution Dynamics on Networks: Cooperative vs Noncooperative Games

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ANNO ACCADEMICO 2022/2023

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Introduction

In this thesis, we investigate a growth model in which the capital stock represents the quantity of "clean air", or more generally of an unpolluted resource, that diminishes over time as pollution occurs. The process of producing the consumption goods that agents use to get their utility has the side effect of emitting polluting gasses which lower the amount of “clean air” in the system.

Additionally, we assume that this capital stock, representing the clean resource, is distributed across various locations, each with varying regeneration capacities. These locations are interconnected, and thus, pollution can spread from one location to the other ones with varying intensities due to the morphological characteristics of the terrain.

A single decision maker has the goal to design which among these subregions are the best to possibly localize such production/pollution, and to determine the optimal intensity of consumption at every site in order to maximize the combined utility of all locations from consumption.

The model structure is inspired by a recent paper authored by by Fabbri *et al.* [8], in which they present a dynamic model involving multiple players engaged in resource harvesting competition. This work represents a novel application of similar techniques and enhances the analysis by contrasting the impacts of competition with those arising from coordinated actions.

The first chapter will be dedicated to a brief enunciation of the main mathematical tools used in the subsequent analysis, to have a clear picture of the background theory employed.

On Chapter two instead, we will focus on an explanation of the base model both in a more prosaic way and a more rigorous version. The last part of the chapter will be allocated to a discussion on some crucial parameters of the model. (The discussion of the most critical parts of the model will be left to the last chapter where there will also be an attempt to propose a more advanced model as a possible way for further development.)

The main body of this work will be comprised by chapters three and four which are entrusted to expose the calculation and the main results of framework one, in the case of chapter three, and the exposition of the known results from the original paper about framework two in chapter four. The second part of chapter four is used to make a comparison between the results of the two different frameworks for what attains utility of agents, and the optimal emission rate of them.

The last chapter will be devoted to an examination of some parts of the model that could be improved or reworked in order to get an advanced model to make more complex simulation of the real behaviour of the agents. These suggestions will be combined with some empirical facts and/or theoretical intuitions to propose and advanced model, which could represent a plausible further development of the original model.

Notation

Throughout the text we will make use of the following notation:

$\mathbb{R}$ set of real numbers

$\mathbb{C}$ set of complex numbers

$Re(z)$ the real part of the complex number z

$\mathbb{R}_+ = [0, +\infty)$ set of nonnegative real numbers

$\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ the inner product of (column) vectors $x, y \in \mathbb{R}^n$

$e_i \in \mathbb{R}^n$ the i-th vector of the canonical base in $\mathbb{R}^n$

$A^\top$ is the transposed of the matrix A

Moreover, we here summarize the notation used in the model to represent the various parameters and variable quantities:

$X_i(t)$ is the "clean" stock present at node i at time t

$X(t)$ the column vector $(X_1(t), X_2(t), ..., X_n(t))^\top$

g_{ij} intensity of flow from node i to node j

Γi natural regeneration rate at node i

G the $n \times n$ adjacency matrix of the network, with entries g_{ij}

r_i net regeneration rates (the regeneration factor Γ_i plus the net carbon dioxide intake at node i flowing from the other nodes)

R diagonal matrix of with entries r_i on the ii-th place

λ Perron-Frobenius eigenvalue of the matrices $R + G^\top$ and $R + G$

$\eta > 0$ dominant eigenvector of $R + G$, associated to the Perron-Frobenius eigenvalue λ

$\zeta > 0$ dominant eigenvector of $R + G^\top$, associated to the Perron-Frobenius eigenvalue λ

$\sigma \in (0,1]$ relative risk aversion parameter of the agents

$\rho > 0$ discount rate employed by the agents for discounting future utilities

$\theta_1 = \frac{\rho - \lambda(1-\sigma)}{\sigma}$ cumulative rate of "impatienty" in framework (F1)

$\theta_2 = \frac{\rho - \lambda(1-\sigma)}{1-(1-\sigma)f}$ cumulative rate of "impatienty" in framework (F2)

x_i amount of the stock of "clean air" present at node i at time 0

x (column) vector $(x_1, x_2, ..., x_n)$

$\langle X(t), \eta \rangle := \sum_{i=1}^n X_i(t)\eta_i$ weighted total mass of "clean air" present in the system at time t

Chapter 1

Mathematical Tools

1.1. Optimal Control Problems

We here present the traits of a general optimal control problem in $\mathbb{R}^n$, and in continuous time consistent with problems treated in Chapter 3. We first introduce a *controlled system*.
We assume that a system evolves continuously in time according to an ordinary differential equation (briefly, ODE) in $\mathbb{R}^n$, namely

$$\dot{X}(t) = f(t, X(t), c(t)), \quad t \geq 0 \tag{1.1}$$

where $X(t) = (X_1(t), X_2(t), ..., X_n(t))^\top \in \mathbb{R}^n$, is called the *state* of the system. The function

$$f : [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

is called the dynamic, and embodies all the features of the evolution that are exogenously imposed on the system, while $c(t) = (c_1(t), c_2(t), ..., c_f m(t))^\top \in \mathbb{R}^m$ is called the *control* or *strategy*, is chosen by a unique decision maker and represents the channel through which they can intervene or interact with the system.

Mathematically speaking, a controlled system is a family of ordinary differential equations, parametrized by the control function: for every choice of the control, one has a different evolution of the system.

Equation (1.1) is usually coupled with an initial condition of type

$$X(0) = x \in \mathbb{R}^m \tag{1.2}$$

The control may be required to attain values $c(t)$ in a proper subset C of $\mathbb{R}^m$, at all times - such set C is called the *control space*, to be a measurable function and to satisfy, jointly with the state, possible additional constraints. For the sake of the present thesis, we will consider only positivity constraints (quite frequent in control theory applied to economic problems), i.e.

$$c_i(t) \geq 0, X_i(t) \geq 0, \quad \forall t \geq 0. \tag{1.3}$$

so that altogether, she set of *admissible controls* is chosen as

$$\mathbb{A} = \{c : [0,T] \to C : c(\cdot) \text{ measurable, } c(t) \geq 0,\ X(t;x,0,c(\cdot)) \geq 0, \forall t \in [0,T]\}$$

where by $X(t;x,s,c(\cdot))$ we mean the trajectory at time t that starts at x at time s and is driven by control $c(\cdot)$. The agent choosing the control function $c(\cdot)$ may act according to different objectives. In *optimal control problems* such agent chooses the control so to maximize (or minimize) a functional call the *objective functional* (or *pay-off*) of type

$$J(c(\cdot),x) = \int_0^T \ell(t,c(t),X(t))dt \tag{1.4}$$

where $\ell : [0,+\infty) \times \mathbb{R}^f \times \mathbb{R}^n \to \mathbb{R}$ is called *instantaneous objective* function and it needs to be at least measurable. When $T < +\infty$ the problem is said to have a finite time horizon, conversely, when $T = +\infty$ the problem is said to have an infinite time horizon. In the latter case, we will assume

$$\ell(t,c,x) = e^{-\rho t}\ell_0(c,x),$$

The constant ρ is often positive, and then called the *discount rate*. This particular dependence of ℓ on time is meant to weigh differently the outcome $\ell_0(X(t),c(t))$ in time.

Definition 1 *A control $c \in \mathbb{A}$ is said* optimal *for the problem described by (1.1)(1.2) and (1.4) if*

$$J(c^*(\cdot),x) \geq J(c(\cdot),x), \quad \forall c \in \mathbb{A}.$$

The corresponding trajectory $X(t;0,x;c^*)$ is called an optimal trajectory *and will be denoted by X^*. The state-control couple(c^*,X^*) will be called an* optimal couple*.*

The *optimal control problem (P)* is then:

Maximize $J(c(\cdot),x)$ over $c \in \mathbb{A}$ and subject to the contraints (1.1)(1.2).

Open-loop vs Closed-loop Control Strategies

An *open-loop strategy* predetermined time-dependent function $c(t)$ lying in the set of admissible controls $\mathbb{A}$, chosen once and for all at the beginning of the time span, regardless the value of the state $X(t)$ in the process. In contrast, a *closed-loop* *strategy* is a responsive function that depends on the observed state $X(t)$, i.e., it takes the form:

$$c(t) = G(t, X(t))$$

where the function $G : [0, T] \times \mathbb{R}^n \to \mathbb{R}^m$ is called a *reaction*, or *feedback law*. Closed-loop controls, also known as feedback controls, are of particular interest because they provide real-time responses to the system's current state, allowing them to adapt to unforeseen disturbances in the system.

When the feedback control is inserted into the state equation, it gives rise to a so-called *closed-loop equation*, when the control falls out of the picture

$$\begin{cases} \dot{X}(t) = f(t, X(t), G(t, X(t))), & t > 0 \\ X(0) = x, \end{cases} \tag{1.5}$$

If the control $c(t) = G(t, X(t))$ is optimal, then the feedback law G is called an *optimal feedback law*.

The assumptions that are made on the data f, ℓ, ℓ_0 or on $\mathbb{A}$ are usually meant to ensure existence and uniqueness of the solution of (1.1)(1.2), that all quantities are well defined, and finally that a suitable techniques of solution of the problem can be put into action. For instance, the Lipschitz-continuity of the dynamic f in x, uniformly in t and c, meaning the existence of a constant $K > 0$ such that

$$|f(t, x, c) - f(t, y, c)| \leq K|x - y|, \quad \forall t \geq 0, \forall c \in C, \forall x, y \in \mathbb{R}^n$$

ensures existence and uniqueness of the solution of (1.1)(1.2), while the other data are generally required to be at least measurable or satisfy bound growth of some kind. We do not go deep into this discussion as we will be considering linear dynamics and differentiable measurable running objectives, and prove all properties in detail at due time.

A few words about the techniques that are available for solving optimal control problems. They are mainly

- Pontryagin's Maximum Principle, an extension, say, of the Lagrange Theorem for dynamic problems, developed by Lev Pontryagin in 1956;
- Bellman's Dynamic Programming, developed by Richard Bellman in the 1950s.

One of the main differences between the outcomes of such techniques is that the former characterizes open-loop optimal controls, while the second characterizes the closed-loop controls in terms of the gradient of the so-called *value function*, that we define in section 1.2. There, we proceed with a brief description of Dynamic Programming method, in order to use it in the following chapters.

1.2. Dynamic Programming

The Dynamic Programming method is based on the immersion of the original problem (1.1)(1.2)(1.4) in a family of subproblems starting at an intermediate time $t \in [0, T]$, hence

$$\begin{cases} \dot{X}(s) = f(s, X(s), c(s)) & s \geq t \\ X(t) = x \end{cases}$$

When necessary, we will refer to the solution of this system as $X(s; t, x, c(\cdot))$ and mean "the value at time s of the trajectory that started at x at time t and was driven by the control function $c(\cdot)$". The functional to be maximized is then

$$J(c(\cdot); t, x) = \int_t^T \ell(s, X(s), c(s))ds.$$

Then one takes the following steps:

- define the *value function* of the problem, that is the maximal (or supremal) value of the payoff $V : [0, T]X\ \mathbb{R}^n \longrightarrow \mathbb{R}^n$

  $$V(t, x) := \sup_{c(\cdot) \in \mathbb{A}(t,x)} J(c(\cdot), t, x)$$

  where $\mathbb{A}(t, x)$ is, in the case that we are interested in, defined as

  $$\mathbb{A}(t, x) = \{c : [t, T] \to C : c(\cdot) \text{ measurable, } c(ts) \geq 0,\ X(s; x, t, c(\cdot)) \geq 0, \forall s \in [t, T]\}$$

  .

- show that the value function can be characterized as a solution (possibly unique) of a partial differential equation called Hamilton-Jacobi-Bellman (briefly, HJB) equation built from the data of the problem as follows. One defines *the current value Hamiltonian function* as

  $$h(t; c; x; p) = \ell(t; x; c) + \langle p, f(t; x; c) \rangle$$

  where $p \in \mathbb{R}^n$ is an adjoint variable, and the (maximal) Hamiltonian function as

  $$H(t; x; p) = max\{h(t; c; x; p)\}$$

  then the HJB equation is

  $$v_t(t; x) = H(t; x; \nabla v(x)), \quad t \geq 0, x \in \mathbb{R}^n$$

- find a solution v, if possible, of the Bellman equation and prove that such solution coincides with the value function V.

- find a relationship between the optimal strategy and the gradient of the value function;
- characterize, via the so called closed-loop equation, both the optimal strategy and trajectory in terms of the value function, finds a feedback optimal map and.

1.2.1 Problems with Infinite Horizon

Dynamic Programming for a problem with an (ideally) infinite time horizon, where the state equation is autonomous (i.e. the dynamic does not depend directly on the time variable), takes a simplified form that we discuss in the sequel, expanding the steps briefly described at the beginning of section 1.2. First of all we assume that the state equation is of type

$$\begin{cases} \dot{X}(s) = f(X(s), c(s)) \quad s \geq 0 \\ X(0) = x \end{cases} \tag{1.6}$$

then we define a current value Hamiltonian function h_0 by means of the following position

$$\begin{aligned} h(t,x,c,p) &= \ell(t,x,c) + \langle p, f(x,x) \rangle \\ &= e^{-\rho t}\ell_0(x,c) + e^{-\rho t}\left\langle e^{\rho t}p, f(x,x) \right\rangle \\ &= e^{-\rho t}\{\ell_0(x,c) + \langle q, f(x,x) \rangle \\ &\equiv e^{-\rho t}h_0(x,c,q) \end{aligned}$$

where

$$q = e^{\rho t}p$$

is the new adjoint variable. Consistently one defines

$$H(t,x,p) = e^{-\rho t}\sup_c h_0(x,c,p) \equiv H_0(x,p).$$

The derivation of a suitable HJB equation requires an intermediate step called dynamic programming principle, of which HJB equation represents an infinitesimal version. Before the enunciation of the Dynamic Programming Principle, we need to define some condition on the admissible control strategies and on the function f, ℓ.

We require the family of admissible control strategies $\mathbb{A}(t,x)$ for varying t and x to satisfy the following assumptions:

(A1) For every $0 \leq t \leq \tau \leq T, x \in \mathbb{R}^n$,

$$c \in \mathbb{A}(t,x) \Rightarrow c|_{[\tau,T]} \in \mathbb{A}(\tau, X(\cdot,t,x,c))$$

(where $c|_{[\tau,T]}$ is the restriction of the control function c to the interval $[\tau, T]$), meaning: *the second part of an admissible strategy is itself admissible*.

(A2) For every $0 \leq t \leq \tau \leq T, x \in \mathbb{R}^n$, $c_1 \in \mathbb{A}(t,x)$, $c_2 \in \mathbb{A}(\tau, X(\cdot,t,x,c_1))$, the control c defined as

$$c(s) = \begin{cases} c_1(s) & \text{if} \quad s \in [t,\tau] \\ c_2(s) & \text{if} \quad s \in [\tau, T] \end{cases} \tag{1.7}$$

belongs to $\mathbb{A}(t,x)$, meaning: *the juxtaposition of two admissible strategies is admissible*.

Lemma 1 (Dynamic Programming Principle, DPP) *The value function $V(x) \equiv V(0,x)$ of the problem with initial time $t = 0$, initial state x, satisfies*

(a) $V(t,x) = e^{-\rho t} V(x), \quad \forall t \geq 0, \ \forall x \in \mathbb{R}^n$;

(b) (Dynamic Programming Principle)

$$V(x) = \sup_{c(\cdot) \in \mathbb{A}(0,x)} \left\{ \int_0^\tau e^{-\rho s} \ell_0(X(s), c(s)) ds + e^{-\rho s} V(X(\tau; x, 0, c(\cdot))) \right\}$$

Theorem 1.1 *When differentiable, the value function $V(x)$ of the described optimal control problem satisfies the following HJB equation, for all $x \in \mathbb{R}^n$*

$$\rho v(x) = H_0(x, \nabla_x v(x)).$$

We do not prove the statements above, for the full demonstration we remand to the book by Bardi and Capuzzo-Dolcetta [2].

We present instead a heuristic derivation of HJB equation to appreciate where it comes from. In the DPP, we move all terms to one side (anything not depending on c can be taken inside the sup) and we multiply each side by $\frac{1}{\tau}$, with $\tau > 0$, obtaining

$$\sup_{c(\cdot)} \left\{ \frac{1}{\tau} \int_0^\tau e^{-\rho s} \ell_0(x(s), c(s)) ds + \frac{e^{-\rho t} V(X(\tau)) - V(x)}{\tau} \right\} = 0 \tag{1.8}$$

Now note that, by means of chain-rule[1] we have

$$
\begin{aligned}
&\lim_{\tau\to 0}\frac{e^{-\rho t}V(X(\tau))-V(x)}{\tau}=\left[\frac{d}{d\tau}e^{-\rho t}V(X(\tau))\right]_{\tau=0}\\
&=-\rho e^{-\rho 0}V(x)+e^{-\rho 0}\left\langle V_x(X(0,x,c(0));\dot{X}(0)\right\rangle\\
&=-\rho V(x)+\left\langle V_x(X(0,x,c(0)),\dot{X}(0)\right\rangle\\
&=-\rho V(x)+\langle V_x(X(0,x,c(0)),f(x,c(0))\rangle
\end{aligned}
$$

and that

$$
\begin{aligned}
\lim_{\tau\to 0}\frac{1}{\tau}\int_0^{\tau}e^{-\rho s}\ell_0(x(s),c(s))ds &= \left[e^{-\rho s}\ell_0(X(s),c(s))\right]_{\tau=0}\\
&= e^{-\rho 0}\ell_0(X(0),c(0))\\
&= \ell_0(x,c(0))
\end{aligned}
$$

so that (1.8) becomes

$$
\sup_{c(0)}\{\ell_0(x,c(0))-\rho V(x)+\langle V_x(x),f(x,c(0))\rangle\}=0
$$

Note that now the sup, originally taken over a set of control functions, depends only on the initial value of such controls, the vector $c(0)\in\mathbb{R}^n$. Then, after taking anything not depending on $c(0)$ out of the sup, we are left with

$$
-\rho V(x)+\sup_{c(0)}\{\ell_0(x),c(0))+\langle\nabla_x V(x),f(x,c(0))\rangle\}=0
$$

that is

$$
-\rho V(x)+H_0(x,\nabla_x V(x))=0
$$

as in the statement of Theorem 1.1.

1.3. Differential Games

Optimal control theory typically deals with problems where a single decision maker maximizes their payoff. We now want to consider a scenario in which multiple agents are engaged in dynamic competition to maximize their individual payoffs. Therefore, it is essential to integrate Dynamic Programming with Game

[1]We recall the chain rule in multiple dimension:

$$
\frac{d}{d\tau}V(X(\tau))=\sum_{i=1}^{n}\frac{\partial V(X(\tau))}{\partial x_i}\cdot\frac{\partial X_i(\tau)}{\partial\tau}=\langle\nabla_x V(X(\tau)),X'(\tau)\rangle
$$

Theory by employing Differential Games.

Differential games are the natural extension of static games in a dynamic framework with a continuous time setting. In our simplified exposition, we consider an autonomous system of type (1.6) , that is

$$\begin{cases} \dot{X}(s) = f(X(s), c(s)) & s \geq 0 \\ X(0) = x \end{cases} \tag{1.9}$$

where $c_i(s) \in \mathbb{R}^m$ are the controls implemented by the agents, and the set of their choices $c(t)$ is called a (complete) *strategy profile*. In a non-cooperative game, the agents try to maximise their own objective function

$$J_i(c_i; c^*_{-i}, x) = \int_0^{+\infty} e^{-\rho t} \ell_i(X(t), c_i(t), c^*_{-i}(t))dt, \tag{1.10}$$

by choosing their control c_i, and *taking as given the control chosen by the other agents*, and denoted by c^*_{-i}.

Definition 2 *We call a complete strategy profile $c^* \in \mathbb{A}$ a* Nash Equilibrium *when*

$$J(c_i; c^*_{-i}, x) \leq J(c^*_i, c^*_{-i}, x)$$

for every admissible strategy profiles $(c_i, c^*_{-i}) \in \mathbb{A}$, and for every player i. That is, when (already) playing c^* it is inconvenient for any agent to change their strategy c_i if the other players are sticking to c^*_{-i}.

To find Nash equilibrium solutions, we thus need to simultaneously solve m optimal control problems, as many as the players; the optimal solution c^*_{-i} of the set of players different from the i-th enters as a parameter in the problem of the i-th player.

Also in differential games there is a distinction between open-loop and closed-loop strategy profiles:

- with an *open-loop strategy profile* agent i is unable to gather information on the state of the system once the game has started, and/or on the strategy implemented by other players; is forced to chose at the nbeginning their strategies, once and for all. Hence open-loop strategy profile depend solely on time.

- with *Markovian Strategies* we assume that every agent i, at each time t, knows the state of the system, and/or on the strategy implemented by other players until time t.

Dynamic Programming helps devising Markovian equilibrium strategies, by the definition of a *value function* V_i for every player, defined for a family of intermediate problem, starting at a time $t \in [0, T]$ at the state x, as

$$V_i(t, x) = \sup_{c_i} J(c_i(\cdot); t, x, c^*_{-i})$$

with players solving simultaneously their own optimal control problem, having each an associated HJB equation also to be solved simultaneously.
Nonetheless, a detailed explanation of Dynamic Programming for Differential Games goes beyond the scope of this thesis.

1.4. The Perron–Frobenius Theorem

The Perron-Frobenius Theorem establishes properties of eigenvalues and eigenvectors of real square matrices which are positive or, in its extended versions, nonnegative and irreducible, or primitive. We are in particular interested in the version dealing with *Metzler* matrices.
We preliminarily recall the following definitions:

- a square matrix is called *positive (respectively, nonegative)* if it has all strictly positive (resp., nonnegative) entries;
- a square matrix is called a *Metzler matrix*, if it has all nonnegative entries except those on the principal diagonal.
- a square matrix A is called *reducible* if there exists a permutation matrix P such that $P^{-1}AP$ is upper block-triangular matrix, that is

$$PAP^{-1} = \begin{pmatrix} O & A_1 & O & ... & O \\ O & O & A_2 & ... & O \\ \vdots & \vdots & \vdots & & \vdots \\ O & O & O & ... & A_{h-1} \\ A_h & O & O & ... & O \end{pmatrix}$$

  The matrix is called *irreducible* otherwise.

Theorem 1.2 *[Perron-Frobenius Theorem for Metzler matrices] Assume A is an irreducible $n \times n$ real Metzler matrix. Then A has a simple real eigenvalue λ whose real part is greater than the real parts of all other eigenvalues. The associated eigenvector can be chosen strictly positive (i.e., with all strictly positive entries) and it is the only eigenvector with such property.*

Some remarks are here useful:

- the positive eigenvector cited in the Theorem is often called the *dominant* eigenvector of A;
- clearly the statement holds also for the transpose $A^\top$ of A, that has the same eigenvalues, but possibly different associated eigenvectors;
- the property stated in the theorem of having a simple real eigenvalue of maximal real part and an associated positive eigenvector is often defined as *the strong Perron-Frobenius property*;
- we will see in the next section that the adjacency matrix of a strongly connected network is irreducible amnd nonnegative, so it enjoys the strong Perron-Frobenius property.

Among the consequences of Perron-Frobenius Theorem more relevant to our work there are those stated in the following Theorem, both associated to Metzler matrices, when they represents the dynamics of linear ODEs. To this extent, we consider the following Cauchy problem

$$\begin{cases} X(t) = AX(t), & t \geq 0 \\ X(0) = x \end{cases} \tag{1.11}$$

Theorem 1.3 *Consider the Cauchy problem* (1.11)*, where A is a Metzler matrix. Then:*

(i) *the trajectory of the system lies in the positive orthant $\mathbb{R}^n_+$ for every initial datum x chosen in the positive orthant; the viceversa also is true: if the latter property is true, then A is a Metzler matrix.*

(ii) *The trajectory of the system converges towards the direction of the dominant positive eigenvector of A.*

A proof of the first statement can be found in the book by Farina and Rinaldi [12], while the second descends from the well-known fact that trajectories of linear systems converge towards the direction of the eigenvector of maximal real part (when simple).

Hence, if A is not a Metzler matrix, one can show that there exist nonnegative initial data x for which the associated trajectory leaves the positive orthant at least for some time.

1.5. Networks

A *network*, often referred to as a "graph," is a mathematical and graphical representation of a set of interconnected elements. It consists of two main components: nodes (vertices) and edges (links or connections).
Nodes are the fundamental units or entities within a network. They represent individual elements, entities, or points in the network. In various applications, nodes can represent a wide range of things, such as people in a social network, routers in a computer network, cities in a transportation network, or any other discrete objects relevant to the system being studied.

On the other hand, *edges* (or links, or connections) are the connections or relationships between pairs of nodes in a network. They represent the interactions, associations, or dependencies between the nodes they connect. Edges can be directed (pointing from one node to another) or undirected (bidirectional or symmetrical). In some contexts, edges may also have associated weights or attributes that provide additional information about the relationships they represent. In the sequel we will in fact define a weighted networks.

Networks are used to model and analyze a wide range of complex systems in various fields, including computer science, sociology, transportation, biology, and more. They are a powerful tool for studying relationships and patterns within interconnected data. Network systems are defined through the use of the mathematical objects called *Graphs*, which have their own specialized branch of mathematical theory called *Graph Theory*.

Definition 3 *An directed and weighted graph (or network) is a triplet $\mathcal{G} = \{V, E, G\}$, is composed by the sets:*

(i) V is the set of vertices (also called nodes) with elements $\{v_1, v_2, ..., v_n\} \neq \{\emptyset\}$;

(ii) E is the set of edges (also called links) defined as a subset

$$E \subseteq \{(v_i, v_j) | v_i, v_j \in V \text{ and } v_i \neq v_j\}$$

When $(v_i, v_j) \in E$ that means that the node v_i is linked (unidirectionally) to the node v_j;

(iii) A set of weights on such link, represented through a matrix G of entries g_{ij} representing the intensity of connection from v_i to v_j. Such matrix is called adjacency matrix *of the network.*

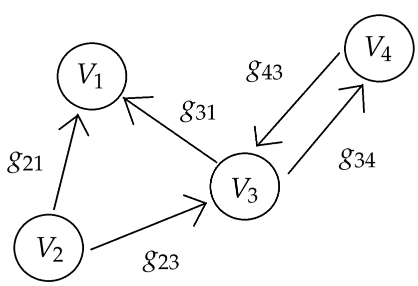
A few remarks:

- multiple edges connecting the same pair of unordered vertices (v_i, v_j) are not allowed to exist;
- a network defined in this way does not contain any loops, which would involve edges connecting a vertex v_i to itself.
- The weights g_{ij} on the links, in the simplest instance of the adjacency matrix, are set equal to 1 when there's an edge connecting v_i and v_j and to zero otherwise.

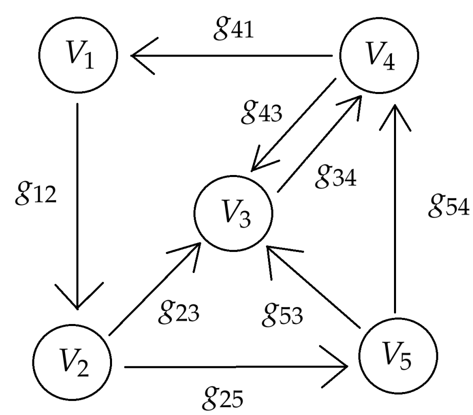
Definition 4 *A network (V, E, G) is said* totally connected *or* complete *when all nodes are linked with positive weights. It is said instead* strongly connected *when there exists always a directed path between two different nodes.*

In a strongly connected network there are no isolated nodes, and every node can be reached from any other node through a sequence of directed edges.

In the figure (a) below we see an example of graph which is not strongly connected. Indeed vertex v_2 is unreachable from the other vertices and if we start in vertex v_1 we cannot go anywhere. In figure (b) instead, we see an example of a strongly connected graph. In this case, we can reach a vertex starting from any other vertex.



(a) A not strongly connected graph



(b) A strongly connected graph

An important property of strongly connected networks, that we will need in the next chapters, is the following.

Lemma 2 *A matrix A is irreducible if and only if its associated graph (or network) is strongly connected.*

This fact will be used to state that the the adjacency matrix of a strongly connected and weighted network, with nonnegative weights is irreducible and nonnegative, and satisfies the assumptions of the Perron-Frobenius Theorem.

Chapter 2

A Spatial Model for Pollution Dynamics and Growth

2.1. Model Explanation

The primary objective of this work is to propose a model for the diffusion and, in suitable extensions, containment of environmental pollution. For instance, we can consider this pollution to be atmospheric pollution caused by CO2 (although the model is applicable to other natural resources as well), with the characteristic of being 'mobile' and capable of spreading from one area to connected areas through flows of pollutant. The various areas are to be regarded as internally homogeneous in terms of pollutant production and containment, and distinct in this regard from the surrounding areas. Hence, the decision to represent them as a network, where each area constitutes a node, nodes in communication are denoted by the links of the node, and the flow intensities are represented by the weights assigned to these links, strictly positive if and only if a positive flow from the source node to the destination node is possible.

2.1.1 Spatial Dynamics on Networks

Mathematically speaking, we describe our "world" through the use of a network $\mathcal{G}$, where each region is represented by a single node n. We denote the set of nodes by $N := \{1, .., n\}$, and with $g_{ij} \geq 0$, the weight upon the edge connecting a source node i and a target node j, with g_{ij} representing the intensity of the outflow from i to j. In the special case when $g_{ij} = 0$ and $g_{ji} = 0$, there are no direct connections between the two nodes.

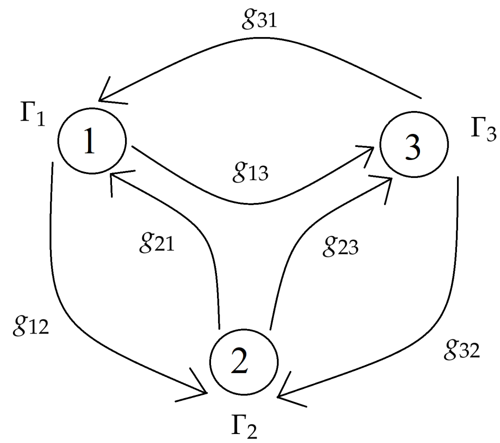


Figure 2.1: Example of a network with three nodes

We assume that $\mathcal{G}$ is *strongly connected*, with $g_{ii} = 0$ for all $i \in N$, that is, there exists in $\mathcal{G}$ a path connecting any two nodes with corresponding strictly positive coefficients g_{ij}, effectively excluding the existence of isolated nodes. $\mathcal{G}$ has for definition no loops.
The $n \times n$ matrix G with elements g_{ij}, and $i, j \in N$ is the so-called *adjacency matrix* of the network. For all $i \in N$, the quantity $X_i(t)$ stands for the non polluted capital stock present in the node i at time t, and $X(t)$ for the vector with components $X_1(t), ..., X_n(t)$. The evolution in time of the residual clean air $X_i(t)$ on region i depends on several factors:

(a) the composite regeneration factor $\Gamma_i X_i(t)$ for the resource at time t at node i, embodied by the (constant) natural growth rate Γ_i. This factor is essentially replicating the compound effect of absorption of CO_2 carried out by plants, trees and water masses (primarily oceans), and natural degradation of the CO_2 molecules in the atmosphere. For renewable resources $\Gamma_i > 0$, while for non-renewable resources $\Gamma_i \leq 0$;

(b) the overall difference between outflow of the resource from region i to a linked region j at time t, given by $g_{ij} X_i(t)$, and the inflow from the same region given by $g_{ji} X_j(t)$. Therefore the net inflow at location i from region j is given by:

$$\left(\sum_{j=1}^{n} g_{ji} X_j(t)\right) - \left(\sum_{j=1}^{n} g_{ij} X_i(t)\right) = \langle Ge_i, X(t)\rangle - \left(\sum_{j=1}^{n} g_{ij}\right) X_i(t)$$

(c) the rate of pollution $c_i(t)$ emitted at time t from region i, which contributes to decrease the capital stock of "clean air".

As a whole, we then have for all i

$$\dot{X}_i(t) = \left(\Gamma_i - \sum_{j=1}^{n} g_{ij}\right) X_i(t) + \langle Ge_i, X(t)\rangle - c_i(t).$$

With additional specification we can rearrange this equation in a more compact form, which will come handy in future calculations.

- We define $R = (r_{ij})$ as the diagonal matrix of the regeneration factors net of the outflows from node i:

$$\begin{cases} r_{ij} = 0 & \text{if } i \neq j \\ r_{ii} \equiv r_i = \Gamma_i - \left(\sum_{j=1}^{j=n} g_{ij}\right), \end{cases}$$

- $c(t)$ is the vector with components $c_1(t), ..., c_n(t)$,
- x_0 is the vector of all initial stocks of the unpolluted air capital stock at the different nodes.

Then, the evolution of the system in vector form can be describes as:

$$\begin{cases} \dot{X}(t) = (R + G^{\top})X(t) - c(t), \quad t > 0 \\ X(0) = x_0 \in \mathbb{R}^n_+. \end{cases} \tag{2.1}$$

In addition, we require the following positivity constraints as conditions of existence for our model :

$$c_i(t) \geq 0, \quad t \geq 0, i \in N \tag{2.2}$$

as well as

$$X_i(t) \geq 0, \quad t \geq 0, i \in N. \tag{2.3}$$

The constraints (2.3) and (2.2) are essential for establishing realistic boundaries within which our model variables can evolve. While these constraints are self-evident for time and the capital stock of 'clean air', given that both time and the quantity of a tangible entity like a gas cannot be negative, the same does not necessarily hold for a variable like pollution emissions. Pollution emissions could potentially be negative in cases where pollutant gases are captured more than they are emitted at a particular node. However, we will not consider this possibility.

The dynamic just enunciated is the core of the model and represent the natural diffusion process of the GHGs, or other types of pollutants, when introduced into a closed system (the atmosphere). They tend to naturally spread from their initial point of entry into all the system in order to achieve a uniform spread across all

the part of the system itself, but such naturally tendency is modulated by natural barriers or accelerators, represented by the weights g_{ij}.
Here the pollution's behaviour emerges by difference with the stock of clean resource, and the "imaginary" quantity of "clean air" mimics the behaviour of its real counterpart (the concentration of GHGs).

2.1.2 Emissions Rules and Utility

The agents we have described earlier are in communication through the network. We can consider them as acting separately in a non-cooperative game, where each of them maximizes their own intertemporal utility, or coordinated by a single social planner who makes decisions for the community by maximizing a cumulative utility (the sum of individual agents' utilities) while choosing consumption for each.

We assume that some regions, among the n available, are not usable by agents for several reasons: the regulator could have decided to designate some nodes as natural reserves where production and consequent emission of pollution are not allowed; some nodes can be occupied by dense forests, deserts or be mountainous and or arctic areas, which all are characterized by a very sparsely or null density of population, and as such we consider these nodes as unavailable for agents occupation; other nodes instead are occupied by sea zones which cannot be occupied by agents. Economic activities, and the consequent consumption and pollution $c_i(t)$, is null in a subset O of N, while every other node i with $i \in F := N \setminus O$ is exclusively assigned to agent i. The total number of agents present in the network is f so that the elements of O will be $n - f$.
Every agent has an instantaneous utility from consumption described by

$$u(c) = \ln(c) \quad or \quad u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \quad \sigma > 0,\, \sigma \neq 1$$

(the case of a logarithmic u stands for the case $\sigma = 1$). Then the following two frameworks are considered:

(F1) agents are coordinated by a unique social planner maximizing the payoff

$$J(\mathbf{c}, x) = \int_0^{+\infty} e^{-\rho t} \sum_{i=1}^{f} u(c_i(t))dt, \quad i \in F \tag{2.4}$$

with respect to $\mathbf{c} = (c_1, c_2, \cdots, c_n)$ (note that x is the initial stock of the resource, and the notation $J(c; x)$ points out that dependence in J) varying in a set of admissible controls encompassing positivity constraints on consumption and stocks, such as

$$\mathbb{A} = \{\mathbf{c} \in L^{1,\rho}_{loc}([0,+\infty[;\mathbb{R}^n_+) \ \ : \ X(t;\mathbf{x};\mathbf{c}(\cdot)) \geq 0, \forall t \geq 0\}$$

where $L^{1,\rho}_{loc}$ indicates the set of locally integrable controls in $[0,+\infty[$ [1] . While

(F2) agents strategically interact in a differential game where agent i maximizes the payoff

$$J_i(c_i, x) = \int_0^{+\infty} e^{-\rho t} u(c_i(t))dt, \quad i \in F \tag{2.5}$$

where, consistently, $c_i \in \mathbb{A}_i$ with:

$$\mathbb{A}_i = \{\mathbf{c}_i \in L^{1,\rho}_{loc}([0,+\infty[;\mathbb{R}^n_+) \ \ : \ X(t;\mathbf{x};\mathbf{c}(\cdot)) \geq 0, \forall t \geq 0\}$$

In both ρ represents a *discount factor* weighing future utilities less than those closer in time.
We will concentrate mostly on framework (F1) (Section 1,2 in Chapter 3), and the parallel results on (F2) will be deduced (see Chapter 4) by the pioneering work by Fabbri*et al.* [8] , as well as on the comparison between the results in the two framework (Section 2,3 in Chapter 4)

2.1.3 Parameters and Primitives of the Network

As a consequence of the Perron-Frobenius Theorem 1.2, we observe that the square matrix $R+G$ (which is irreducible and non-negative) has one eigenvalue that is simple, real, and strictly greater that the real parts of the other eigenvalues, with a *positive* associated normalized eigenvector. We can order the eigenvalues of the square matrix $R+G$ as

$$\lambda > Re(\lambda_2) \geq Re(\lambda_3) \geq \cdots \geq Re(\lambda_n). \tag{2.6}$$

and define η an eigenvector of $R+G$ associated with λ, and ζ the respective eigenvector of $R+G^\top$ associated to λ. Both have strictly positive coordinates.

Trajectories in the long run

When the agents decide to opt for strategies with no emissions ($c \equiv 0$) the trajectories of the system converge to the direction of the eigenvector ζ associated to the Perron-Frobenius eigenvalue λ. This fact is a well-known result in the theory of linear ordinary differential equations (ODEs) in $\mathbb{R}^n$ (see for instance the book by Hirsch *et al.* [11]). Additionaly, the trajectories that at the initial time are on the positive ray through ζ, will forever remain on the positive ray through ζ.

[1] $L^{1,\rho}_{loc}(I,\mathbb{R})^n_+ = \{f \ : \ I \subseteq \mathbb{R} \to \mathbb{R}^n_+ \ : \ \int_a^b (e^{-\rho t}\|f(t)\|dt < +\infty \ : \ \forall \ [a,b] \ \subseteq I, bounded\}$

Weighted Total Mass

The total stock of "clean air" present in the system at every time is $\sum_{i=1}^{n} X_i(t)$. We will make use instead of a *weighted* total mass of the stock, where weights are the components of the eigenvector η, that is

$$\langle X(t), \eta \rangle := \sum_{i=1}^{n} X_i(t)\eta_i.$$

which, basically, is the scalar product of $X(t)$ and η.

Natural Growth Rate of the System

When there are no emissions in the system made by the agents (essentially a "natural" state without agents), namely $c_i = 0$; λ represents the *total mass growth rate* of the system. In fact, if we multiply (2.1) for η we get the following equation

$$\langle \dot{X}(t), \eta \rangle = \lambda \langle X(t), \eta \rangle$$

which can be solved as an ordinary ODE, and its result shows us what is the growth dynamics of the stock into our model, which is:

$$\langle X(t), \eta \rangle = e^{\lambda t} \langle x, \eta \rangle.$$

Moreover, if we look at the expansion in rows of the equality $(R + G)\eta = \lambda\eta$, we see that it is equal to:

$$(\lambda - r_i)\eta_i = \sum_{j=1, j\neq i}^{n} g_{ij}\eta_j, \tag{2.7}$$

From the description of the model, we note that the right hand side of the equation must be strictly positive consequently to the hypothesis that the network is strongly connected (which implies that at least one of the g_{ij} is strictly positive. Thus the following conditon must hold to respect this derivation of the strongly connected hypothesis:

$$r_i < \lambda, \quad \forall i \in F. \tag{2.8}$$

where r_i are the net reproduction rates of the stock of "clean air".

Meaning of the eigenvector η

In our model there are three interpretations of the components η_i of eigenvector η. They are derived directly from the interpretation studied in the original paper by Fabbri *et al.* [8], thus we will enunciate them accordingly:

1. The component η_i of η measures the *long-term carbon intake capabilities of the system at node i*. One way to establish this is to consider an initial amount of the stock of "clean air" starting with a unitary mass concentrated in the i-th node, namely $x = e_i$. Then, if we are in the case of null emissions, we would get
$$\langle x, \eta \rangle = \eta_i,$$
which implies that the total mass, in the long run, is maximized when such unitary mass is allocated in the node where η_i is maximal.

2. If every player choose its emission strategy proportionally to the (weighted) total mass, that is, $c_i = I_i(t)\langle X(t), \eta \rangle$, with $I(t) = (I_i(t))_i$ denoting the intensities of emission at time t, then the resulting evolution of the system is going to be:
$$\dot{X}(t) = (A + G^\top)X(t) - \langle X(t), \eta \rangle I(t)$$
which implies $\langle \dot{X}(t), \eta \rangle = g\langle X(t), \eta \rangle$ with
$$g = \lambda - \langle I(t), \eta \rangle = \lambda - \sum_{i=1}^{n} I_i(t)\eta_i$$
Here g represent the new growth rate of the system after the new emission strategy $I(t)$ of the agents is implemented. In particular, the rate g is a decreasing function of I_i with
$$\frac{\partial g}{\partial I_i} = -\eta_i, \tag{2.9}$$
This derivative thus suggest us that in order to lower the negative impact to the growth rate of the stock of "clean air" agents should concentrate their emissions in the node or nodes where η_i is minimal.

3. The η_i's represents what network theory terms the *eigencentrality* of node i, not of the original $\mathcal{G}$ but of a related network $\mathcal{G}'$ whose adjacency matrix is $A + G$. Note that since $(R + G)\eta = \lambda\eta$, and the matrix $\lambda I - R$ is diagonal with all positive diagonal coefficients $\lambda - r_i$, one can rewrite
$$\begin{aligned} (R + G)\eta &= \lambda\eta \\ G\eta &= (\lambda I - R)\eta \\ (\lambda I - R)^{-1}G\eta &= \eta \end{aligned} \tag{2.10}$$
Then η is the dominant eigenvector (of eigenvalue 1) also of the *migration* network with adjacency matrix $(\lambda I - R)^{-1}G$, that is, where the coefficients of the original adjacency matrix $\mathcal{G}$ are magnified by reproduction rates: the i-th row of G is multiplied by $1/(\lambda - r_i)$, and flows are magnified by such factor.

Discount rate ρ

The discount rate has a crucial role in determining how the agents weigh their present and future utilities. The model just described in the previous sections has, as one of its features, an infinite time horizon, and the agent(s) decides their emission strategy continuously in time. A way to differentiate between present and future utilities is achieved through the use of the discount factor $e^{-\rho t}$ (which is the standard way to discount future values in the presence of continuous time).

When the discount factor is applied to financial products it is used to represent the opportunity cost of investing a given sum of money into possible alternative investment products, usually these alternative investment products are used to represent possible benchmarks (riskless, if using AAA bond of risk free countries like USA for dollar denominated assets and Germany for euro denominated assets, or with some part of risk, if using stock index like the $S\&P500$ etc...) to which compare the performance of the chosen financial product.

Our case is a little-bit different from this standard case, in the sense that, the agents of our model are not in a set up where they have alternative class of assets from which to generate their utility to choose looking at the expected returns. Instead, they are tasked to choose an emission strategy c_i^* that allows them to maximize their own payoff functions (or the cumulated payoffs of all agents in the case of a single planner). Thus, the discount rate utilized in our model takes the role of signalling the opportunity cost of choosing between strategy with different temporal structures.

If the discount rate applied is positive and too large the contribution to the overall payoff functions of the agents of future utilities tends to zero very rapidly, therefore incentivizing them to adopt emission strategy heavily focused on the short-term production of high amount of air pollution, completely ignoring the "future" part of the infinite time horizon in which they are. Instead, if the discount rate applied is negative, the discount factor turns into an amplifier factor and thus, it would give the agents an incentive to continuously ramp up their pollution emission the more the time frame goes on. From this analysis we can clearly see that if we want the agents to decide their future behaviour in a more realistically way, we need to restrict the possible values of the discount rate at least into the interval $]0;1[$ with a possible further restriction to the interval $]0;0,05]$.

Relative Risk Aversion (RRA) σ

For what attains the parameter (σ), which appears in the characterisation of the utility function chosen for the model, it represent the relative risk aversion (RRA) of the agents and, at the same time, the reciprocal of the elasticity of intertemporal substitution (EIS). We are mainly interested on the effect of σ on the value of the utility function ($u(c_i(t))$) for the agents of the model.

Given the fact that we have used a characterisation of the utility function

$$u(c_i) = \frac{c_i^{1-\sigma}}{1-\sigma}, \quad \text{for } \sigma \neq 1$$

And we have also assumed that the emission strategies of the agents in order to be admissible have to be higher or equal to zero, then the sign of the utility functions for the agents fall entirely upon the interval on which we decide to define the parameter (σ).

If we decide to consider the condition $\sigma \in]0,1[$ then the object $(1-\sigma)$ is positive and the resulting utilities functions are positive (higher or equal to zero). Instead, if we decide to consider the condition $\sigma \in]1,+\infty[$, the object $(1-\sigma)$ becomes negative and has to be interpreted as follows:

- If the consumption level chosen by agent i is equal to zero then the relative utility function is $-\infty$;
- if the consumption level chosen by agent i is higher than zero then the relative utility function is negative, but nonetheless increasing, with $\lim_{c_i \to 0^+} u(c_i) = -\infty, \ \lim_{c_i \to +\infty} u(c_i) = 0^-$.

Given all these specific reasons we decide to restrict the choice of σ to the interval $]0,1[$, when making use of the power function utilities. In the case $\sigma = 1$ the chosen utility remains logarithmic.

Chapter 3

Single Planner Framework

In the following chapter we analyse the framework were a single planner make choices for all agents, so to maximise J given by (2.4) subject to (2.1),(2.2),(2.3). In particular, in the forthcoming Theorem 4.1 and the subsequent remarks, we are going to establish the existence of an optimal policy of consumption/pollution, computing an explicit formula for it, the welfare of players, and other relevant quantities through the use of the dynamic programming. Specifically we will use Bellman's dynamic programming approach. In doing so, we will describe the dynamic programming procedure, and calculations will follow.

Dynamic Programming

To apply the dynamic programming method described in Chapter 2.1.1, we will proceed as follows:

(a) we define the value function V of the problem as

$$V(x) = \max_{c \in \mathbb{A}} J(c; x)$$

(b) we associate the problem to the HJB equation

$$\rho v(x) = H(x, \nabla v(x))$$

where the Hamiltonian function H is defined by

$$H(x, p) = \max_{c \geq 0} \left\{ \sum_{i=1}^{f} u(c_i) - \langle p, (R + G^\top)x - c \rangle \right\} \tag{3.1}$$

(c) we compute a solution of the HJB equation and show that it coincides with the value function of the problem;

(d) we simultaneously compute a feedback formula for the optimal control, written in terms of the gradient of the value function, and of the optimal trajectory.

The HJB Equation

LEMMA 3 *In* (3.1), *the Hamiltonian function can be rewritten as follows:*

(i) *when* $\sigma \neq 1$*,*

$$H(x,p) = \frac{\sigma}{1-\sigma} \sum_{i=1}^{f} p_i^{1-\frac{1}{\sigma}} + \left\langle p, (R+G^\top)x \right\rangle,$$

with maximum attained at $c_i^* = p_i^{-\frac{1}{\sigma}}$

(ii) *when the utility is logarithmic* ($\sigma = 1$)*,*

$$H(x,p) = -\sum_{i=1}^{f} \ln(p_i) - f + \left\langle p, (R+G^\top)x \right\rangle,$$

with maximum attained at $c_i^* = \frac{1}{p_i}$

Proof. We want to find the maximiser for

$$\varphi(c) = \sum_{i=1}^{f} u(c_i) - \langle p, (R+G^\top)x - c\rangle$$

We note that φ is a concave function of c, so that stationary points are global maximizers. To find them, we compute the solutions to

$$\frac{\partial\varphi(c)}{\partial c_i} = 0 \iff u'(c_i) - p_i = 0$$

and since in both cases (of power function and logarithmic utility) u' is invertible on the positive real axis, the previous relationship is equivalent to

$$c_i^* = (u')^{-1}(p_i).$$

That implies:

(i) For $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, one has $c_i^* = p_i^{-\frac{1}{\sigma}}$ so that

$$
\begin{aligned}
H(x,p) &= \varphi(c^*) \\
&= \sum_{i=1}^{f} \left[\frac{(p_i^{-\frac{1}{\sigma}})^{1-\sigma}}{1-\sigma} - p_i (p_i^{-\frac{1}{\sigma}}) \right] + \left\langle p, (R+G^\top)x \right\rangle \\
&= \frac{\sigma}{1-\sigma} \sum_{i=1}^{f} p_i^{1-\frac{1}{\sigma}} + \left\langle p, (R+G^\top)x \right\rangle
\end{aligned}
$$

(ii) For $u(c) = \ln(c)$, one has $c_i^* = p_i^{-1}$ so that

$$
\begin{aligned}
H(x,p) &= \varphi(c^*) \\
&= \sum_{i=1}^{f} \left[\ln\left(\frac{1}{p_i}\right) - p_i \frac{1}{p_i} \right] + \left\langle p, (R+G^\top)x \right\rangle \\
&= \sum_{i=1}^{f} \left[\ln\left(\frac{1}{p_i}\right) - 1 \right] + \left\langle p, (R+G^\top)x \right\rangle \\
&= -\sum_{i=1}^{f} \ln(p_i) - f + \left\langle p, (R+G^\top)x \right\rangle
\end{aligned}
$$

□

Remark 1 Note that coupling $c^* = p_i^{-\frac{1}{\sigma}}$; $c^* = \frac{1}{p_i}$ with (HJB), we establish the relationship

$$u'(c_i^*) = \frac{\partial V}{\partial x_i}(x)$$

between the candidate optimal control c^* and the *gradient* of value function $\nabla V(x)$. This has an interesting interpretation in economic terms: at optimum, the marginal utility from the polluting emissions equals to the marginal cost of a diminishing stock of clean air at node i at every moment.

Moreover the same relationship rewritten as

$$c_i^* = (u')^{-1} \left(\frac{\partial V}{\partial x_i}(x) \right) \tag{3.2}$$

becomes a candidate feedback-law for the optimal feedback control c_i^*, once V is known.

3.1. Optimal Policies for Power Function Utility

For the case of power utility, we have proven that the HJB equation is

$$\rho v(x) = \frac{\sigma}{1-\sigma} \sum_{i=1}^{f} \left(\frac{\partial v(x)}{\partial x_i} \right)^{1-\frac{1}{\sigma}} + \left\langle \nabla v(x), (R+G^\top)x \right\rangle \tag{3.3}$$

with maximum of the current value Hamiltonian attained at

$$c_i^* = \left(\frac{\partial v}{\partial x_i}(x)\right)^{-\frac{1}{\sigma}}$$

We also define the constants θ_1 and M that will be used in the sequel

$$\theta_1 := \frac{\rho - \lambda(1-\sigma)}{\sigma}; \quad M = \sum_{j=1}^{f} \eta_j^{1-\frac{1}{\sigma}} \tag{3.4}$$

We want to prove the following theorem.

Theorem 3.1 *Assume $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, with $\sigma > 0, \sigma \neq 1$, $\theta_1 > 0$. Assume that the feedback control defined by*

$$c_i^*(t) = \frac{\theta_1}{M}\eta_i^{-\frac{1}{\sigma}} \langle X(t), \eta \rangle \text{, for all } i \in F, \qquad c_i^*(t) = 0\text{, for all } i \notin F. \tag{3.5}$$

are admissible at the initial state x, i.e., $c^* \in \mathbb{A}(x)$. Then:

(i) $c^*(t)$ is an optimal control strategy for the problem;

(ii) the cumulative welfare along such strategy is

$$V(x) = \frac{\theta_1^{-\sigma}}{1-\sigma} M^{\sigma} \langle x, \eta \rangle^{1-\sigma}; \tag{3.6}$$

(iii) $\langle X^*(t), \eta \rangle = e^{gt} \langle x, \eta \rangle$, with

$$g = \lambda - \theta_1 = \frac{\lambda - \rho}{\sigma}, \tag{3.7}$$

Proof. Firstly, we show that a function of type

$$v(x) = \frac{B}{1-\sigma} \langle x, \eta \rangle^{1-\sigma}$$

is a solution of the HJB equation, for a suitable choice of the constant B. We now proceed with the calculation

$$\frac{\partial v(x)}{\partial x_i} = B \langle x, \eta \rangle^{-\sigma} \eta_i, \quad \nabla v(x) = B \langle x, \eta \rangle^{-\sigma} \eta$$

We rewrite the HJB substituting $v(x)$ and its derivatives

$$\begin{aligned}
\rho\frac{B}{1-\sigma}\langle x,\eta\rangle^{1-\sigma} &= \frac{\sigma}{1-\sigma}\sum_{i=1}^{f}(B\langle x,\eta\rangle^{-\sigma}\eta_i)^{1-\frac{1}{\sigma}} + \left\langle B\langle x,\eta\rangle^{-\sigma}\eta,(R+G^{\top})x\right\rangle \\
&= \frac{\sigma}{1-\sigma}B^{1-\frac{1}{\sigma}}\langle x,\eta\rangle^{1-\sigma}\sum_{i=1}^{f}\eta_i^{1-\frac{1}{\sigma}} + B\langle x,\eta\rangle^{-\sigma}\langle (R+G)\eta,x\rangle \\
&= \frac{\sigma}{1-\sigma}B^{1-\frac{1}{\sigma}}\langle x,\eta\rangle^{1-\sigma}M + B\lambda\langle x,\eta\rangle^{1-\sigma}
\end{aligned}$$

we gather the $\langle x,\eta\rangle^{1-\sigma}$ that is in common on the L.H.S.

$$\rho\frac{B}{1-\sigma}\langle x,\eta\rangle^{1-\sigma} = \left[\frac{\sigma}{1-\sigma}B^{1-\frac{1}{\sigma}}M + \lambda B\right]\langle x,\eta\rangle^{1-\sigma}$$

This equality needs to be true for every value of x in order for the above to be an identity. That happens if and only if

$$\rho\frac{B}{1-\sigma} = \left[\frac{\sigma}{1-\sigma}B^{1-\frac{1}{\sigma}}M + \lambda B\right]$$

We bring all the elements on the R.H.S., and after we rearrange the equation by grouping what we can

$$\begin{aligned}
\rho\frac{B}{1-\sigma} - \frac{\sigma}{1-\sigma}B^{1-\frac{1}{\sigma}}M - \lambda B = 0 \\
(\frac{\rho}{1-\sigma} - \lambda)B - \frac{\sigma}{1-\sigma}B^{1-\frac{1}{\sigma}}M = 0 \\
B\left[\frac{\rho}{1-\sigma} - \lambda - \frac{\sigma}{1-\sigma}B^{-\frac{1}{\sigma}}M\right] = 0
\end{aligned}$$

which is true in two cases: the first one when $B = 0$, which is not very interesting; and when $[\cdots] = 0$, which is what we are going to analyse.

$$\begin{aligned}
\frac{\rho}{1-\sigma} - \lambda - \frac{\sigma}{1-\sigma}B^{-\frac{1}{\sigma}}M = 0 \\
B^{-\frac{1}{\sigma}}M = (\frac{\rho}{1-\sigma} - \lambda)\frac{1-\sigma}{\sigma} \\
B = \left[\frac{\frac{\rho-\lambda(1-\sigma)}{\sigma}}{M}\right]^{-\sigma} = \theta_1^{-\sigma}M^{\sigma}
\end{aligned}$$

As a consequence the solution to the HJB equation is

$$v(x) = \frac{B}{1-\sigma}\langle x,\eta\rangle^{1-\sigma} = \frac{1}{1-\sigma}\left(\frac{\theta_1}{M}\right)^{-\sigma}\langle x,\eta\rangle^{1-\sigma}$$

and the associated maximizer is

$$c^*(t) = \left(\frac{\partial v(x)}{\partial x_i}\right)^{-\frac{1}{\sigma}} = \left(B\left\langle x,\eta\right\rangle^{-\sigma}\eta_i\right)^{-\frac{1}{\sigma}} = \frac{\theta_1}{M}\left\langle x,\eta\right\rangle\eta_i^{-\frac{1}{\sigma}}$$

To prove that actually $v = V$ we need a standard verification theorem (see for instance Fleming and Rishel [14]), the uniqueness of the optimal control follows by the concavity of the problem (see Acemoglu, [1]). This proves (i) and (ii).

To prove (iii), we multiply both sides of the CLE by η and get

$$\begin{aligned}
\left\langle \dot{X}(t),\eta\right\rangle &= \left\langle (R+G^\top)X(t),\eta\right\rangle - \frac{\theta_1}{\langle\eta,\xi\rangle}\left\langle X(t),\eta\right\rangle\left\langle \xi,\eta\right\rangle \\
&= \left\langle X(t),(R+G)\eta\right\rangle - \frac{\theta_1}{\langle\eta,\xi\rangle}\left\langle X(t),(\eta\xi)^\top\eta\right\rangle \\
&= \lambda\left\langle X(t),\eta\right\rangle - \frac{\theta_1}{\langle\eta,\xi\rangle}\left\langle\eta,\xi\right\rangle\left\langle X(t),\eta\right\rangle \\
&= (\lambda-\theta_1)\left\langle X(t),\eta\right\rangle
\end{aligned}$$

By solving this linear ODE we find (iii). □

Admissibility

Next we discuss admissibility of the optimal control described by (3.5). We introduce some additional notation to rewrite the optimal feedback rule (3.5) and write the associated closed-loop equation.
We can rewrite the optimal control

$$c^*(t) = \frac{\theta_1}{M}\left\langle X(t),\eta\right\rangle\sum_{i=1}^{f}\eta_i^{-\frac{1}{\sigma}}\mathbf{e}_i$$

by setting

$$\xi_i := \begin{cases}\eta_i^{-\frac{1}{\sigma}} & i\in F\\ 0 & i\notin F\end{cases} \quad\Rightarrow\quad \xi = \sum_{i=1}^{f}\eta_i^{-\frac{1}{\sigma}}\mathbf{e_i}$$

and

$$E := \xi\eta^\top$$

so that

$$\left\langle\eta,\xi\right\rangle = \eta\xi^\top = \sum_{i=1}^{n}\eta_i\xi_i = \sum_{i=1}^{f}\eta_i^{1-\frac{1}{\sigma}} = M$$

and

$$c^*(t) = \frac{\theta_1}{\langle \eta, \xi \rangle} EX(t).$$

Then the associated closed-loop equation is

$$\begin{cases} \dot{X}(t) = \left(R + G^\top - \frac{\theta_1}{\langle \eta, \xi \rangle} E\right) X(t), \quad t > 0 \\ X(0) = x. \end{cases} \tag{3.8}$$

Then the closed-loop equation appears as a linear system ruled by a matrix that correspond to $R + G^\top$ modified by the extraction. Such matrix, expanded, read as

$$\begin{pmatrix} r_1 - \frac{\theta_1}{\langle \eta, \xi \rangle}\xi_1\eta_1 & g_{21} - \frac{\theta_1}{\langle \eta, \xi \rangle}\xi_1\eta_2 & \cdots & g_{n1} - \frac{\theta_1}{\langle \eta, \xi \rangle}\xi_1\eta_n \\ g_{12} - \frac{\theta_1}{\langle \eta, \xi \rangle}\xi_2\eta_1 & r_2 - \frac{\theta_1}{\langle \eta, \xi \rangle}\xi_2\eta_2 & \cdots & g_{n2} - \frac{\theta_1}{\langle \eta, \xi \rangle}\xi_2\eta_n \\ \vdots & \vdots & & \vdots \\ g_{1n} - \frac{\theta_1}{\langle \eta, \xi \rangle}\xi_n\eta_1 & g_{2n} - \frac{\theta_1}{\langle \eta, \xi \rangle}\xi_2\eta_n & \cdots & g_{nn} - \frac{\theta_1}{\langle \eta, \xi \rangle}\xi_n\eta_n \end{pmatrix}$$

Note that the above matrix is a Metzler matrix if an only if for every i and j, $i \neq j$ one has

$$g_{ji} - \frac{\theta_1}{\langle \eta, \xi \rangle}\xi_i\eta_j \geq 0 \iff \theta_1 \leq \frac{g_{ji}\langle \eta, \xi \rangle}{\xi_i\eta_j} = g_{ji}\langle \eta, \xi \rangle \eta_i^{-\frac{1}{\sigma}}\eta_j^{-1} \tag{3.9}$$

That suggest a simple condition under which the optimal control described in Theorem 3.1 is admissible.

Theorem 3.2 (Admissibility) *If θ_1 satisfies*

$$\theta_1 \leq g_{ji}\langle \eta, \xi \rangle \eta_i^{-\frac{1}{\sigma}}\eta_j^{-1}, \quad \forall i \neq j \tag{3.10}$$

and the initial stock k belongs to the positive orthant $\mathbb{R}^n_+$, then the entire trajectory $X^*(t)$ lies in the positive orthant and the control (3.5) is admissible and hence optimal for the problem.

The statement is a simple consequence of Theorem 1.3.

Long-run Stocks

We here intend to briefly discuss whether the optimal trajectories of the system converge towards a particular direction when time tends to infinity, namely, if they enjoy some asymptotic stability property. Since the optimal trajectory is diverging in general, one can introduce the *detrended optimal trajectory*, namely the optimal trajectory net of the trend

$$Y(t) = e^{-(\lambda - \theta_1)X^*(t)}.$$

We will need also the following Lemma.

Lemma 4 *Consider the base $\{\zeta, v_2, ..., v_n\}$ of generalized eigenvectors of $R + G^\top$, associated to the eigenvalues $\{\lambda, \lambda_2, ..., \lambda_n\}$ derived from the Perron-Frobenius Theorem, as the one introduced after 2.6. Then, $R + G^\top - \frac{\theta_1}{\langle \eta, \xi \rangle} E$ has eigenvalues $\{\lambda - \theta_1, \lambda_2, ..., \lambda_n\}$ associated respectively with eigenvectors $\{\hat{\zeta}, v_2, ..., v_n\}$.*

The proof of the above lemma can be found in Fabbri *et al.*, Lemma 1. Note that it states that the optimal consumption modifies only the direction of the dominant eigenvector of the original matrix $R + G^\top$, changing it from ζ to $\hat{\zeta}$, and decreasing its associated eigenvalue from λ to $\lambda - \theta_1$.

Theorem 3.3 *In the assumptions of Theorem 3.5, and for*

$$0 < \theta_1 < \lambda - \operatorname{Re} \lambda_2,$$

where λ_2 is the eigenvalue with greatest real part among $\{\lambda_2, ..., \lambda_n\}$, the optimal trajectory $X^*(t)$ converges towards the direction of $\hat{\zeta}$. Said differently

$$\lim_{t \to +\infty} Y(t) = \alpha \hat{\zeta} \tag{3.11}$$

for a suitable positive constant α

Proof. Since by assumption $\lambda - \theta_1 < \operatorname{Re}(\lambda_2)$ and hence $\lambda - \theta_1$ is still the dominant eigenvalue of the matrix of the system $R + G^\top - \frac{\theta_1}{\langle \xi, \eta \rangle} E$, associated to the dominant eigenvector $\hat{\zeta}$, the trajectory of the system converges towards the direction of $\hat{\zeta}$ by Theorem 1.3.

□

3.2. Optimal Policies for Logarithmic Utility

For the case of power utility, we have proven that the HJB equation is

$$\rho v(x) = -\sum_{i=1}^{f} \ln\left(\frac{\partial v(x)}{\partial x_i}\right) - f + \left\langle \nabla v(x), (R + G^\top) x \right\rangle \tag{3.12}$$

with maximum of the current value Hamiltonian attained at

$$c_i^* = \left(\frac{\partial v}{\partial x_i}(x)\right)^{-1}$$

We want to prove the following theorem.

Theorem 3.4 *Assume $u(c) = \ln c$, with $\sigma = 1$. Assume that the feedback control defined by*

$$c_i^*(t) = \frac{\rho}{f\eta_i}\langle X(t), \eta\rangle, \text{ for all } i \in F, \qquad c_i^*(t) = 0, \text{ for all } i \notin F. \tag{3.13}$$

are admissible at the initial state x, i.e., $c^* \in \mathbb{A}(x)$. Then:

(*i*) *c^* is an optimal control strategy for the optimal control problem ()()();*

(*ii*) *the cumulative welfare of all agents i along such strategy is*

$$V(x) = \frac{f}{\rho}\ln\langle x, \eta\rangle + \frac{1}{\rho}\left[-\sum_{i=1}^{f}\ln\eta_i - f + f\ln\frac{\rho}{f} + \frac{\lambda f}{\rho}\right]; \tag{3.14}$$

iii $\langle X^*(t), \eta\rangle = e^{gt}\langle , \eta\rangle$, *with*

$$g = (\lambda - \rho), \tag{3.15}$$

Proof. Firstly, we show that a function of type

$$v(x) = B\ln(\langle x, \eta\rangle) + A$$

is a solution of the HJB equation, for a suitable choice of the constant A,B. We now proceed with the calculation

$$\frac{\partial v(x)}{\partial x_i} = \frac{B\eta_i}{\langle x, \eta\rangle}, \quad \nabla v(x) = \frac{B\eta}{\langle x, \eta\rangle}$$

We rewrite the HJB substituting $v(x)$ and its derivatives

$$\begin{aligned}\rho(B\ln(\langle x, \eta\rangle) + A) &= \sum_{i=1}^{f}\left[\ln\left(\frac{\langle x, \eta\rangle}{B\eta_i}\right)\right] - f + \left\langle \frac{B\eta}{\langle x, \eta\rangle}, (R + G^\top)x \right\rangle \\ &= (\ln\langle x, \eta\rangle - \ln B)\sum_{i=1}^{f}(1) - \sum_{i=1}^{f}\ln\eta_i - f + \frac{B}{\langle x, \eta\rangle}\langle (R+G)\eta, x\rangle\end{aligned} \tag{3.16}$$

we rearrange the equation by bringing all the terms with $\ln\langle x, \eta\rangle$ on the L.H.S and all the other terms on the R.H.S da qui in poi i calcoli vanno sistemati

$$(\rho B - f)\ln\langle x, \eta\rangle = -\sum_{i=1}^{f}\ln\eta_i - f - f\ln B - \rho A + B\lambda$$

In order to have such equation satisfied for all x in $\mathbb{R}^n_+$, one has to annihilate the coefficient of $\ln\langle x, \eta\rangle$ and the constant on the RHS, so that

$$\begin{cases}\rho B - f = 0 \\ -\sum_{i=1}^{f}\ln\eta_i - f - f\ln B - \rho A + B\lambda = 0\end{cases}$$

giving

$$B = \frac{f}{\rho}, \quad A = \frac{1}{\rho}\left[-\sum_{i=1}^{f} \ln \eta_i - f + f \ln\left(\frac{\rho}{f}\right) + \frac{f\lambda}{\rho}\right]$$

As a consequence

$$\begin{aligned} v(x) &= B \ln \langle x, \eta \rangle + A \\ &= \frac{f}{\rho} \ln \langle x, \eta \rangle + \frac{1}{\rho}\left[-\sum_{i=1}^{f} \ln \eta_i - f + f \ln\left(\frac{\rho}{f}\right) + \frac{f\lambda}{\rho}\right] \end{aligned} \tag{3.17}$$

The maximizer in the Hamiltonian function gives a formula for the candidate feedback optimal control

$$c_i^*(t) = \left(\frac{\partial v(X(t))}{\partial x_i}\right)^{-1} = \left(\frac{f}{\rho}\frac{\eta_i}{\langle X(t), \eta \rangle}\right)^{-1} = \frac{\rho}{f\eta_i} \langle X(t), \eta \rangle$$

To prove that actually $v = V$ we need a standard verification theorem (see for instance Fleming and Rishel [14]), the uniqueness of the optimal control follows by the concavity of the problem (see Acemoglu, [1]). This proves (i) and (ii).

To prove (iii), we multiply both sides of the CLE by η and get

$$\begin{aligned} \left\langle \dot{X}(t), \eta \right\rangle &= \left\langle (R + G^\top) X(t), \eta \right\rangle - \frac{\rho}{f} \langle X(t), \eta \rangle \langle \xi, \eta \rangle \\ &= \langle X(t), (R+G)\eta \rangle - \frac{\rho}{f} \left\langle X(t), (\eta\xi)^\top \eta \right\rangle \\ &= \lambda \langle X(t), \eta \rangle - \frac{\rho}{f} f \langle X(t), \eta \rangle \\ &= (\lambda - \rho) \langle X(t), \eta \rangle \end{aligned} \tag{3.18}$$

By solving this linear ODE we find (iii). □

Admissibility

Next we discuss admissibility of the optimal control described by (3.5). We introduce some additional notation to rewrite the optimal feedback rule (3.5) and write the associated closed-loop equation.
We can rewrite the optimal control

$$c^*(t) = \frac{\rho}{f} \langle X(t), \eta \rangle \frac{1}{\eta_i} \mathbf{e}_i$$

by setting

$$\xi_i := \begin{cases} \eta_i^{-1} & i \in F \\ 0 & i \notin F \end{cases} \quad \Rightarrow \quad \xi = \sum_{i=1}^{f} \eta_i^{-1} \mathbf{e_i}$$

and

$$E := \xi \eta^\top$$

so that

$$\langle \eta, \xi \rangle = \eta \xi^\top = \sum_{i=1}^{n} \eta_i \xi_i = \sum_{i=1}^{f} \eta_i \frac{1}{\eta_i} = \sum_{i=1}^{f} 1 = f$$

and

$$c^*(t) = \frac{\rho}{f} \langle \eta, \xi \rangle E X(t).$$

Then the associated closed-loop equation is

$$\begin{cases} \dot{X}(t) = \left(R + G^\top - \frac{\rho}{f} E \right) X(t), \quad t > 0 \\ X(0) = x. \end{cases} \tag{3.19}$$

Then the closed-loop equation appears as a linear system ruled by a matrix that correspond to $R + G^\top$ modified by the extraction. Such matrix, expanded, read as

$$\begin{pmatrix} r_1 - \frac{\rho}{f}\xi_1\eta_1 & g_{21} - \frac{\rho}{f}\xi_1\eta_2 & \cdots & g_{n1} - \frac{\rho}{f}\xi_1\eta_n \\ g_{12} - \frac{\rho}{f}\xi_2\eta_1 & r_2 - \frac{\rho}{f}\xi_2\eta_2 & \cdots & g_{n2} - \frac{\rho}{f}\xi_2\eta_n \\ \vdots & \vdots & & \vdots \\ g_{1n} - \frac{\rho}{f}\xi_n\eta_1 & g_{2n} - \frac{\rho}{f}\xi_2\eta_n & \cdots & g_{nn} - \frac{\rho}{f}\xi_n\eta_n \end{pmatrix}$$

Note that the above matrix is a Metzler matrix if an only if for every i and j, $i \neq j$ one has

$$g_{ji} - \frac{\rho}{f}\xi_i\eta_j \geq 0 \iff \rho \leq \frac{g_{ji} f}{\xi_i \eta_j} = g_{ji} f \eta_i^{-1} \eta_j^{-1} \tag{3.20}$$

That suggest a simple condition under which the optimal control described in Theorem 3.1 is admissible.

Theorem 3.5 (Admissibility) *If ρ satisfies*

$$\rho \leq g_{ji} f \eta_i^{-1} \eta_j^{-1}, \quad \forall i \neq j \tag{3.21}$$

and the initial stock k belongs to the positive orthant $\mathbb{R}^n_+$, then the entire trajectory $X^*(t)$ lies in the positive orthant and the control (3.5) is admissible and hence optimal for the problem.

The statement is a simple consequence of Theorem 1.3.

Long-run Stocks

We here intend to briefly discuss whether the optimal trajectories of the system converge towards a particular direction when time tends to infinity, namely, if they enjoy some asymptotic stability property. Since the optimal trajectory is diverging in general, one can introduce the *detrended optimal trajectory*, namely the optimal trajectory net of the trend

$$Y(t) = e^{-(\lambda-\rho)X^*(t)}.$$

We will need also the following Lemma.

LEMMA 5 *Consider the base $\{\zeta, v_2, ..., v_n\}$ of generalized eigenvectors of $R + G^\top$, associated to the eigenvalues $\{\lambda, \lambda_2, ..., \lambda_n\}$ derived from the Perron-Frobenius Theorem, as the one introduced after 2.6. Then, $R + G^\top - \frac{\rho}{f}E$ has eigenvalues $\{\lambda - \rho, \lambda_2, ..., \lambda_n\}$ associated respectively with eigenvectors $\{\hat{\zeta}, v_2, ..., v_n\}$.*

The proof of the above lemma can be found in Fabbri *et al.*, Lemma 1. Note that it states that the optimal consumption modifies only the direction of the dominant eigenvector of the original matrix $R + G^\top$, changing it from ζ to $\hat{\zeta}$, and decreasing its associated eigenvalue from λ to $\lambda - \rho$.

THEOREM 3.6 *In the assumptions of Theorem 3.5, and for*

$$0 < \rho < \lambda - \mathrm{Re}\,\lambda_2,$$

where λ_2 is the eigenvalue with greatest real part among $\{\lambda_2, ..., \lambda_n\}$, the optimal trajectory $X^*(t)$ converges towards the direction of $\hat{\zeta}$. Said differently

$$\lim_{t\to+\infty} Y(t) = \alpha\hat{\zeta} \tag{3.22}$$

for a suitable positive constant α

Proof. Since by assumption $\lambda - \rho < \mathrm{Re}(\lambda_2)$ and hence $\lambda - \rho$ is still the dominant eigenvalue of the matrix of the system $R + G^\top - \frac{\rho}{f}E$, associated to the dominant eigenvector $\hat{\zeta}$, the trajectory of the system converges towards the direction of $\hat{\zeta}$ by Theorem 1.3.

□

3.3. Eco-friendly Production Design

In settings where a network of flows connects the stock of clean air in different sites, do different regeneration rates in the various sites and different intensities of flow map into a specific hierarchy of the sites? Does this hierarchy affect how the access of agents should be regulated and, in particular, in which regions pollution should be avoided, when possible?

In the previous sections of this chapter we derived (case $\sigma \neq 1$)

$$V(x) = \frac{\theta_1^{-\sigma}}{1-\sigma} \left(\sum_{j=1}^{f} \eta_j^{-\frac{1-\sigma}{\sigma}} \right)^{\sigma} \langle x, \eta \rangle^{1-\sigma}$$

We recall that by the sake of simplicity we assumed that the first f sites are occupied by production, although it is sufficient to rename nodes to obtain a different subset of f nodes over the n available. That said, note that under the assumption $\sigma \in]0,1[$, we have

$$\frac{1-\sigma}{\sigma} > 0$$

so that V is maximal when

$$\sum_{j=1}^{f} \eta_j^{-\frac{1-\sigma}{\sigma}}$$

is minimal, and that correspond to possibly select, among the available, the f nodes corresponding to the minimal η_j's.
Said differently, the production of the consumption good is best when placed in the most "peripheral" nodes in the sense of eigencentrality η.
We recall also that eigencentralities η_j's combine the two different effects of regeneration factors and diffusion flows.

Thus, the nodes with a lower "centrality" with respect to the rest of the network (meaning a low regeneration rate and a low connection with the more "central nodes") are the one were the emissions and the following exploitation of the stock of "clean air" have the lowest impact on the overall system.

Chapter 4

Cooperative vs Non-Cooperative Frameworks

After having stated the results of the two separate frameworks, an important task is to compare the results coming from the two cases both from a cumulative and individual perspective. The reason behind this comparison is to determine if the single planner is effectively able to coordinate the agents and thus achieving an overall better exploitation of the "clean air" resource and higher utilities for the agents with respect to the case where agents are totally free to compete for the exploitation of the resource.

4.1. The Non-Cooperative Pollution Game

We take into consideration a network where players make independent decision in order to maximize their own utility, thus we set the problem in the framework that in Section 2.1.1 we named (F2). The results in this chapter are known and extensively discussed in the paper by Fabbri *et al.*. A simplified version of some of those results is reported here for comparison with the framework (F1).

The value function or wealth of player i is defined as

$$W_i(x) = \max_{c_i \in \mathbb{A}_i} J_i(c_i; x), \quad i \in F$$

where the intertemporal utility J and the set of admissible controls $\mathbb{A}_i$ are those defined in Chapter 2. Then W_i represents the highest possible utility achieved by agent i by choosing its optimal emission rate c_i in response to the emission decision of the other agents.

We set

$$\theta_2 := \frac{\rho + (\sigma - 1)\lambda}{1 + (\sigma - 1) f}. \tag{4.1}$$

Theorem 4.1 *Assume* $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, *with* $\sigma > 0, \sigma \neq 1$, $\theta_2 > 0$. *Assume also that*

$$\hat{c}_i(t) = \frac{\theta_2}{\eta_i} \langle X(t), \eta \rangle \text{, for all } i \in F, \qquad \hat{c}_i(t) = 0, \text{ for all } i \notin F. \tag{4.2}$$

is an admissible strategy profile, namely $c_i^* \in \mathbb{A}_i$. *Then:*

(*i*) $c^*(t)$ *is an optimal Markovian equilibrium for the game;*

(*ii*) *the welfare of agent i along such equilibrium is*

$$W_i(x) = \frac{\theta_2^{-\sigma} \eta_i^{\sigma-1}}{1-\sigma} \langle x, \eta \rangle^{1-\sigma}; \tag{4.3}$$

(*iii*) *If* $\hat{X}(t)$ *is the trajectory at the equilibrium then*

$$\left\langle \hat{X}(t), \eta \right\rangle = e^{gt} \langle x, \eta \rangle \tag{4.4}$$

with

$$g = \lambda - \theta_2 f = \frac{\lambda - f\rho}{1 + (\sigma - 1) f}, \tag{4.5}$$

Remark 2 (a) In the case of logarithmic utility, the Markovian equilibrium is obtained by setting $\theta_2 = \rho$ in (4.2) while the value function W_i is

$$W_i(x) = \frac{1}{\rho} \left[\ln \left(\frac{\rho}{\eta_i} \langle x, \eta \rangle \right) + \lambda - f\rho \right].$$

(b) The case deemed a "regular regime" in Fabbri et al. [8] is that in which θ_2 is positive with both positive numerator and denominator. This is necessarily the case when we assume θ_1 positive, as necessarily $\rho - (1-\sigma)\lambda > 0$. As for θ_2, a set of conditions in which it is also positive is the following

$$0 < \sigma < 1, \quad 1 \leq f < \frac{1}{1-\sigma}. \tag{4.6}$$

As an alternative

$$\sigma \geq 1, f \geq 1.$$

We will always assume, in this chapter, that (4.6) is satisfied.

4.2. Comparison in emission/consumption

We expect that in the game agents tend to overpollute with respect to the case when a unique decision maker is choosing their policies, and we intend to confirm this hypothesis.

Power Function Utility

We want to compare the effects of the feedback strategies in the two framewors, namely

$$c_i^*(t) = \frac{\theta_1}{M}\eta_i^{-\frac{1}{\sigma}} \left\langle X^*(t), \eta \right\rangle, \quad \hat{c}_i(t) = \frac{\theta_2}{\eta_i} \left\langle \hat{X}(t), \eta \right\rangle. \tag{4.7}$$

To this extent, we start by showing that

$$\theta_2 \geq \theta_1. \tag{4.8}$$

Indeed, since by Remark 2 we have $\rho - (1-\sigma)\lambda > 0$, then

$$\begin{aligned} \theta_2 \geq \theta_1 &\iff \frac{\rho - \lambda(1-\sigma)}{1-(1-\sigma)f} \geq \frac{\rho - \lambda(1-\sigma)}{\sigma} \\ &\iff \frac{1}{1-(1-\sigma)f} \geq \frac{1}{\sigma} \\ &\iff \sigma \geq 1-(1-\sigma)f \\ &\iff (f-1)(1-\sigma) \geq 0 \end{aligned}$$

which is true under assumptions (4.6). Note that, as a consequence, the coefficients multiplying the weighted total mass in (4.9) satisfy Thus, in the case of power function utility, agents in the game overpollute if and only if

$$\begin{aligned} \frac{\theta_2}{\eta_i} \geq \frac{\theta_1}{M}\eta_i^{-\frac{1}{\sigma}} &\iff \frac{\theta_2}{\eta_i} \geq \frac{\theta_1}{\sum_j^f \left(\eta_j^{1-1/\sigma}\right)}\eta_i^{-1/\sigma} \\ &\iff \frac{\theta_2}{\theta_1} \geq \frac{\eta_i^{1-1/\sigma}}{\sum_j^f \left(\eta_j^{1-1/\sigma}\right)} \end{aligned}$$

but this is true as the RHS is less than 1, while the LHS is greater than 1 by (4.8). That can be interpreted as follows: proportionally to the weighted total mass (which is different in time in the two frameworks), the agents in the game tend to overpollute.

Now we compare *open-loop* formulas for the two controls under study, and obtained by combining the closed-loop formulas with the evolution of the weighted total mass expressed in (iii) respectively in Theorems 3.1 and 4.1, namely

$$c_i^*(t) = \frac{\theta_1}{M}\eta_i^{-\frac{1}{\sigma}} \langle x, \eta \rangle e^{(\lambda-\theta_1)t}, \quad \hat{c}_i(t) = \frac{\theta_2}{\eta_i} \langle x, \eta \rangle e^{(\lambda-\theta_1)t}. \tag{4.9}$$

At time 0, clearly $\hat{c}_i(0) > c_i^*(0)$, but as time goes by, since in the game the weighted total mass is overexploited, the inequality reverses. In detail

$$\begin{aligned} c_i^*(t) \leq \hat{c}_i(t) &\iff \frac{\theta_1}{M}\eta_i^{-\frac{1}{\sigma}} \langle x, \eta \rangle e^{(\lambda-\theta_1)t} \leq \frac{\theta_2}{\eta_i} \langle x, \eta \rangle e^{(\lambda-\theta_1)t} \\ &\iff e^{(\theta_2-\theta_1)t} \leq \frac{\theta_2}{\theta_1} \frac{\sum_j \eta_j^{1-\frac{1}{\sigma}}}{\eta_i^{1-\frac{1}{\sigma}}} \\ &\iff t \leq \frac{1}{\theta_2-\theta_1}\left[\ln\left(\frac{\theta_2}{\theta_1}\right) + \ln\left(\frac{\sum_j \eta_j^{1-\frac{1}{\sigma}}}{\eta_i^{1-\frac{1}{\sigma}}}\right)\right] \end{aligned}$$

(where all quantities in the square brackets are positive).

Logarithmic Utility

In this case the comparison is even simpler as

$$c_i^*(t) = \frac{\rho}{f\eta_i} \langle X^*(t), \eta \rangle, \quad \hat{c}_i(t) = \frac{\rho}{\eta_i} \langle \hat{X}(t), \eta \rangle,$$

and clearly

$$\frac{\rho}{\eta_i} \geq \frac{\rho}{f\eta_i}$$

for $f \geq 1$, whereas the comparison of open-loop formulas

$$c_i^*(t) = \frac{\rho}{f\eta_i} \langle x, \eta \rangle e^{(\lambda-\rho)t}, \quad \hat{c}_i(t) = \frac{\rho}{\eta_i} \langle x, \eta \rangle e^{(\lambda-f\rho)t},$$

leads to

$$\begin{aligned} c_i^*(t) \leq \hat{c}_i(t) &\iff \frac{\rho}{f\eta_i} e^{(\lambda-\rho)t} \leq \frac{\rho}{\eta_i} e^{(\lambda-f\rho)t} \\ &\iff e^{(f-1)\rho t} \leq f \\ &\iff t \leq \frac{1}{(f-1)\rho} ln(f), \end{aligned}$$

and the conclusion is the same.

4.3. Comparison of Welfares

For all choices of the functionj $u(c)$, we define the *cumulative utility function* for the game as

$$W(x) = \sum_{i=1}^{f} W_i(x)$$

Clearly

$$W(x) = \sum_{i=1}^{f} J_i(x; \hat{c}_i(\cdot)) = \sum_{i=1}^{f} \int_0^\infty e^{-\rho t} u(\hat{c}_i(t)) dt \tag{4.10}$$

$$= \int_0^\infty e^{-\rho t} \sum_{i=1}^{f} u(\hat{c}_i(t)) dt = J(x; \hat{c}(\cdot)) \tag{4.11}$$

$$\leq J(x; c^*(\cdot)) = V(t, x) \tag{4.12}$$

as c^* was the optimal control for the problem with functional J. Hence, the total utility with a single decision maker is greater than the sum of welfares W_i at equilibrium in the game. Now we want to check if every single player is better off with the choice of the decision maker c_i^* in comparison to their choices in the noncooperative game, that is if

$$W_i(x) \leq J_i(x; c_i^*(\cdot)) = \int_0^\infty e^{-\rho t} u(c_i^*(t)) dt.$$

Power Function Utility

We start by computing an explicit formula for $J_i(x; c_i^*(\cdot))$:

$$\begin{aligned} J_i(x; c_i^*(\cdot)) &= \int_0^\infty e^{-\rho t} u(c_i^*(t)) dt \\ &= \int_0^\infty e^{-\rho t} \frac{1}{1-\sigma} \left(\frac{\theta_1}{M} \eta_i^{-\frac{1}{\sigma}} \langle x, \eta \rangle e^{(\lambda - \theta_1)t} \right)^{1-\sigma} dt \\ &= \frac{1}{1-\sigma} \left(\frac{\theta_1}{M} \eta_i^{-\frac{1}{\sigma}} \langle x, \eta \rangle \right)^{1-\sigma} \int_0^\infty e^{[-\rho + (\lambda - \theta_1)(1-\sigma)]t} dt \end{aligned}$$

Now note that

$$\begin{aligned} -\rho + (\lambda - \theta_1)(1-\sigma) &= -\rho + (\lambda - \frac{\rho - \lambda(1-\sigma)}{\sigma})(1-\sigma) \\ &= -\rho + (\frac{\lambda\sigma - \rho + \lambda - \lambda\sigma}{\sigma})(1-\sigma) \\ &= \frac{-\rho\sigma - \rho + \rho\sigma + \lambda - \lambda\sigma}{\sigma} \\ &= -\frac{\rho - \lambda(1-\sigma)}{\sigma} \\ &= -\theta_1 \leq 0 \end{aligned}$$

thus we solve the integral as

$$\begin{aligned}\int_0^{+\infty} e^{-\theta_1 t} dt &= \lim_{T\to+\infty} \int_0^T e^{-\theta_1 x} dx \\ &= \lim_{T\to+\infty} -\frac{1}{\theta_1}[e^{-\theta_1 T} - e^{-\theta_1 0}] \\ &= -\frac{1}{\theta_1}[0-1] \\ &= \frac{1}{\theta_1}\end{aligned}$$

so that

$$J_i(x; c_i^*(\cdot)) = \frac{1}{1-\sigma}\left(\frac{\theta_1}{M}\eta_i^{-\frac{1}{\sigma}}\right)^{1-\sigma} \frac{1}{\theta_1}\langle x, \eta\rangle^{1-\sigma}$$

Then

$$\begin{aligned}W_i(x) \le J_i(x; c_i^*(\cdot)) &\iff \frac{\theta_2^{-\sigma}\eta_i^{\sigma-1}}{1-\sigma}\langle x, \eta\rangle^{1-\sigma} \le \frac{1}{1-\sigma}\left(\frac{\theta_1}{M}\eta_i^{-\frac{1}{\sigma}}\right)^{1-\sigma}\frac{1}{\theta_1}\langle x, \eta\rangle^{1-\sigma} \\ &\iff \theta_2^{-\sigma}\eta_i^{\sigma-1} \le \left(\frac{\theta_1}{M}\eta_i^{-\frac{1}{\sigma}}\right)^{1-\sigma}\frac{1}{\theta_1} \\ &\iff \left(\frac{\theta_1}{\theta_2}\right)^{\sigma} \le \left(\frac{\eta_i^{1-\frac{1}{\sigma}}}{M}\right)^{1-\sigma} \\ &\iff M\left(\frac{\theta_1}{\theta_2}\right)^{\frac{\sigma}{1-\sigma}} \le \eta_i^{\frac{\sigma-1}{\sigma}} \\ &\iff \eta_i^{\frac{1-\sigma}{\sigma}} \le \frac{1}{M}\left(\frac{\theta_2}{\theta_1}\right)^{\frac{\sigma}{1-\sigma}} \\ &\iff \eta_i \le \frac{1}{M^{\frac{\sigma}{1-\sigma}}}\left(\frac{\theta_2}{\theta_1}\right)^{\frac{\sigma^2}{(1-\sigma)^2}}\end{aligned}$$

This last line gives a necessary and sufficient condition on the single centrality measure η_i to establish whether player i is better off in the game or under the guidance of a unique planner: η_j needs to be under a certain common threshold. Clearly, the inequality is more easily satisfied for low η_i's, meaning that the agents taking more advantage from a coordinated actions are those residing in the most peripheral nodes.

Chapter 5

Advanced model

The model explained and used in this thesis is, for what attains some of its core hypothesis, a moderately simple model; focused on using low complexity hypothesis in order to retain manageable computations and intuitively results. Although these choices, the model has a discrete potential to be further "upgraded" with several more advanced and complex hypothesis. We will proceed to give an overview of the main possible future research paths that we would like to implement in order to get a progressively more accurate and realistic model to describe the real world.

5.1. Utility Function

The utility function used in the base model is characterized in a way such that the utility of the agents depends only on their emission decision (which where a proxy for the production and consumption decision of the agents seen the perfect correlation between output gap[ΔGDP] and the emission gap [ΔCO_2]. Therefore, one of the things that we would like to introduce would be a more complex utility function with the following form:

$$u(GDP_i, CO_{2i}) = \alpha f(GDP_i(t)) - \beta g(CO_{2i}(t))$$

This reworked utility function will positively depend on the level of GDP reached by agent ß at time t and negatively impacted by the level of excess concentration of CO_2 reached at node i with respect to a fixed reference point (like the pre-industrial era CO_2 gas concentration in the atmosphere). The parameters $\alpha \wedge \beta$ will be used to characterise the different weights that agents can assign to the two different part of the utility function (e.g. a specific type of agents could value more the gains deriving from an increase in their GDP with respect to the corresponding potential damage due to an increase in the CO_2 concentration caused by the increase emissions linked to the increase in the GDP).

5.1.1 Agents Characterization

Base model agents

Each agent of the model occupies one of the f inhabited nodes of the network, they are used to symbolize all the people living in the territories represented by the node. Effectively they represent individual countries or part of large and/or highly populated countries which, for convenience, can be splinted into multiple neighbouring nodes. We define the agents of the model as utility maximisers like their real world counterparts.

Real world agents get their utility from the consumption of goods and services of various nature. The production and the consumption processes of those goods and services cause the emission of GHGs into the atmosphere through the generation of the energy required to feed these same processes. Thus, we can say that agents, in order to be able to satisfy their needs, decide to emit a certain amount of polluting emission into the atmosphere. Now we want to expand on the relationship describe above and thanks to the "*Kaya identity*" proposed by Kaya,Keiichi(1997)[15], we are able to decompose the total production of CO_2 emissions into its core components.

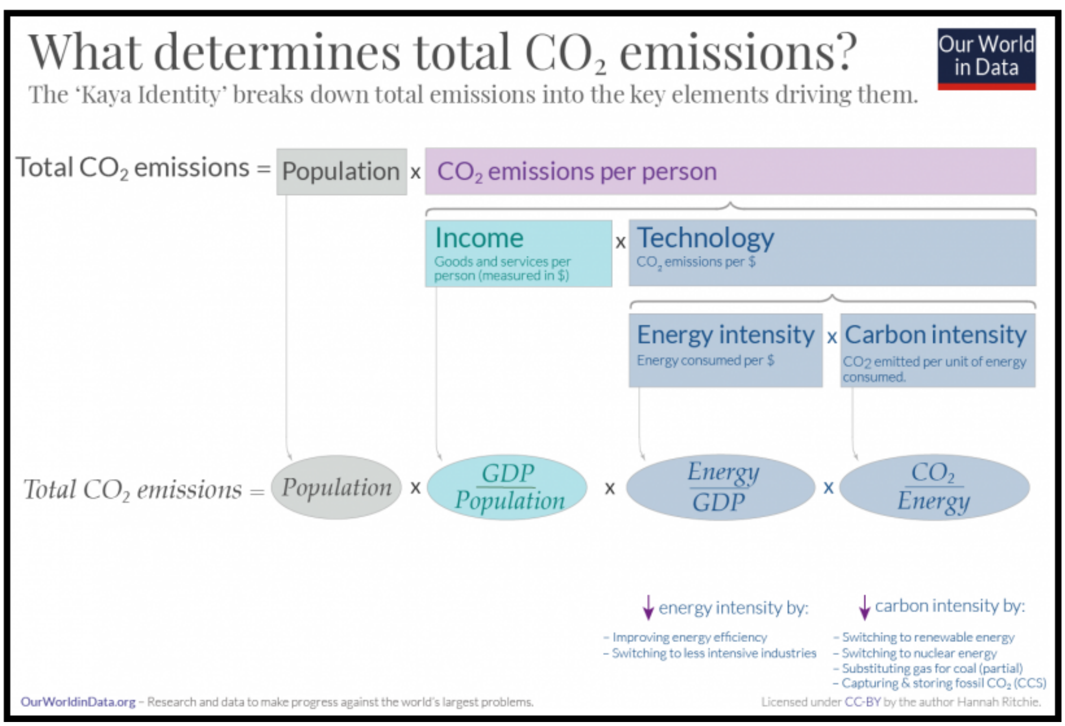


Figure 5.1: The Kaya identity[13]

Applying this concept to our model we get that the total emissions in a given point in time of an agent in its node i would be describe by:

$$c_i(t) = Pop_i(t) \times \frac{GDP_i(t)}{Pop_i(t)} \times \frac{Energy_i(t)}{GDP_i(t)} \times \frac{c_i(t)}{Energy_i(t)}$$

As a consequence of the model hypotheses, the value of the population in any node of the system is capped at a maximum of one. Therefore the above relationship will become equal to:

$$c_i(t) = GDP_i(t) \times E_{int\,i}(t) \times C_{int\,i}(t)$$

$$c_i(t) = GDP_i(t) \times \ \gamma_i(t)$$

Where:

i $E_{int\,i}(t)$ represent the energy intensity, that is the amount of energy used per unit of wealth consumed in the node i at time t;

ii $C_{int\,i}(t)$ represent the carbon intensity, that is the amount of CO_2 (or any other GHGs) emitted per unit of energy consumed in node i at time t;

iii $\gamma_i(t)$ represent the technology employed in node i at time t and is described as the product of carbon intensity and the energy intensity of the production method employed in node i at time t.

Which means that the emission of agent i are equal to the amount of wealth accumulated by the agent i until time t ($GDP_i(t)$) and the technology he use to produce and consume($\gamma_i(t)$). For simplicity, we suppose that the technology used by all the agents in the base model has a value equal to one, thus implying the existence of a perfect correlation between the changes of the emission levels of the agents and the changes of their accumulated wealth. This hypothesis will be relaxed in the advanced model.

With this procedure we have found a way to express the relationship between the emissions produced by the agents and their GDP which represent the total amount of goods and services produced in their node, which are instrumental to the agents utilities.

Advanced model agents

From the results of development economics, we know that even when confronted with the same phenomenon agents will tend to respond and adapt to them in different ways according to their previous particular conditions such as: their level of education, their level of wealth, their risk aversion level, their life expectancy, their expected value of life, their countries institutions, etc...

Thus, seen the different behaviour that we have witnessed looking at the data of the changes occurred in the last thirty years at the GDP and the CO_2 emission of the world countries, we would like to create differentiate the agents of the advanced model into two subtypes: the high-income countries agents i_{HI} and the middle/low-income countries agents i_{LI}. This differentiation will be made through the use of the parameters α, β described above. The HI agents will be characterized with an higher weight on the damage function while the LI agents will overweight the gains attainable through a growth in their GDP with respect to the potential adverse effects of the damage function. The resulting utility function will then become:

$$u_{HI}(GDP_i, CO_{2i}) = \alpha_{HI} f(GDP_i(t) - \beta_{HI} g(CO_{2i}(t))$$

$$u_{LI}(GDP_i, CO_{2i}) = \alpha_{LI} f(GDP_i(t) - \beta_{LI} g(CO_{2i}(t))$$

5.2. Decoupling between Carbon emissions and Economic growth

As previously said in section (2.2.2) the hypothesis of perfect correlation between the output gap (ΔGDP) and the emissions gap (ΔCO_2) is a simplification of reality made to keep the base model as simple as possible. In the advanced version we would like to introduce a more realistic relation between output gap and the emission gap.

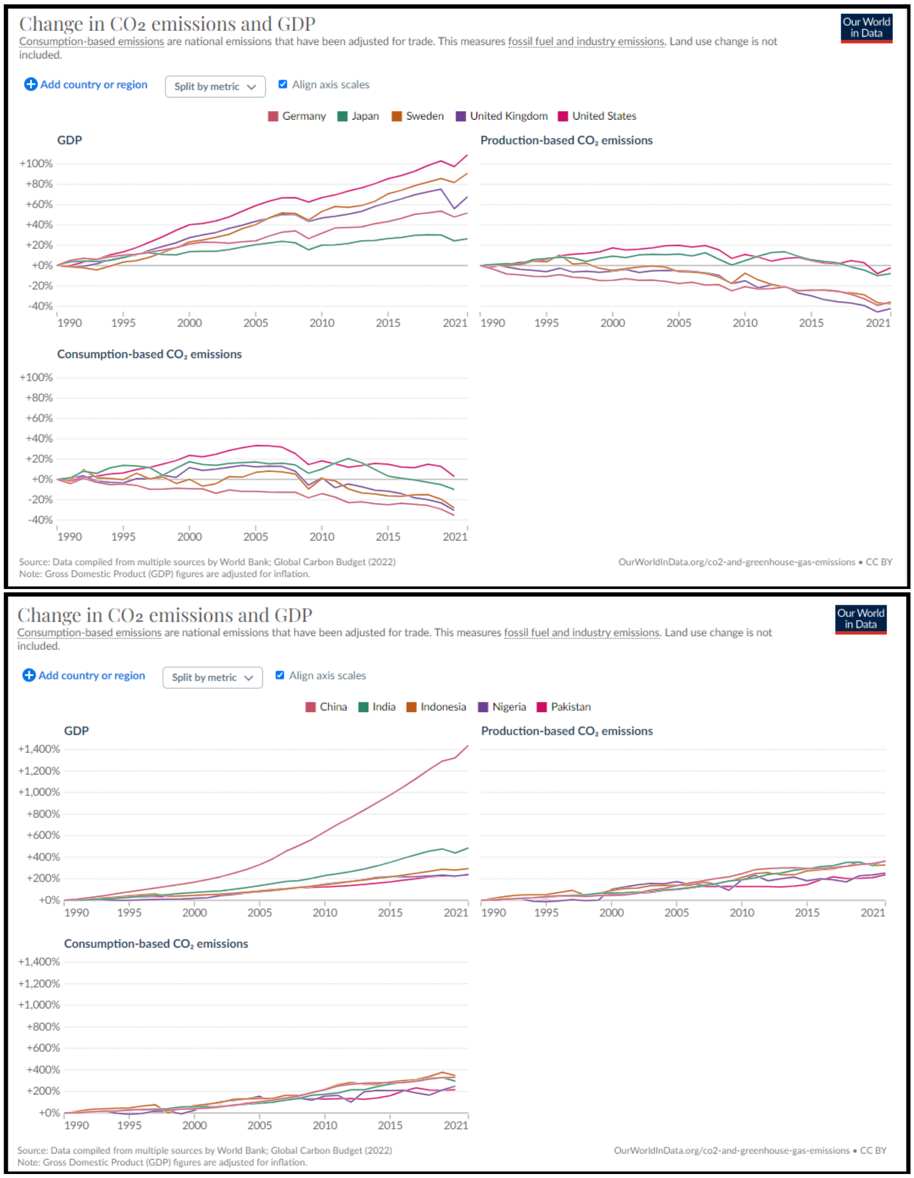


Figure 5.2: Changes in CO_2 emissions and GDP between 1990-2021[13]

Where:

- Production-based emissions are the emissions produced in a given nation or region, excluding land-use change and the emissions embedded in traded goods.
- Consumption-based emissions are national or regional emissions which have been adjusted for trade (i.e. territorial/production emissions minus emissions

embedded in exports, plus emissions embedded in imports). If a country's consumption-based emissions are higher than its production emissions it is a net importer of carbon dioxide.

If we look at the data concerning the changes registered in the last thirty years to the GDP and the CO_2 emission of the countries across the world we can see some interesting phenomenon. The amount of correlation between the growth rate of GDPs and the relative emission levels varies across different countries and in some cases we can even found examples of negative correlation among those quantities. This imply that some countries were able to achieve a positive growth of their GDP and at the same time decrease their annual emission of polluting gasses. A possible way to characterize the emission made by the agents in response to this could be:

$$GHG_i(t) = GHG_i(t_0) + \gamma_i(t)\Delta GDP_i$$

With this functional form, even with a negative $\gamma_i(t)$ (which is the factor trying to capture the technology factor which bonds the emission and the economic activities of the agents) which would capture the action of a possible decarbonisation of the economy, the total emission of the economy would not turn to zero immediately. The $\gamma_i(t)$ could also be further developed into a function of the fraction of investment directed at the improvement of the carbon efficiency of the plants and equipment responsible for the production and consumption of goods and services for the agents in order to capture a possible dynamical technological factor.

5.3. Growth model

In the base model the system is characterised by one dynamics that explain the evolution over time of the stock of "clean air" given the "regeneration" rate Γ and the emission produced by the agents. A possible option to expand the model would be to add one other dynamics into it to describe the economic growth experienced by the agents. One of the possible ways to make this addition could be to use a sort of Solow growth model. With this dynamics we could link the function γ (the technology employed by the agents) to a fraction of the investment, causing it to enter into the capital dynamics equation. Thus, we could model the fact that improving the energy intensity (energy consume per \$) and the carbon intensity (CO_2 emitted per unity of energy consumed) to decarbonise the production and consumption processes is a costly process that bring benefit to the agents through a reduction of their damage function (which depends on the carbon dioxide concentration in the atmosphere) at the cost of a decrease of the funds available to fuel a growth of the overall capital (GDP). Thus the decision of the agents would turn from an optimal emission level (the base model) to a decision of how much resource assign to the general growth and how much on the reduction of emissions (the advanced model).

5.4. Climate Carbon Cycle Model

The "regeneration" parameter Γ of the base model was used to model the natural intake/uptake of atmospheric CO_2 carried out by the oceans and the land ([5],[9]) is such that the evolution of the stock of "clean air" evolves following a logistic function. But, as we can see from the measurements shown in the IPCC reports, these natural processes are unable to keep up with the amount of GHGs emitted by human activities forcefully leading to a progressive increase of the concentration of the GHGs concentration in the atmosphere and the consequent increase of the greenhouse effect.

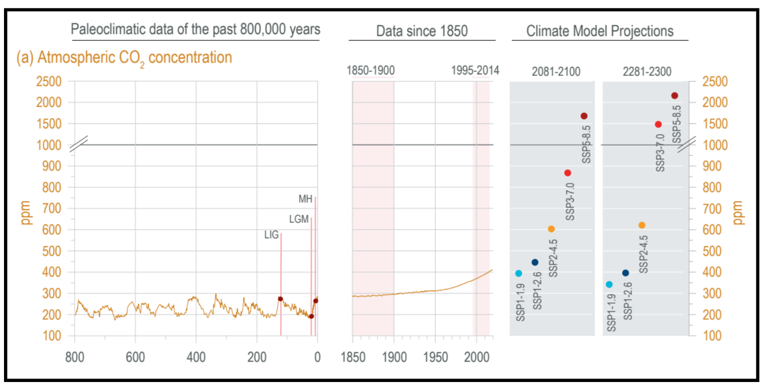


Figure 5.3: Temperature of the world in the last 800 000 years [4]

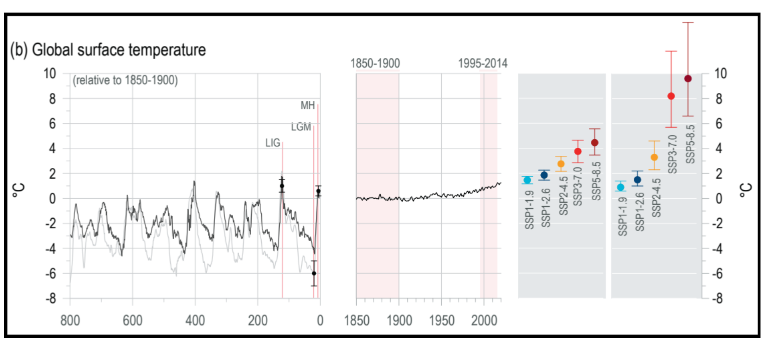


Figure 5.4: Carbon dioxide concentration in the atmosphere in the last 800 000 years [4]

Thus, to render the evolution of the stock of “clean air” in the model more realistic we would need to implement a *Climate- Carbon Cycle* model, in order to fully characterise the mechanism of intake/uptake of carbon dioxide of the oceans and land masses. A model such that could be like the one proposed by Gommem,Hajj,Puri (2012)[7], where the capacity of carbon intake of the natural sequestration reservoirs like oceans and land masses is influenced not only by the concentration of carbon dioxide in the atmosphere but also by the temperature of the planet itself; or the one proposed by Zickfeld,Azevedo,Mathesius and Matthews (2021)[16] which is more focused on the effect of the concentration of carbon dioxide on the intake/uptake capability of the natural sequestration reservoirs.

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