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SPECULATION LEVEL-K REASONING AND INVESTORS' WEALTH DYNAMIC

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Abstract

The aim of this thesis is to consider the effect of different traders' knowledge of the market system on wealth and security prices. To this purpose, I investigate with different level-k reasoning if the category of investors that has less information about the market is able in the long run to reach the truth. Heterogeneity arises because traders have different distribution assumptions about an informed trader's private signal. I construct a model based initially on a two-period exchange economy with complete markets and heterogeneous prior beliefs, in which it is possible identify two different categories of traders: the informed ones who know the probability distributions of random shocks and the uninformed who are not able to determine the probability. I use the binomial tree structure to analyze investor choices and to understand the influence that previous choices have on subsequent ones using, as variables for this analysis in the different time intervals, the price of the securities and the probabilities associated with the realizations of the random variables. Moreover, I realize subsequently a dynamic economic financial model in which I deepen the market selection under level-k reasoning optimization. In the end, I report some examples, generated using some codes in Matlab, to investigate the ability of the uninformed traders to make up for the lack of information and to reach the truth, increasing one's wealth.

Contents

Introduction

The heterogeneity of beliefs captures how the single trader interprets or has access to differing information sets. Do heterogeneous beliefs matter for Asset Pricing? How important are heterogeneous beliefs in the choice of the optimal portfolio by a generic investor? Can they condition fundamental and non-fundamental aspects of the economy? And, finally, How much do they influence optimal monetary policies? These are some of the questions that, over the years, economists have adressed and tried to solve by developing different economic models. Even though there are many obstacles that prevent the emergence of heterogeneity within macroeconomics and finance, since the paradigm of the representative agent is still preferred.

In this work, firstly, I develop a model based on a two-period exchange economy with complete markets. Moreover, I consider a financial dynamic economy model in which I study the market selection under level-k reasoning optimization. In particular, I consider two categories of agents that differ from each other when it comes to heterogeneous beliefs. One category is embodied by the so-called informed agents, or also called "fundamentalists", the other is represented by the uninformed ones, or "speculators". The main difference between the two types of agents is that the informed ones can identify the probability distributions of random shocks. Relying on more detailed information, they are aware of the value of the probability of the realization of a particular state of the world in the future. On the other hand, the uninformed traders are not able to recognize the value of the probability. Their goal is to try to extract the probabilities from the observed prices. The ability of "speculators" consists in the skill of being able to gather information from the price of the securities provided by "fundamentalists". Taking into consideration the level-k reasoning, it is possible that they themselves distort prices and must therefore also take into account their influence.
Are "speculators" able to learn? How quickly can "speculators" learn? How much does the performance of "speculators" in the market depend on how much they learn? How much does price change depend on this learning process?

The thesis is structured into five chapters. In the first chapter, I give a general introduction to the studies regarding heterogeneous beliefs and the influences they have on the asset pricing model, monetary policy and the creation of a portfolio by an investor. Furthermore, I explain the main differences with information asymmetry and the representative agent paradigm. In the second part of the work, taking as a reference point the model treated in the paper [15], I introduce a model based on a two-period exchange economy with complete markets and heterogeneous priors, characterized by uncertainty. More specifically, I analyse the optimal demands of the two different types of traders mentioned above, giving particular importance to general price equation associated with the probability, for any value of k. In chapter 3, I introduce the alpha function, which depends on the respective demands and wealth of both agents and on the price of the title. A detailed analysis of the relationship that exists between three of the four variables just mentioned: alpha, wealth and price, subsequently applying the final wealth to all the states of the world will also be considered. In the following chapter, the most relevant within the work, I try to understand how much the uninformed traders are able to learn from the information transmitted by the other agents. Here I analyze, within a system, the relationship that exists between the market equilibrium, the marginal wealth of both investors, the respective functions alpha, and the price. Following the imprint of the previous chapter, I represent the function of $Q_k(\pi)$ as a function of the marginal wealth of the two different agents. Finally, in the fifth and last chapter, I create some codes in Matlab that contain the equations present within the system previously mentioned, discussing the results obtained. To discuss the results, I use some graphs and tables to understand if the uninformed traders are able to discern information by the informed traders in the quickest way.

Chapter 1

The heterogeneous beliefs and asset pricing

1.1 Difference between heterogeneous prior beliefs and asymmetric information

Generally it is talked about asymmetric information in some specific financial arrangements or business deals. There is one party who has the so called competitive advantage because before entering the transaction in the first place it has more information than the other party.[1]
Making a step ahead I want to introduce the phenomenon of heterogeneous prior beliefs.
Heterogeneity arises because traders have different distribution assumptions about an informed trader's private signal; that is, the traders agree to disagree with the precision of the signal. [1]
Specifically a trader will be underconfident if his distribution of the signal is too loose, while it will be overconfident if it is too tight.
As Ricardo quoted in the paper [1], in a world in which there are some rational individuals that respectively hold different prior beliefs, a sender could have an impact on the behaviour of a receiver by controlling the informativeness of an experiment, which is represented by a public signal.
Always quoting Alonso and Camara, it is known that decision makers are influenced by individuals, who are able to change the information available. In fact, they are able to achieve and communicate hard evidence, or signal soft information.
An alternative way to affect decision makers' learning is given by the so called strategic experimentation (for example: a central bank shapes the informativeness of a market index observed by households, like inflation, by determining which information is collected and how to compute the index).

[1] https://corporatefinanceinstitute.com/resources/wealth-management/asymmetric-information/. Accessed February 1, 2023.

In this sense if the characteristics of the experiment is changed, also the education of the decision makers won't remain the same.
Going into more detail, I can take into consideration the model developed in the paper [12], which, in addition to the heterogeneous prior beliefs, it also takes into consideration the asymmetric information.
In this specific case the different distribution assumptions about a specific informed trader's private signal give raise to the heterogeneity, which creates a dilemma in trading for the informed trader beacuse of the co-existence with the asymmetric information.
The model created by Wang is able to solve the dilemma; in fact the informed trader trades on information first and on heterogeneous prior beliefs last.

1.2 Heterogeneous beliefs and the paradigm of the representative agent

One of the most relevant questions of recent years within economic finance is if heterogeneous beliefs of traders matter for asset pricing.
In fact, it is known that one of the characteristics that most financial markets have in common is the presence of heterogeneous beliefs among different investors, which, as demonstrated by various economic models, it is able to significantly influence the formation and dynamics of securities prices and plays a fundamental role in creating exchanges between the various traders in the financial market.
Within the economic literature, there are various models that deal with the influence of heterogeneous investor beliefs on fundamental and non-fundamental aspects of the economy.
I can cite different economists who analyzed in their papers the heterogeneous beliefs of traders through using different models.
Some examples are given by the earlier single- or multiple-period discrete-time works of Harrison and Kreps (1978), Varian (1985, 1989), Abel (1990), De Long et al. (1990), Harris and Raviv (1993), the continuous-time works of Williams (1977), Wang (1994), and the subsequent continuous-time developments of Detemple and Murthy (1994), Zapatero (1998), and Basak (2000).
Although various economists have dealt with the subject of heterogeneous investor beliefs in their writings with the intention of including heterogeneity in macroeconomics and finance, the paradigm of the representative agent [2] is still the leading structural approach to asset pricing, as reported

[2] An economic model is said to have a representative agent if all agents of the same type are identical. A particular model also has a representative agent when investors differ, but act in such a way that the sum of their choices is mathematically equivalent to the decision of one individual or many identical individuals.

in the paper [2]?".

This fact is explained by the economists just cited above identifying three main causes.

First of all, although much progress has been made over the years, in many situations it is still difficult to derive certain predictions in asset pricing models with the presence of heterogeneous investors.

To all this I must add the fact that there is a lack of data which regards heterogeneity in consumption, beliefs of agents and endowments. The last element underlined by the economists is the similarity between the formulations of heterogeneous agent models and the representative agent models.

Even though there are lots of elements which hamper the inclusion of heterogeneity into finance and macroeconomics, Anderson, Ghysels and Juergens tried to focus on the heterogeneity of beliefs as a factor which can capture how the single traders interpret different sets of information.

The three economists analyzed an asset pricing model with optimizing agents using as a proxy for the beliefs of the traders the publicy stated forecast of financial analysts. This is a new element if all the other papers which deal with the heterogeneous beliefs of traders are considered.

The main subject into the model is a representative agent whose beliefs are the composite of the ones of individual traders, even if the beliefs of the individual agents embody useful information.

The paper I am dealing with shows through an empirically testable theoretical prediction that heterogeneous beliefs influence the expected returns and explain a portion of the volatility.

1.3 The influence of heterogeneous beliefs on asset pricing

Going into more detail, I can also cite the model analyzed by Suleyman Basak into the paper [4] which takes as point of reference the subsequent continuous-time developments models treated by the economists Detemple and Murthy (1994), Zapatero (1998), and Basak (2000) just cited in the section above.

In the paper [7] the main focus is on a production economy with logarithmic agents which have heterogeneous beliefs about the unobservable growth of the production process.

On the other hand, Zapatero in his work [14] treated about the heterogeneous beliefs of investors regarding the growth of the aggregate endowment process.

The last point of reference for the realization of the model is a paper written by Basak himself [3], in which the author developed a similar pure-exchange economy with arbitrary utility function agents having heterogeneous beliefs about both nonfundamentals and the fundamental [3] aggregate

[3]https://www.investopedia.com/terms/f/fundamentals.asp. Accessed February 1, 2023.
Fundamentals include the basic qualitative and quantitative information that contributes to the financial or economic

endowment process.

Looking in particular to the volatility, I can take into consideration [14].

It is possible to notice how the heterogeneity of beliefs of the different type of investors in the financial market is the cause of the volatility of the equilibrium interest rate, which is called "market volatility".

The heterogeneity presents in particular an indirect and a direct effect on the interest rate: the first one concerns volatility of the distribution of wealth, or the shares of aggregate consumption, and it is a wealth-weighted average, or a consumption shares-weighted average of the conditional means; the second one concerns the volatility of the conditional means of the traders, which is tied to the common sensitivity of agents to news, update of beliefs.

As reported in the paper of Zapatero, the discussion of the model begins with the specific case of incomplete information into incomplete markets setting. At the beginning investors receive a specific signal, which embodies additional information and also in this case traders have heterogeneous beliefs concerning this information, which will affect the way in which they update their beliefs.

It is shown that the additional information will cause higher and higher local volatility and additionally it is known that, given the presence of more information, the traders will update their beliefs faster and will converge in the shortest time to the true value.

Moreover, Zapatero underlines that after some time volatility might be lower than it would have been without the signal.

Going into more detail and considering a complete market setting, the economist takes into account an additional security, which has the characteristics not to provide more information if compared to the signal presented in the additional information context.

Knowing that the focus of the paper cited above is on the market volatility,it can be noticed that Zapatero indicates that the additional security has an effect on the volatility which can be decomposed in two distinct parts: the first part in particular refers to the fact that an increase in the volatility of the share of wealth of the individuals causes an increase of the existing volatility and the second one concerns the effect of the change in wealth share in the covariance between information update and wealth share.

The covariance can be positive or negative.

If the security and the additional information are bundled together, the volatility goes up.

If I want to emphasize even more the importance of heterogeneous beliefs on asset pricing, I must refer to the paper [6].

As reported by the economists in the paper, thanks to the results of their empirical analysis, we are able to tell that the heterogeneous beliefs on monetary policy have important implications on

well-being of a company, security, or currency, and their subsequent financial valuation.
In business and economics, fundamentals represent the primary characteristics and financial data necessary to determine the stability and health of an asset. This data can include macroeconomic, or large-scale factors, and microeconomic, or small-scale factors to set a value on securities or businesses.

asset pricing because they make agents' consumptions and stochastic discount factors more volatile and correlated with money supply.
Moreover, they are able in certain situations to break the neutrality of money.
Furthermore, stock returns, inflation and bond yields become more volatile.

1.4 The effect of heterogeneous beliefs on the optimal portfolio choice

During the years lots of papers have treated about the question of how agents' different information is able to influence the value of the prices.
Way back in 1976, the economist Sanford Grossman in the paper [9] proved that the investors might not have been able to get a return on their investment in information if the competitive prices had revealed some information.
In his model Grossman introduced two distinct figures of traders: the informed traders which, thanks to the information, are able to take a position in the market and the uninformed ones which quite the opposite do not invest in information, but they are aware that the current prices reflect the information of the other category of traders.
The uninformed traders form their beliefs about future prices starting from the information held by the informed ones.
These two types of investors are the same that I take into account in the discussion of our model during the thesis.
Following the model devised by Grossman, Martin F. Hellwig expanded the model just cited in the paper [10].
He underlined that the price of equilibrium is not fully revealing, in the case in which the supply of the risky asset is random and showed that the heterogeneity in beliefs remains in equilibrium with agents drawing information from the market price as well as their own private sources.
Moreover, Hellwig highlighted as in large market the importance of the information available depends on the preferences of the traders and also that the equilibrium price can reflect only those parts of information which are shared by a large number of investors.
This is the proof that the market is a good aggregator of information when there is the presence of many agents that embody many independent sources of information.
Going into more detail, I take into consideration the paper [16], who analyses the behaviour of a specific investor who is confident that he can beat the market.
The model, which considers two distinct forms of heterogeneous beliefs of the investors (confidence

and relative optimism), shows the factors which are important for the agents in the choice of the optimal portfolio under heterogeneous beliefs.

Turning back to the two types of beliefs, we have to make a distinction between them: when uncertainty of both the market and the investor about their respective estimates of future dividend growth differ, we can talk about confidence.

On the other hand, the relative optimism characterizes a situation in which the assessment of future dividend growth by the single trader and the market differ.

Naturally, the two different systems will influence differently if compared between themselves the composition of the portfolios for the investors. Ziegler showed that the confidence is able to influence the hedging demand while the relative optimism exerts influence on the tangency component of the optimal asset demand.

An additional paper that has its main focus on the heterogeneous beliefs is Joseph T.Williams' paper [13].

In this paper, it is underlined how the traders are able to estimate with accuracy the unknown variances and covariances, but on the other hand they do not manage to know the means.

For this specific reason investors should adjust their optimal portfolios in order to hedge against the risk generated from unpredictable shifts in subjective estimates of unknown means.

Given the fact that the expected returns are unknown, the investors replace these with their conditional expectations, which embody the best estimate of the unknown expected returns linked to the history of asset prices.

Turning back to the model treated by Ziegler, I notice some differences if I compare this one with the model presented by Williams.

It considers a model in which a single investor has some expectations which are not the same as the ones of the market. Moreover, the asset price is totally determined by the market's assessment of future dividend growth, given the fact that the single trader has no power to influence the market prices.

Knowing this, the rational investor has to be aware of the fact that the market dynamics and the prices are "driven" by the market and for this reason his aim must be to anticipate the market's reaction to dividend news.

In previous years some economists constructed models in which the traders were able to exploit private information from the markets and vice versa. Instead, in this specific case neither the single investor nor the market tries to exploit information from each other. Furthermore there is no private information.

In the construction of his optimal portfolio the single agent takes as given some variables of the market: the beliefs, the evolution and the pricing function.

In particular, the investor has the aim of discovering the expected return of the risky assets. To achieve this he has to be able to adopt the market's valuation function together with his own

assessment of the dynamics of the dividend process.
Going into more detail, he has to hedge against both unfavorable random shift between his own beliefs and those of the market and random changes in the valuation of the securities by the market.

1.5 The influence of heterogeneous expectations on the optimal monetary policy

In addition to having influence on asset pricing models, volatility and expected returns, heterogeneous beliefs can play a primary role in the implementation of an optimal monetary policy.
The main purpose of the monetary policies of the banks is to have the control of inflationary pressure and in addition to reduce unemployment, only after having controlled inflation, and also to promote moderate interest rates in the long term.
I focus in particular on the paper [8] in which it can be observed that the monetary policy effectiveness is directly correlated with central bank's credibility.
In the paper, two points regarding the central bank's credibility are underlined. In particular, it is said that thanks to the adoption of the monetary policy instrument the central banks are able to reply to private sector changes.
Moreover, it is important to indicate that the behaviour of private sector depends on the ability of the economic traders to perceive the course of monetary policy and in addition a fundamental role is played by the expectations which are formed about future monetary policy.
In the model discussed by Orlando, as in other models of previous year, the monetary policy has been treated in a framework which embodies short run economic conditions.
Going into detail, the system considers the inflation rate and the output gap, which are both endogenous variables. In addition, these variables cannot be determined by any public authority, while they are linked with the economic conditions of the private sector.
On the other hand, public authorities and in particular the monetary authorities can have some influence on the time trajectory of the two states variables using the nominal interest rate.
The monetary authority faces an optimal control problem, which concerns the maximization of a specific function, in which the main variable of study is the low inflation rate, but importance is also given to the output gap.
The maximization of the optimal control problem just cited leads to an optimal interest rate rule, which is, to a time path for the interest rate in the future which is the one that best contributes to attain the policy goal.

In the model taken into consideration for any initial values assumed by the inflation rate and by the output gap, the optimal interest rate presents always a steady state result.
It is important to underline that the result just cited is also conditioned by expectations about the output gap and future inflation.
Going into more detail, I must underline that in the model considered the traders are called fundamentalists. This category of individuals believe that both the future inflation and the output gap will converge to a steady state value given some velocity of convergence parameters.
At this point, the figure of fundamentalist is flanked by the so called chartist, which are investors that create some expectations about the attention taking into account the past history if the inflation changes and not the convergence of the steady state mechanism as the other category of traders.
The main question is whether the optimal interest rate continues to be stable even if there is the introduction of this new type of investors.
The economist Orlando Gomes explained how in both cases the optimal interest rate continues to be stable even if traders continuously change the way in which they form their expectations.
All this happens in a bounded rationality way and for this reason the changes to the best performance expectation rule are not immediate and definitive.
There is only one difference between the case in which only the fundamentalists are considered and the other one. Indeed, in the second situation there are periods of large fluctuations in the inflation rate and for this specific reason the time price stability disappears for some time.
It is important to conclude by saying that these periods of instability are not persistent and the steady state result recovers quickly.

Chapter 2

A two-period financial economy model

2.1 Introduction of the model

In this model I consider a basic, two-period exchange economy with complete markets [1] and heterogeneous priors, characterized by uncertainty.

The state space for the future period is $\Sigma = \{1,2\}$. In this specific economy there is only one commodity and consumption takes place only in the second period.

I consider two different types of traders in the market. There are the "fundamentalists", that can also be called the informed, who are able to identify the probability distributions of random shocks; they know that the probability of the realization in the future of the state of the world $\sigma = 1$ is $(\pi, 1-\pi) \in \Delta$, where π is comprised between 0 and 1.

The second category is represented by the so called "speculators", or uninformed, who hold different prior beliefs if compared with the informed traders and they are not able to recognize the value of π, like the fundamentalists. In fact, they initially are able to understand only that π is a realization of the random variable Π, whose support is Δ.

The super-index a which belongs to {F,S} is used to stand for the different types of agents.

Both "fundamentalists" and "speculators" differ in their future wealth; if an agent of type a is considered, in state σ, it will be endowed with a "wealth" ω^a_σ.

It is important to underline that the final wealth is the one I will consider the most and it will be represented by Ω for the informed traders and by ω for the uninformed.

[1] https://ebrary.net/7102/business-finance/what-meant-complete-and-incomplete-markets. Accessed February 1, 2023.

A complete market is the one in which the number of linearly indipendent securities is the same of the states of the world in the future. Making an example, I can consider the binomial model (the binomial option pricing model uses an iterative procedure, allowing for the specification of nodes, or points in time, during the time span between the valuation date and the option's expiration date) in which at the next time step there are boht two states of the world and two securities,in particular cash and stock.

The agents, who are characterized by respective masses μ^F, which stays for the fundamentalists, and μ^S, which is attributed to the "speculators", have expected-utility preferences with type-dependent Bernoulli utility υ^a : R → R.

Given the fact that y^a_σ represents the holdings of the security paid in state $\sigma = 1$ by the agents of type a, the ex-ante utility of a fundamentalist is given by the subsequent expression:

$$\pi\upsilon^F(\omega^F_1 + y^F_1) + (1 - \pi)\upsilon^F(\omega^F_2 + y^F_2). \; \mathbf{(1)}$$

On the other hand, the ex-ante utility of a speculator, who is able to discern or receive information J ⊆ ∆, is:

$$E[\Pi\upsilon^S(\omega^S_1 + y^S_1) + (1 - \Pi)\upsilon^S(\omega^S_2 + y^S_2) \mid J], \; \mathbf{(2)}$$

where E represents the conditional mean with respect to J.

Naturally, the partial information J, used by the "speculators", will depend on the ability of these traders to understand the surrounding market.

As hypothesis, at the beginning of the model I state that I consider a two-period exchange economy with uncertainty, so I have to take into consideration two different prices of the security: the first at state σ=1 and the second at σ=2, respectively represented by two vectors: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

I assume that the price of the security for σ=2 is equal to 1; on the other hand, for σ=1, I denote by q the price.

Taking into consideration this information, the only constraint that an agent deals with when he has to choose her portfolio is:

$$qy^a_1 + y^a_2 = 0. \; \mathbf{(3)}$$

Using the last equation, I can solve for the holdings of the second security, and I can reconsider the expressions representing the ex-ante utility both of a "fundamentalist" and of a "speculator" considering the first security only.

Taking into account all these elements I can state that the optimal demands for the first security are:

- $Y^F(q; \pi)$ for the "fundamentalists".
- $Y^S(q; J)$ for the "speculators".

Considering the market clearing, which is the process by which the supply of something that's traded is equated to the demand, so that there's no leftover supply or demand, it is possible to report the following equation:

$$\mu^F Y^F(q; \pi) + \mu^S Y^S(q; J) = 0, \; \mathbf{(4)}$$

where the first term represents the mass of fundamentalists multiplied by the respective optimal demand and the second term the mass of the other type of traders multiplied by the own optimal demand.

Note that in equation **(3)** y^a, in certain situations, could assume negative values.

It is possible to state this knowing the basic laws of supply and demand [2]:

- Prices fall when supply increases and demand remains constant;
- Prices fall when demand decreases and supply remains constant;
- Prices rise when supply decreases and demand remains constant;
- Prices rise when demand increases and supply remains constant.

2.2 The ability of "speculators" to learn

The concept of rational expectations equilibrium states that the "speculators" know the equilibrium and exploit it to figure out information they lack; this concept is so similar to the approach of Nash equilibrium in game theory [3].

This last concept can be applied to the model I am describing; starting from the fact that the "fundamentalists" are able to understand the information, the "speculators" should be able to learn the mapping from fundamental to prices through observation, in order to reach the level of knowledge of the other category of trader.

The possibility for one trader (in this model recognized in the figure of the uninformed trader) to make inferences from market prices, that I have identified as q, about the information J possessed by other traders (represented by the categories of the informed traders), rests upon his having a "model" or "expectations" of how equilibrium prices are determined, specifically how equilibrium prices are related to the information which is at the beginning owned by the "fundamentalists".

The relationship between prices and the information J is endogenous [4] to the market system. I am able to speak about REE (rational expectations equilibrium) knowing the fact that if uninformed traders have the opportunity to compare the models created by themselves with the results of the operation of the market, then there must exists a suitable equilibrium concept, which would

[2]https://www.netsuite.com/portal/resource/articles/erp/law-of-supply-demand.shtml. Accessed February 1, 2023.

[3]It is a concept that determines the optimal solution in a non-cooperative game, which is a game with competition between the individual players, in which each component lacks any incentive to change his/her own initial strategy. Under the Nash equilibrium, assuming the other players also keep their strategies unchanged, players do not gain anything from deviating from their initially chosen strategy.

[4]An endogenous factor in economics is something that is explained or calculated from within the model being studied.

require that their models not be obviously controverted by the observations of the market.

The particular rational expectations equilibrium that an investor would obtain depends upon the investor's model or expectations of the relationship between investor's initial information and equilibrium prices.

Going into detail, I can refer to the paper [11], in which with the rational expectations equilibrium all information that is possessed by individual traders is made known to others.

Watching carefully, I can state that this property of equilibrium might cast doubt on the incentives for an uninformed traders, which wants to discern information in the market, to obtain information prior to entering the market, knowing that he could count on other "speculators" achieving the same information, which later would be revealed thanks to market prices, identified in the model with q.

Following the reasoning, it is possible to affirm that if each uninformed trader reason in this way, then no trader would obtain prior information, and so there would be no prior information for market prices to reveal.

Moreover there is the possibility that a particular equilibrium could not exist. A specific example regarding the non-existence of a specific equilibrium is given by the situation in which a trader has perfect knowledge of the relationship between equilibrium prices and initial information, so in this case, even with "standard" assumptions about preferences,endowments, the equilibrium does not need to exist.

I consider $\overline{Q} : \Delta \to \mathrm{R}$ the function of the rational expectations equilibrium; in this way, for all π I can substitute, in equation (4), q with $\overline{Q}(\pi)$. On the other hand, the information J, which the agents ("speculators") are able to find out, is equal to:

$$\mathrm{J} = \overline{Q}^{-1}(q). \ \mathbf{(5)}$$

The equilibrium will be fully revealing if and only if:

$$\overline{Q}^{-1}(\overline{Q}(\pi)) = \pi, \ \mathbf{(6)}$$

for all π.

Equation **(5)** requires that the categories of "speculators" discern the totality of the information, which is transmitted by prices correctly at equilibrium.

Further equation **(6)** shows how prices transmit all of the information available to the class of fundamentalists.

2.3 Different types of "speculators"

I distinguish between different types of "speculators", that differ between each other because of the capability to comprehend the information received by the "fundamentalists".

The so called clueless "speculators" are not able to understand that the price q depends on the information transmitted by the other category of traders. So, the demand of "level-0 understanding" traders can be defined as $Y_0^S(q) = Y^S(q; \Delta)$.

Taking into account that they use unconditional expectations to solve portfolio problem, the ex-ante utility is:

$$Y_0^S(q) = argmax[E[\Pi]\nu(\omega_1^S + y) + (1 - E[\Pi])\nu(\omega_2^S - qy)]$$

In this way equation **(4)** can be rewritten as:

$$\mu^F Y^F(Q_0(\pi); \pi) + \mu^S Y_0^S(Q_0(\pi)) = 0,$$

for all π, taking into consideration that $Q_0 : \Delta \to R$.

On the other hand, I can take into account that a specific "speculator" is able to understand the strong correlation between the prices and the information transmitted by the "fundamentalists", clarified by the price function $Q_{k-1} : \Delta \to R$.

In this specific case, the trader has a "level-k understanding" if, at prices q, she adopts information $J = Q_{k-1}^{-1}(q)$ to construct the optimal portfolio.

So, the optimal demand for the first security can be defined as:

$$Y_k^S(q) = Y^S(q; Q_{k-1}^{-1}(q)),$$

where the two terms between the parenthesis represent Π.

The new pricing functions is the following:

$$\mu^F Y^F(Q_k(\pi); \pi) + \mu^S Y_k^S(Q_k(\pi)) = 0.$$

The category of "speculators" is not able to comprehend that the usage of the function Q_{k-1} transforms the equilibrium prices at each value of π, that leads to the creation of a new map Q_k.[5]

2.4 An example with log preferences

Initially, I have supposed that both the "fundamentalists" and the "speculators" present the same Bernoulli utility $v(x) = ln(x)$.

[5] Specifically, models of level-k reasoning assume that individuals have an exogenous type which corresponds to the number of rounds of iterated reasoning they perform. In these models, a level-0 individual represents a non-strategic type that follows some exogenously specified behavior, while a level-1 individual best responds to level-0, and so forth.

Moreover, I consider that in states $\sigma = 1$ and $\sigma = 2$ the following relations are true: $\omega_\sigma^F = \Omega > \omega = \omega_\sigma^S$ and $\mu^F = \mu^S = 1$.
Lastly, I obtain that $E(\Pi) = 1/2$ and $\Delta = [\pi, \overline{\pi}] \subset (\omega^2/2\Omega^2, 1 - \omega^2/2\Omega^2)$.
Taking into consideration the constraint that an agent faces when he has to choose the optimal portfolio (equation **(3)**), I can reconsider the first two equations only considering the first security. So, it is possible to write:

$$\pi \upsilon^F(\omega_1^F + y^F) + (1-\pi)\upsilon^F(\omega_2^F - qy^F), \textbf{ (1')}$$

for the "fundamentalists" and

$$E[\Pi \upsilon^S(\omega_1^S + y^S) + (1-\pi)\upsilon^S(\omega_2^S - qy^S) \mid J], \textbf{ (2')}$$

for the "speculators".
Starting from the categories of the "speculators", I can derive the individual demand $Y^F(q;\pi)$. This is obtained deriving the expression **(1')** with respect to y_1^F and then maximizing it by setting it equal to zero.
Considering the initial hypothesis that $\upsilon(x) = ln(x)$, the following equation results:

$$\frac{d\textbf{(1')}}{dY^F} = 0 \rightarrow \pi(\frac{1}{\omega_1^F + Y^F}) + (1-\pi)(\frac{1}{\omega_2^F - qY^F})(-q) = 0.$$

Now, the individual demand (Y^F) has to be derived. Below, the mathematical steps to obtain it are presented:

$$\pi(\frac{1}{\omega_1^F + Y^F}) = (1-\pi)(\frac{1}{\omega_2^F - qY^F})(q);$$

$$\pi(\omega_2^F - qY^F) = q(1-\pi)(\omega_1^F + Y^F);$$

$$\pi\omega_2^F - \pi qY^F = q(1-\pi)Y^F + q(1-\pi)\omega_1^F;$$

$$Y^F[q(1-\pi) + \pi q] = \pi\omega_2^F - q(1-\pi)\omega_1^F;$$

$$Y^F q = \pi\omega_2^F - q(1-\pi)\omega_1^F;$$

Finally, the individual demand for the "fundamentalists" will be equal to:

$$Y^F(q;\pi) = \frac{1}{q}[\pi\omega_2^F - q(1-\pi)\omega_1^F],$$

which can also be presented as:

$$Y^F(q;\pi) = \frac{\pi}{q}(q\omega_1^F + \omega_2^F) - \omega_1^F. \textbf{ (7)}$$

Knowing the hypothesis that $\omega_\sigma^F = \Omega > \omega = \omega_\sigma^S$ equation **(7)** can be rewritten as:

$$Y^F(q;\pi) = (\pi\frac{(q+1)}{q} - 1)\Omega. \textbf{ (7)}$$

Using the same procedure even for the categories of "speculators", I can obtain the individual demand also for this type of traders:

$$Y^S(q;\pi) = \frac{\pi}{q}(q\omega_1^S + \omega_2^S) - \omega_1^S, \textbf{ (8)}$$

which is equal to:

$$Y^S(q;\pi) = (\pi\frac{(q+1)}{q} - 1)\omega. \textbf{ (8)}$$

Using the assumption that $\mu^F = \mu^S = 1$, and substituting it into equation **(4)** we can solve for q which is equal to $\overline{Q}(\pi)$.

Below, I report the mathematical steps to obtain the final equation:

$$\mu^F Y^F(q;\pi) + \mu^S Y^S(q;J) = 0;$$

$$\frac{\pi}{q}(q\omega_1^F + \omega_2^F) - \omega_1^F + \frac{\pi}{q}(q\omega_1^S + \omega_2^S) - \omega_1^S = 0;$$

$$\pi\omega_1^F + \frac{\pi}{q}\omega_2^F - \omega_1^F + \pi\omega_1^S + \frac{\pi}{q}\omega_2^S - \omega_1^S = 0;$$

$$\frac{\pi}{q}(\omega_2^F + \omega_2^S) = \omega_1^F + \omega_1^S - \pi(\omega_1^F + \omega_1^S);$$

$$q(1-\pi)(\omega_1^F + \omega_1^S) = \pi(\omega_2^F + \omega_2^S);$$

in the end I obtain:

$$\overline{Q}(\pi) = \frac{\pi}{1-\pi}\frac{\omega_2^F + \omega_2^S}{\omega_1^F + \omega_1^S}. \textbf{ (9)}$$

For hypothesis, like I said previously, certain conditions are fixed. In particular that $\omega_\sigma^F = \Omega > \omega = \omega_\sigma^S$.

Given this condition I can state that $\omega_1^F + \omega_1^S = \omega_2^F + \omega_2^S$ and for this reason equation **(9)** can be semplified as:

$$\overline{Q}(\pi) = \frac{\pi}{1-\pi}. \textbf{ (9)}$$

Moreover, I can construct the equation above because for hypothesis I state that there is no aggregate risk.

Aggregate risk is defined as the total amount of an institution's exposure to foreign exchange counterparty risk deriving from a single client. For example, the foreign exchange contracts, (spot and forward), involve a counterparty who is responsible for holding up their side of an agreement. Moreover, an institution may suffer losses if it has made too many agreements with a specific client, who in the future turns out to be insolvent. The aim of the institution, to prevent big losses, is to diversify the portfolio.

[6] The summation of $\omega_1^F + \omega_1^S$ can be also represented as $\overline{\omega_1}$ and $\omega_2^F + \omega_2^S$ as $\overline{\omega_2}$. Naturally if there was aggregate risk $\omega_1^F + \omega_1^S$ will not be equal to $\omega_2^F + \omega_2^S$ and I could not explicit $\overline{Q}(\pi)$ as $\frac{\pi}{1-\pi}\frac{\omega_2^F + \omega_2^S}{\omega_1^F + \omega_1^S}$.

Given the fact that $\overline{Q}$ is bijective, I confirm that the REE is fully revealing. Its inverse, which I represent as $\overline{\Pi}$ is equal to:

[6] https://www.investopedia.com/terms/a/aggregate-risk.asp. Accessed February 1, 2023.

$$\overline{\Pi}(q) = \frac{q(\omega_1^F + \omega_1^S)}{q(\omega_1^F + \omega_1^S) + \omega_2^F + \omega_2^S}, \ \mathbf{(10)}$$

which for the same reason of equation **(9)** can be written as:

$$\overline{\Pi}(q) = \frac{q}{q+1}. \ \mathbf{(10)}$$

Shown below the steps to achieve the final equation starting from the hypothesis that $\mu^F = \mu^S = 1$, the same used to obtain $\overline{Q}(\pi)$:

$$\mu^F Y^F(q;\pi) + \mu^S Y^S(q;J) = 0;$$

$$\frac{\pi}{q}(q\omega_1^F + \omega_2^F) - \omega_1^F + \frac{\pi}{q}(q\omega_1^S + \omega_2^S) - \omega_1^S = 0;$$

$$\pi\omega_1^F + \frac{\pi}{q}\omega_2^F - \omega_1^F + \pi\omega_1^S + \frac{\pi}{q}\omega_2^S - \omega_1^S = 0;$$

$$\pi(\omega_1^F + \frac{\omega_2^F}{q} + \omega_1^S + \frac{\omega_2^S}{q}) = \omega_1^F + \omega_1^S;$$

$$\pi(\frac{\omega_1^F q + \omega_2^F + \omega_1^S q + \omega_2^S}{q}) = \omega_1^F + \omega_1^S;$$

$$\overline{\Pi}(q) = \frac{q(\omega_1^F + \omega_1^S)}{q(\omega_1^F + \omega_1^S) + \omega_2^F + \omega_2^S}. \ \mathbf{(10)}$$

I have demonstrated that the inverse of $\overline{Q}$ is $\overline{\Pi}$.

2.5 Learning of the "speculators"

In addition to the assumptions made previously, I can assume that mapping Q_{k-1} is bijective. Using its inverse function, it is possible to pin down the beliefs of level-k "speculators" upon observation of price $q : \hat{\Pi}_k(q) = Q_{k-1}^{-1}(q)$.

Given this information, and knowing that the "speculators" try to infer the probability distributions of random shocks from asset prices, I can rewrite equation **(8)** as:

$$Y_k^S(q) = \frac{\hat{\Pi}_k(q)}{q}(q\omega_1^S + \omega_2^S) - \omega_1^S,$$

while for the other traders (fundamentalists) the individual demand remains the same as the equation **(7)**:

$$Y^F(q;\pi) = \frac{\pi}{q}(q\omega_1^F + \omega_2^F) - \omega_1^F.$$

I define the price q of the security as $Q_k(\pi)$, which is the next pricing function. Since $\mu^F = \mu^S = 1$, equation **(4)** can be redifined as:

$$Y^F(Q_k(\pi);\pi) + Y_k^S(Q_k(\pi)) = 0;$$

$$Y^F(Q_k(\pi);\pi) = -Y_k^S(Q_k(\pi)).$$

Substituting into the last equation the demands of the two classes of investors and considering the new pricing function Q_k, I will have the following expression:

$$\frac{\pi}{Q_k(\pi)}(Q_k(\pi)\omega_1^F + \omega_2^F) - \omega_1^F = \omega_1^S - \frac{\hat{\Pi}_k(Q_k(\pi))}{Q_k(\pi)}(Q_k(\pi)\omega_1^S + \omega_2^S). \textbf{ (11)}$$

Knowing the assumption that $\omega_1^F = \omega_2^F = \Omega > \omega = \omega_1^S = \omega_2^S$, like I have made for equation **(7)** and equation **(8)**, I can rewrite the expression **(11)** as:

$$[\pi\frac{Q_k(\pi)+1}{Q_k(\pi)} - 1]\Omega = [1 - \hat{\Pi}_k(Q_k(\pi))\frac{Q_k(\pi)+1}{Q_k(\pi)}]\omega. \textbf{ (12)}$$

Considering that $\hat{\Pi}_0(q) = E(\Pi) = 1/2$ from the equation above, I can derive the formula for $Q_0(\pi)$. I report the mathematical steps to obtain the ultimate expression:

$$[\pi(1 + \frac{1}{Q_0(\pi)}) - 1]\Omega = [1 - \frac{1}{2}(1 + \frac{1}{Q_0(\pi)})]\omega;$$

$$\pi\Omega + \frac{\pi}{Q_0(\pi)}\Omega - \Omega = \omega - \frac{\omega}{2} - \frac{\omega}{2Q_0(\pi)};$$

$$\frac{2Q_0(\pi)\pi\Omega + 2\pi\Omega - 2Q_0(\pi)\Omega - Q_0(\pi)\omega + \omega}{2Q_0(\pi)} = 0;$$

$$Q_0(\pi)(2\Omega + \omega - 2\pi\Omega) = 2\pi\Omega + \omega;$$

In the end, the equation will be:

$$Q_0(\pi) = \frac{2\pi\Omega + \omega}{2(1-\pi)\Omega + \omega}.$$

Adopting mathematical induction over k:

$$Q_k(\pi) = \frac{2\pi\Omega^{k+1} + (-1)^k\omega^{k+1}}{2(1-\pi)\Omega^{k+1} + (-1)^k\omega^{k+1}}, \textbf{ (13)}$$

for any value of $\pi \in \Delta$.

It is possible to notice that if I substitute k = 0 into equation **(13)**, I obtain $Q_0(\pi) = \frac{2\pi\Omega + \omega}{2(1-\pi)\Omega + \omega}$, which is the same equation that we have demonstrated before.

It is possible to conclude that this mapping is bijective.

2.6 Asymptotics of learning

The last equation I have realized, using the mathematical induction [7], can also be written as:

$$Q_k(\pi) = \frac{2\pi + (-1)^k(\frac{\omega}{\Omega})^{k+1}}{2(1-\pi) + (-1)^k(\frac{\omega}{\Omega})^{k+1}}.$$

Taking into account the initial hypothesis that $\Omega > \omega > 0$, I can demonstrate that for all $\pi \in \Delta$, $Q_k(\pi) \to \overline{Q}(\pi)$. I prove it taking into consideration the difference between $Q_k(\pi)$ (equation **(13)**) and $\overline{Q}(\pi)$ (equation **(9)**). Below the mathematical steps are reported:

$$Q_k(\pi) - \overline{Q}(\pi) = \frac{2\pi + (-1)^k(\frac{\omega}{\Omega})^{k+1}}{2(1-\pi) + (-1)^k(\frac{\omega}{\Omega})^{k+1}} - \frac{\pi}{1-\pi};$$

$$Q_k(\pi) - \overline{Q}(\pi) = \frac{2\pi(1-\pi) + (-1)^k(\frac{\omega}{\Omega})^{k+1}(1-\pi) - 2\pi(1-\pi) - (-1)^k(\frac{\omega}{\Omega})^{k+1}\pi}{2(1-\pi)^2 + (-1)^k(1-\pi)(\frac{\omega}{\Omega})^{k+1}},$$

which takes to the final equation:

$$Q_k(\pi) - \overline{Q}(\pi) = \frac{(-1)^k(\frac{\omega}{\Omega})^{k+1}(1-2\pi)}{2(1-\pi)^2 + (-1)^k(1-\pi)(\frac{\omega}{\Omega})^{k+1}}. \quad \textbf{(14)}$$

I can state that for $k \geq 2$ the difference between the two terms is continuous on π over Δ, and for this reason $Q_k \to \overline{Q}$ is not just pointwise but uniformly. Because of the term $(-1)^k$ on the right hand-side of equation **(14)**, the sign of the difference $Q_k(\pi) - \overline{Q}(\pi)$ will oscillate between successive levels of understanding; the difference will be null only when π will be equal to $E(\Pi)$, in particular when k will tend to infinite.

2.7 The concept of background risk

Previously, I have demonstrated that equation **(12)** can help to solve equation **(11)**.
If I suppose that Q_k is bijective and we continue to assume that both categories of traders have logarithmic Bernoulli utilities, the inverse equation of **(11)** will have the following form:

$$\frac{\hat{\Pi}_{k+1}(q)}{q}(q\omega_1^F + \omega_2^F) - \omega_1^F = \omega_1^S - \frac{\hat{\Pi}_k(q)}{q}(q\omega_1^S + \omega_2^S). \quad \textbf{(15)}$$

[7] https://www.tutorialspoint.com/discrete-mathematics/discrete-mathematical-induction.htm. Accessed February 1, 2023.
Mathematical induction is a concept that helps to prove mathematical results and theorems for all natural numbers. The principle of mathematical induction is a specific technique that is used to prove certain statements in algebra which are formulated in terms of n, where n is a natural number. Any mathematical statement, expression is proved based on the premise that it is true for n = 1, n = k, and then it is proved for n = k + 1.

Starting from this last equation, I am able to derive $\hat{\Pi}_k(q)$, which represents the difference equation, that at given prices (q), guides the beliefs of "speculators" as their level of understanding k evolves. So, I will have:

$$\hat{\Pi}_k(q)(q\omega_1^F + \omega_2^F) = (\omega_1^S + \omega_2^S)q - \hat{\Pi}_{k-1}(q)(q\omega_1^S + \omega_2^S);$$

$$\hat{\Pi}_k(q) = \frac{(\omega_1^S + \omega_2^S)q}{q\omega_1^F + \omega_2^F} - \frac{q\omega_1^S + \omega_2^S}{q\omega_1^F + \omega_2^F}\hat{\Pi}_{k-1}(q),$$

from which I can obtain the final equation:

$$\hat{\Pi}_k(q) = min[max[\frac{(\omega_1^F + \omega_2^S)q}{q\omega_1^F + \omega_2^F} - \frac{q\omega_1^S + \omega_2^S}{q\omega_1^F + \omega_2^F}\hat{\Pi}_{k-1}(q), \underline{\pi}], \overline{\pi}], \textbf{(16)}$$

where $\underline{\pi}$ represents the lower bound, quite the opposite $\overline{\pi}$ embodies the upper bound. As explained before, π shows the probability of the realization of the state of the world σ in the future.
So, the equation **(16)** tells that $\hat{\Pi}_k(q)$ depends on $\hat{\Pi}_{k-1}(q)$ and should be included between the lower bound and the upper bound of the probability π. Looking at equation **(16)**, there is a convergence of each level-k belief to the rational expectations beliefs. This is true even if such convergence is non-monotonic [8].
In the end, taking into consideration the assumption that the nominal wealth of "speculators" is lower if compared to the one of the categories of fundamentalists, I can state the following inequality:

$$\frac{q\omega_1^S + \omega_2^S}{q\omega_1^F + \omega_2^F} < 1,$$

for any value that q assumes. Starting from this inequality, I conclude that:

$$\hat{\Pi}_k(q) \rightarrow \frac{q(\omega_1^F + \omega_2^S)}{q(\omega_1^F + \omega_2^S) + \omega_2^F + \omega_2^S} = \overline{\Pi}(q),$$

where $\overline{\Pi}(q)$ is represented by equation **(10)**.

[8]The monotone convergence theorem is any of a number of related theorems proving the convergence of monotonic sequences (sequences that are non-decreasing or non-increasing) that are also bounded. The theorems state that if a sequence is increasing and bounded above by a supremum (which is the least upper bound), then the sequence will converge to it; in the same way, if a sequence is decreasing and is bounded below by an infimum (which is the greatest lower bound), it will converge to the infimum.

Chapter 3

A financial dynamic economy model

3.1 Introduction of the function α

Previously, I have stated that Y^F and Y^S represent respectively the optimal demand for informed and uninformed traders.
Now, I want to introduce the function α, which is positively correlated with the demand Y^a.
α is given by the subsequent formula for the informed traders:

$$\alpha = \frac{Yq}{q\Omega + (1-q)\Omega}. \quad \textbf{(17)}$$

Furthermore, I can present the same function calculated for the categories of the uninformed traders, whose final value is given by ω:

$$\alpha = \frac{Yq}{qw + (1-q)w}. \quad \textbf{(18)}$$

It can be noticed that α, as mentioned before, is positively correlated to the demand and also to the price of security q. On the other hand, it is negatively correlated with the final wealth, which is represented by Ω for the categories of "fundamentalists" and by ω for the other category of traders.
In both cases, as it can be seen from the formulas, the price of securities at state $\sigma = 1$ and $\sigma = 2$ are respectively q and (1 - q), differently from what I have stated at the beginning of the model, in which I have assigned a price q for the security in state $\sigma = 1$ and a price equal to 1, which characterizes the security at state $\sigma = 2$.
Using the inverse formula, I am able to derive the value of Y as a function of the value of α as follows:

$$Y = \frac{\alpha(q\Omega + (1-q)\Omega)}{q},$$

and simplifying the member $q\Omega$ into the parenthesis obtain:

$$Y = \frac{\alpha\Omega}{q}. \textbf{(19)}$$

Naturally, I can do the same for the categories of "speculators" obtaining the following equation:

$$Y = \frac{\alpha w}{q} \textbf{ (20)}$$

3.2 The ex-ante utilities applied to σ_T

In the previous chapter, I have taken into account the ex-ante utility for the two separate categories of traders, only considering two states of the world, in which I have also considered two different prices for the security.

As it will be seen later, I will change the two prices of the security, assigning them different values if compared with the ones considered in the previous chapter.

Now, I take a step forward considering in the model not only the two states, $\sigma = 1$ and $\sigma = 2$, but all the states in the world until time T, in which finally I have Ω and ω, the final wealth for the two classes of traders.

σ_T is equal to σ_1, σ_2, σ_3, ..., $\sigma_{(T-1)}$.

Taking into consideration all the states of the world, I can represent the ex-ante utility both for the informed and the uninformed traders, as made previously with equation **(1)** and **(2)**.

Differently from what has been done before, I use the summation to show the two functions.

Starting from the informed traders, the following expression is the following:

$$\textstyle\sum_{\sigma_1}^{\sigma_T} \pi\upsilon^F(\omega_T^F + y_T^F) + (1-\pi)\upsilon^F(\omega_T^F + y_T^F).\textbf{(21)}$$

As it can be observed from the equation above, the summation goes from $\sigma = 1$ until $\sigma = T$ and all the variables within the expression no longer have the values 1 and 2 as subscripts, but T.

It represents the final state of the world in which traders will own the final wealth, respectively Ω (for the informed) and σ (for the uninformed).

In the same way, always using the ambiguity neutrality model and the conditional mean with respect to J, which is the quantity of information that the informed are able to discern unlike the other category of traders, I can calculate also the ex-ante utility of a "speculator" similarly to equation **(2)**.

The expression follows:

$$\textstyle\sum_{\sigma_1}^{\sigma_T} E[\Pi(\sigma_T)\upsilon^S(\omega_T^S + y_T^S) + (1-\pi)\upsilon^S(\omega_T^S + y_T^S) \mid J]. \textbf{ (22)}$$

As explained for the ex-ante utility of the "fundamentalists", even in this case there is the summation that goes from $\sigma = 1$ to $\sigma = T$ of the expectations of the ex-ante utility conditional to the past.

3.3 A comparison between α and π

After having introduced the function α, which as I have shown is positively correlated with the demand Y^a, I want to compare between each other the same α and π.

π, as discussed before, represents the probability of the realization of a specific state of the world σ.

The first step to make a comparison between the two variables is to take into consideration the equation **(7)**, that is the individual demand for the categories of the informed traders.

Specifically, it is $Y^F(q;\pi) = \frac{\pi}{q}(q\omega_1^F + \omega_2^F) - \omega_1^F$.

For hypothesis, I can state that in this specific case I do not consider the background risk, so I can affirm that $\omega_1^F = \omega_2^F = \omega^F$.

Following this, I can rewrite equation **(7)** as follows:

$$Y^F(q;\pi) = \frac{\pi}{q}(q\omega^F + \omega^F) - \omega^F.$$

Bringing the value ω^F to the left of the equal, we have that:

$$Y^F(q;\pi) + \omega^F = \frac{\pi}{q}(q\omega^F + \omega^F).$$

To be more specific, always considering the categories of the "fundamentalists", I can take into account time $t = 1$ and the state of world $\sigma = 1$.

Thus, in $w_{t=1}(\sigma = 1)$ I can show the equation below:

$$Y^F(q;\pi) + \omega_1^F = \frac{\pi}{q}(q\omega_1^F + \omega_2^F),$$

where the summation inside the last parenthesis can be rewritten as w_0;

$$Y^F(q;\pi) + \omega_1^F = \frac{\pi}{q}w_0. \textbf{ (21)}$$

Turning back to the ex-ante utility of the informed traders embodied in equation **(1)**, I can also represent the same as a function of α, treated in the section "Introduction of the function α".

I will get:

$$\pi\upsilon(\frac{\alpha w_0}{q}) + (1-\pi)\upsilon(\frac{(1-\alpha)w_0}{1}), \textbf{ (22)}$$

where $\frac{\alpha w_0}{q}$ is equal to Y.

In the same way, the following is another representation of the same function, but in this specific case without considering the α size.

$$\pi\upsilon(Y^F + \omega_1^F) + (1-\pi)\upsilon(Y^F + \omega_2^F). \textbf{ (23)}$$

I underline that the purpose of all these steps is to see if there is a correlation between α and π.
I consider the members into the first parenthesis of both equation **(22)** and equation **(23)**, so respectively $\frac{\alpha w_0}{q}$ and $Y^F + \omega_1^F$.
Looking to equation **(21)** I am able to rewrite $Y^F + \omega_1^F$ as $\frac{\pi}{q} w_0 = \frac{\pi}{q}(q\omega_1^F + \omega_2^F)$.
After doing this, I have all the elements to compare between them the two variables I am interested in.
I show that:

$$\frac{\alpha w_0}{q} = \frac{\pi}{q} w_0;$$

simplifying the terms w_0 and q I obtain that:

$$\alpha = \pi. \; \textbf{(24)}$$

I have demonstrated that the two variables assume the same value.

3.4 An analysis of wealth in time $t = 0$ and $t = 1$

At the beginning of this section, I need to make two assumptions in order to move forward in the discussion of my model. In particular, I must give a detailed definition of both σ_T and σ^T.
It follows that:

- σ_t is defined as the state of the world realized in t which belongs to the summation of all the t, $\sum^t$;
- σ^t, on the other hand, is described as the "history" of all the states up to t, with t belonging to $\sum^t$.

Going specifically, and referring in particular to the summation until time t, I have to make a distinction between two different cases: the first in which there is $\sum(1, 2)$, so in this case I can state that the summation until time T has cardinality [1] 2^T.
On other hand, in the second case having that $\sum(1, ..., N)$, the summation until T has cardinality N^T.
Going back and talking about the wealth of investors, in particular referring to the σ_T states and the previous one (σ_{T-1}), I can present it as a function of both price and alpha, each calculated at time $T - 1$ in state σ_T.
In particular, I can show that:

[1] Cardinality is a mathematical term that refers to the number of elements in a given set.

$$w(\sigma_{T-1}, \sigma T) = \frac{\alpha_{(T-1)}(\sigma_T)\omega_{(T-1)}(\sigma_{T-1})}{q_{(T-1)}(\sigma_T)}. \textbf{ (25)}$$

It is possible to notice that $w(\sigma_{T-1}, \sigma T)$ is directly proportional to $\alpha_{(T-1)}$ calculated in σ_T and to $\omega_{(T-1)}$ calculated in the previous state of the world σ_{T-1}, and inversely proportional to the price q_{T-1} calculated in σ_T.

Going into detail, and in particular considering the two states of the world ($\sigma = 1$ and $\sigma = 2$), I can focus on the denominator, that represents the price of the securities in different states.

If I refer to the two states of the world, which I have taken into consideration previously, I obtain two distinct values: $q_{(T-1)}(\sigma_{T-1})$ for $\sigma_T = 1$ and 1 for $\sigma_T = 2$.

As said before, the most important wealth that I take into account is the final wealth, which is known and it is embodied by Ω for the categories of the informed traders and by ω for the uninformed traders.

By following the binomial tree pattern and going backwards in the calculation of stock prices and investor wealth, I am able to calculate the value of ω_T and Ω_T for the two categories of traders.

First of all, I dwell on the term $t = 0$ and in particular on the relationship between α_0 and q_0.

Introducing the probability π of the realization of the state of the world σ_T and the logarithmic function, using the summation of all the states σ_T I can show equation **(26)** that is:

$$\sum_{\sigma_T \in \sum^T} \pi(\sigma^T) log \frac{\alpha_{0,\sigma_1}}{q_{0,\sigma_1}} = (\sum_{1,\sigma_2,\sigma_3,...} \pi(\sigma^T)) log \frac{\alpha_{0,\sigma_1=1}}{q_{0,\sigma_1=1}} + (\sum_{2,\sigma_1,\sigma_2,...} \pi(\sigma^T)) log \frac{\alpha_{0,\sigma_1=2}}{q_{0,\sigma_1=2}}.$$

I have broken down the term on the left of the equal into two terms which I then added together.

Looking to the structure of the binomial tree, it is possible to notice that in the state of the world σ_1, that I am analyzing in this specific case, the traders have to make a choice: whether to choose $\sigma_1 = 1$ which is assigned a probability π_1, or $\sigma_1 = 2$ which has a probability equal to π_2.

Naturally the sum of π_1 and π_2 is equal to 1.

For this reason $(\sum_{1,\sigma_2,\sigma_3,...} \pi(\sigma^T)) log \frac{\alpha_{0,\sigma_1=1}}{q_{0,\sigma_1=1}}$ can also be rewritten as $\pi(\sigma_1 = 1) log \frac{\alpha_{0,\sigma_1=1}}{q_{0,\sigma_1=1}}$.

I remove the summation of all the σ_t because I am only considering the realization of $\sigma_1 = 1$. Using the same procedure, I rewrite the second term $(\sum_{2,\sigma_1,\sigma_2,...} \pi(\sigma^T)) log \frac{\alpha_{0,\sigma_1=2}}{q_{0,\sigma_1=2}}$ as $\pi(\sigma_1 = 2) log \frac{\alpha_{0,\sigma_1=2}}{q_{0,\sigma_1=2}}$.

Moreover, I can define $\pi(\sigma_1 = 1)$ as π_1 and $\pi(\sigma_1 = 2)$ as π_2.

Having all this information, I present equation **(26)** in a more restricted form:

$$\pi_1 log \frac{\alpha_{0,\sigma_1=1}}{q_{0,\sigma_1=1}} + \pi_2 log \frac{\alpha_{0,\sigma_1=2}}{q_{0,\sigma_1=2}}.$$

If I want to be more detailed and referring to equation **(24)**, I can state that $\alpha_{0,\sigma_1=1} = \pi_1$ and $\alpha_{0,\sigma_1=2} = \pi_2$.

In the end, I obtain $\pi_1 log \frac{\pi_1}{q_{0,\sigma_1=1}} + \pi_2 log \frac{\pi_2}{q_{0,\sigma_1=2}}$.

So, the wealth of the trader in time $t = 1$ will depend on the probability of the realization of

$\sigma_1 = 1$ and $\sigma_1 = 2$ and on the prices of the security $q_{0,\sigma_1=1}$ and $q_{0,\sigma_1=2}$.
After having treated the case in which $t = 0$, I want to analyze the situation in which the traders are in $t = 1$, so they have already taken a decision in time $t = 0$.
As told before, they have chosen either $\sigma_1 = 1$ ore $\sigma_1 = 2$.
In $t = 1$, both the informed traders and the uninformed have to choose between two different choices.
The following function is obtained: $\sum_{\sigma_T \in \sum^T} \pi(\sigma^T) log \frac{\alpha_{1,\sigma_2}}{q_{1,\sigma_2}}$.

In this case, differently form what I have done in equation **(26)**, I have to break down this specific function into four parts, since the investors' choice in the state of the world σ_2 will depend on the one made previously in the state σ_1.
The function written above can be expressed as **(27)**:

$$(\sum_{1,1,\sigma_3,...} \pi(\sigma^T)) log \frac{\alpha_{1,\sigma_2=1}}{q_{1,\sigma_2=1}} + (\sum_{1,2,\sigma_3,...} \pi(\sigma^T)) log \frac{\alpha_{1,\sigma_2=2}}{q_{1,\sigma_2=2}} + (\sum_{2,1,\sigma_3,...} \pi(\sigma^T)) log \frac{\alpha_{2,\sigma_2=1}}{q_{2,\sigma_2=1}} + (\sum_{2,2,\sigma_3,...} \pi(\sigma^T)) log \frac{\alpha_{2,\sigma_2=2}}{q_{2,\sigma_2=2}}.$$

I can rewrite the terms into the parenthesis in another way, having that:

- $\sum_{1,1,\sigma_3,...} \pi(\sigma^T) = \pi(\sigma_2 = 1|\sigma_1 = 1)$;
- $\sum_{1,2,\sigma_3,...} \pi(\sigma^T) = \pi(\sigma_2 = 2|\sigma_1 = 1)$;
- $\sum_{2,1,\sigma_3,...} \pi(\sigma^T) = \pi(\sigma_2 = 1|\sigma_1 = 2;)$
- $\sum_{2,2,\sigma_3,...} \pi(\sigma^T) = \pi(\sigma_2 = 2|\sigma_1 = 2)$.

Every decision of the investors in σ_2 depends on the choice made in σ_1. Finally we will have:

$$\pi(\sigma_2 = 1|\sigma_1 = 1) log \frac{\alpha_{1,\sigma_2=1}}{q_{1,\sigma_2=1}} + \pi(\sigma_2 = 2|\sigma_1 = 1) log \frac{\alpha_{1,\sigma_2=2}}{q_{1,\sigma_2=2}} + \pi(\sigma_2 = 1|\sigma_1 = 2 log \frac{\alpha_{2,\sigma_2=1}}{q_{2,\sigma_2=1}} + \pi(\sigma_2 = 2|\sigma_1 = 2) log \frac{\alpha_{2,\sigma_2=2}}{q_{2,\sigma_2=2}}.$$

I can repeat the same process until $\sigma_t = T$, always taking into account that α is equal to π and that each choice, made by the informed or by the uninformed traders, depend on the one made in the previous state of the world.

3.5 The relationship between the price q, α and the marginal wealth

Turning back to the previous chapter and in particular to the section "Introduction of the model", I want to underline that, considering the phenomenon of the market clearing, I have constructed the equation: $\mu^F Y^F(q;\pi) + \mu^S Y^S(q;J) = 0,$.

This one is subsequently reported as $Y^F(Q_k(\pi);\pi) + Y_k^S(Q_k(\pi)) = 0$ because I have assumed for hypothesis that $\mu^F = \mu^S = 1$.

The last equation tells that the sum of the optimal demands of the informed and the uninformed traders is equal to 0.

I can state that the optimal demand of a specific trader is equal to the sum of the holdings of the security paid in a specific state and to the wealth owned in that same state σ.

Taking into consideration this and considering the state $\sigma = 1$, I express the equation above as:

$$y^F + \omega_1^F + y^S + \omega_1^S = 0. \;\mathbf{(29)}$$

Assuming that I am considering the state of the world $\sigma = 1$ and continuing with the discussion above, I can express the sum of the holdings of the security and the wealth owned by a specific trader as the final wealth in that particular state.

It follows that:

$$w_1^F + w_1^S = 0.\mathbf{(30)}$$

Moreover, at the beginning of this chapter I have introduced the function which I said to be positively correlated with the optimal demand and also equal to the probability π of the realization of the state of the world.

To be more detailed, I have to take into account the fact that Y is also directly proportional to w and inversely proportional to the price of security q.

I have all the information to represent the optimal demand of both the "fundamentalists" and the "speculators" as a function of α:

$$\frac{\alpha_1^F w_0^F}{q_1} + \frac{\alpha_1^S w_0^S}{q_1} = 0.$$

Now, I have to introduce an assumption which involves both the informed and the uninformed traders.

I infer that the final wealth that the trader possesses in a specific state is the same for all others:

- $w^F = w_0^F = w_1^F = w_2^F = w_3^F = ... = w_t^F$;
- $w^S = w_0^S = w_1^S = w_2^S = w_3^S = ... = w_t^S$.

Given these two suppositions and considering also that in each state $q_1 + q_2 = 1$, I can express a new equation considering both equation **(29)** and (30).

Finally, I can express the equation in the following way:

$$\frac{\alpha_1^F w^F}{q_1} + \frac{\alpha_1^S w^S}{q_1} = w^F + w^S. \;\mathbf{(31)}$$

The aim of this mathematical procedure is to find out what variables q depends on.

Below are the mathematical steps to derive the price q_1 of the security starting from equation **(31)**:

$$\frac{\alpha_1^F w^F + \alpha_1^S w^S}{q_1} = w^F + w^S;$$

$$\alpha_1^F w^F + \alpha_1^S w^S = (w^F + w^S) q_1;$$

$$q_1 = \frac{\alpha_1^F w^F + \alpha_1^S w^S}{w^F + w^S}.$$

$$q_1 = \frac{\alpha_1^F w^F}{w^F + w^S} + \frac{\alpha_1^S w^S}{w^F + w^S}.$$

Going into detail, I can analyze the term $\frac{w^F}{w^F + w^S}$ and $\frac{w^S}{w^F + w^S}$, which represent the marginal wealth and I identify it with ϕ; in particular, ϕ^F for the categories of the "fundamentalists" and ϕ^S for the class of "speculators".

The final equation is:

$$q_1 = \alpha_1^F \phi^F + \alpha_1^S \phi^S. \textbf{ (32)}$$

Looking to equation **(32)**, I can show that the price of the security q_1 is based on the sum of two terms: the first is the product of α_1 and the marginal wealth ϕ each of which computed for the cateogories of the informed traders.

The second one is the product of α_1 and ϕ in this case calculated for the uninformed.

The last step I need to compute is to find out the numeric range within which q_1 is comprised.

Even in this situation I need to make one assumption, in particular that:

$$\phi^F + \phi^S = 1. \textbf{ (33)}$$

Furthermore, I have to take into consideration equation **(24)** which allows me to report the following mathematical statements **(34)**:

- $\alpha_1^F = \pi^F$
- $\alpha_1^S = \pi^S$.

It is known that, as discussed in the previous sections, $\pi^S + \pi^F = 1$.

Taking into account the equations **(33)** and **(34)** and applying this knowledge to equation **(32)**, I can state that the price of the security q_1 is comprised between 0 and 1, $q_1 \in [0, 1]$ **(34)**.

After having identified the numerical interval in which the price q is contained, I can make some observations regarding the nominal and real interest rate.

I start from the assumption that $\forall t$ in the set σ^T, I have two different prices ($q_t, \sigma(t+1) = 1$ and $q_t, \sigma(t+1) = 2$) depending on whether trader chooses to invest on one side or the other.

Taking into consideration these two elements, I can show that $q_t, \sigma(t+1) + q_t, \sigma(t+2) = 1$.

Moreover, considering the definition of real interest rate, which is the rate of interest an investor, saver or lender receives after allowing for inflation, I can present it as:

$$r_{t,t+1}(\sigma^t) = \frac{1}{q_{t,1} + q_{t,2}} = 1. \textbf{ (35)}$$

Taking into consideration that $q_t, \sigma(t+1) + q_t, \sigma(t+2) = 1$ and applying this mathematical statement, the subesequent expression is gained:

$$r_{t,t+1}(\sigma^t) = 1. \textbf{ (36)}$$

Given this strong assumption that the real interest rate in each interval of time $t, t+1$, for the whole history of the states up to $t \in \sum^t$, is equal to 1, I can affirm that the final wealth for both the informed and the uninformed traders is equal to the initial one.

3.6 Final wealth applied to all the states T of the world

In the last section of this chapter, I continue my analysis concerning the final wealth of traders in the various states of the world.

Taking into consideration equation **(25)**, treated in the section "An analysis of wealth in time $t = 0$ and $t = 1$" and considering only two periods of time, the subsequent mathematical statement is obtained:

$$w_T(\sigma^T = (\sigma_1, \sigma_2)) = \frac{\alpha_{(T-1),\sigma_T}}{q_{(T-1),\sigma_T}} \frac{\alpha_{(T-2),\sigma_{T-1}}}{q_{(T-2),\sigma_{(T-1)}}} \omega_{(T-2),\sigma(T-2)}. \textbf{ (37)}$$

In the same way, I can determine the specific value of ω_T considering infinite states of the world, until σ_t.

It follows that:

$$w_T(\sigma^T = (\sigma_1, ..., \sigma_t)) = \frac{\alpha_{(T-1)}}{q_{(T-1)}} \frac{\alpha_{(T-2)}}{q_{(T-2)}} \frac{\alpha_{(T-3)}}{q_{(T-3)}} \frac{...}{...} \frac{\alpha_0}{q_0} \omega_0, \textbf{ (38)}$$

where ω_0 is the initial holding available to the traders.

Going into more detail, I must consider the summation until σ_T (which represents the history of all the states up to $t \in \sum_{\sigma_T}$), also taking into account the probability π of all the states in the world σ_T.

Taking into consideration also w_T, calculated as equation **(38)** and applying to it the logarithmic function, I will have:

$$\textstyle\sum_{\sigma_T \in \sum^T} \pi(\sigma^T) log(\frac{\alpha_{(T-1),\sigma_T}}{q_{(T-1),\sigma_T}} \frac{\alpha_{(T-2),\sigma_{(T-1)}}}{q_{(T-2),\sigma_{(T-1)}}} \frac{\alpha_{(T-3),\sigma_{(T-2)}}}{q_{(T-3),\sigma_{(T-2)}}} \frac{...}{...} \frac{\alpha_0, \sigma_1}{q_0, \sigma_1} \omega_0) \textbf{ (39)}$$

Using the property of logarithms that affirms that the logarithm of a product is equal to the sum of the logarithms of the factors that make up the argument, provided however that all factors are greater than zero, I can rewrite the equation above in the following way:

$$\textstyle\sum_{\sigma_T \in \sum^T} \pi(\sigma_T) (log \frac{\alpha_{(T-1),\sigma_T}}{q_{(T-1),\sigma_T}} + log \frac{\alpha_{(T-2),\sigma_{(T-1)}}}{q_{(T-2),\sigma_{(T-1)}}} + log \frac{\alpha_{(T-3),\sigma_{(T-2)}}}{q_{(T-3),\sigma_{(T-2)}}} + log \frac{...}{...} + log \frac{\alpha_0, \sigma_1}{q_0, \sigma_1} + log \omega_0).$$

(40)

Looking at the equation above, it is possible to see that within it there are infinitesimal logarithmic sums.

I can say that the sum of the infinite values is a number that tends to 0, given the fact that the logarithm of each fraction $(\frac{\alpha_{(T-1)}}{q_{(T-1)}}, \frac{\alpha_{(T-2)}}{q_{(T-2)}})$ assumes an almost zero value.

Taking into account this statement, it is possible to show that:

$$w_T = \frac{\alpha_{(T-1)}, \sigma_T}{q_{(T-1), \sigma_T}} \frac{\alpha_{(T-2), \sigma_{(T-1)}}}{q_{(T-2), \sigma_{(T-1)}}} \omega_{(T-2), \sigma_{(T-2)}} \approx$$

$$w_T = \frac{\alpha_{(T-1), \sigma_T}}{q_{(T-1), \sigma_T}} \frac{\alpha_{(T-2), \sigma_{(T-1)}}}{q_{(T-2), \sigma_{(T-1)}}} \frac{\alpha_{(T-3), \sigma_{(T-2)}}}{q_{(T-3), \sigma_{(T-2)}}} \frac{\cdots}{\cdots} \frac{\alpha_0, \sigma_1}{q_0, \sigma_1} \omega_0. \quad \mathbf{(41)}$$

Chapter 4

How much information ”speculators” are able to discern?

4.1 The functions representing w_t and α_t for each categories of traders

In this chapter, the aim is to find out how good the uninformed traders are in discerning the information J, which is embodied into price q, that is given by the informed traders.
First of all, I have to take a step back and take in consideration again wealth.

I must dwell on equation **(25)** $(w(\sigma_{T-1}, \sigma T) = \frac{\alpha_{(T-1)}(\sigma_T)\omega_{(T-1)}(\sigma_{T-1})}{q_{(T-1)}(\sigma_T)})$.

This mathematical statement can refer to both ”fundamentalists (F)” and ”speculators (S)”. For this reason, to simplify things, I use $i = F, S$ to indicate the two categories of traders.
Assuming that $t = 1, 2, 3, ...$, the following function is obtained:

$$w_t^i(\sigma^t) = \frac{\alpha^i_{(t-1),\sigma_t}}{q_{(t-1),\sigma_t}} \omega^i_{(t-1)}(\sigma^{t-1}), \textbf{ (42)}$$

where $\sigma^t = \sigma_1, ..., \sigma_{t-1}, \sigma_t$.
If I want to be more detailed, I have to specify that $\sigma_1, \sigma_2, ..., \sigma_{t-1} = \sigma^{t-1}$. As previously mentioned, the wealth for each category of traders in a specific time t is directly proportional to α and ω calculated in time $t-1$ and inversely proportional to the price q also identified in $t-1$.
Analyzing the equation above, I have to make a discussion related to the function $\alpha^i_{(t-1),\sigma_t}$.
Differently from what I have stated regarding to $w_t^i(\sigma^t)$, the value of $\alpha^i_{(t-1),\sigma_t}$ is not the same for the two different categories of traders. Firstly, I consider the category of the informed traders.
As discussed in the section 3.3 ”A comparison between α and π”, it is known that $\alpha = \pi$, and going into a more detailed analysis I can show that:

$$\alpha^i_{t,\sigma_{t+1}} = \pi^i(\sigma_{t+1}|\sigma^t) = \pi^i(\sigma_{t+1}), \text{ with } i = F, \textbf{ (43)}$$

with $t = 0, 1, 2, ...$

On the other hand, I can not say the same thing for the categories of the "speculators".

Firstly, in this specific case I have to propose a distinction between two cases: the first with $k = 0$ and the second with $k > 0$.

In the first case, it is necessary to consider the assumption made in the chapter "A static model in economy", in particular in the section "An example with log preferences", according to which $E(\Pi) = 1/2$, where Π is the unknown probability associated with the uninformed traders.

Taking into consideration the second case with $k > 0$, it is necessary to underline what was said previously regarding equation **(10)** and the discussion made on $\overline{\Pi}(q)$ and $\hat{\Pi}_k(q)$.

I focus in particular on the fact that $\hat{\Pi}_k(q) \to \dfrac{q(\omega^F_1 + \omega^S_2)}{q(\omega^F_1 + \omega^S_2) + \omega^F_2 + \omega^S_2} = \overline{\Pi}(q).$

Considering what has just been treated, I can affirm that for the uninformed traders, with $K >$ have the following statement is obtained:

$$\alpha^i_{t,\sigma_{t+1}} = \hat{\Pi}_k(q), \text{ with } i = S \text{ and } k > 0.$$

After having discussed the two cases, we can report the function of $\alpha^i_{t,\sigma_{t+1}}$ associated the categories of "speculators" which is represented by the following system:

$$\alpha^i_{t,\sigma_{t+1}} = \begin{cases} \frac{1}{2}, E(\Pi) = \frac{1}{2}, i = S, k = 0 \\ \hat{\Pi}_k(q), i = S, k > 0 \end{cases} . \textbf{ (44)}$$

Now that I have distinguished the two cases according to the value of k for the category of the uninformed, it is possible to report in a system the alpha functions for both the categories of investors.

It follows that: $\alpha^i_{t,\sigma_{t+1}} = \begin{cases} \pi^i(\sigma_{t+1}|\sigma^t) = \pi^i(\sigma_{t+1}), i = F \\ \frac{1}{2}, E(\Pi) = \frac{1}{2}, i = S, k = 0 \\ \hat{\Pi}_k(q), i = S, k > 0 \end{cases}$ $\quad$.**(45)**

4.2 The market equilibrium

After having treated the wealth w for every instant t in the whole "history" of all the states up to t and α_t for all σ_{t+1}, each of which for both the categories of investors, it is possible to move the attention on the so called "market equilibrium".

This is given by the equation **(4)**, $\mu^F Y^F(q; \pi) + \mu^S Y^S(q; J) = 0$, with the assumption that $\mu^F =$

$\mu^S = 1$.

Taking into consideration the supposition, it is possible to write that $Y^F + Y^S = 0$.

From the previous chapters, it has been shown that the optimal demand Y for both the uninformed and the informed traders can be made explicit in various ways.

Firstly, I consider the optimal demand as a function of α, q and w.

Taking equation **(20)**, $Y = \dfrac{\alpha w}{q}$, as a reference point, I can express the equation of equilibrium of the optimal demands of traders as:

$$\frac{\alpha^F_{t,\sigma_t}\omega^F_{t-1}}{q_{t,\sigma_t}} + \frac{\alpha^S_{t,\sigma_t}\omega^S_{t-1}}{q_{t,\sigma_t}} = 0.$$

It is also known from the beginning of the chapter "A static model in economy" that the optimal demand for each category of trades is equal to $y + \omega$. Following the reasoning, it is possible to write:

$$y^F + \omega^F_t + y^S + \omega^S_t = 0,$$

where, I underline, ω represents the wealth with which the investors are endowed and y the holdings of the security paid in state σ_t.

Continuing on this speech and looking newly to equation **(25)**, treated in the section "An analysis of wealth in time $t = 0$ and $t = 1$", I can show another way to present the equation $Y^F + Y^S = 0$:

$$w^F_t + w^S_t = 0.$$

Considering equation **(36)** ($r_{t,t+1}(\sigma^t) = 1, \forall t$) It is possible to create further equality:

$$w^F_t + w^S_t = w^F_{t-1} + w^S_{t-1},\ \forall t.$$

After having presented three different ways that allow me to express the function of the optimal questions of the two traders, I report a further equivalent way of representing the function, which I will later use together with the mathematical statements of $\alpha^i_{t,\sigma_{t+1}}$ and $w^i_t(\sigma^t)$ calculated previously.

Going into more detail and considering equation **(32)** analyzed in section "3.5" ($q_1 = \alpha^F_1 \phi^F + \alpha^S_1 \phi^S$), it can be noticed that if I divide both members at the right of the equal for q_t, σ_t I will have the following form of the equation of equilibrium of the optimal demands:

$$\frac{\alpha^F_{t,\sigma_t}\phi^F_{t-1}}{q_{t,\sigma_t}} + \frac{\alpha^S_{t,\sigma_t}\phi^S_{t-1}}{q_{t,\sigma_t}} = 1,\ \textbf{(46)}$$

where $\phi^F_{t-1} + \phi^S_{t-1} = 1$.

Equation **(46)** with the other two calculated previously, (**(44)** and **(45)**), will result fundamental for the calculation of the ex-ante utility for a "speculator".

4.3 The maximization of the ex-ante utility function for a "fundamentalist"

In the two previous sections, I have treated all the elements that are required to maximize the utility function for an uninformed trader.

Now, I leave aside the discussion of "speculators" for a moment and focus the attention on the maximization of the utility for an informed traders.

Considering that the "fundamentalists" are able to identify the probability distributions of random shocks, and knowing that the probability of the realization in the future of the state of the world σ is $(\pi, 1-\pi) \in \delta$, it will be easier to maximize their utility function, discovering the link which exists belong π and $1-\pi$.

Below, I report the mathematical expression presented as a logarithmic function:

$$max(\pi^F log\frac{\alpha_1^F}{q} + (1-\pi)^F log\frac{\alpha_2^F}{1-q}), \ \mathbf{(47)}$$

with the following constraints:

- $\alpha > 0$
- $\alpha < 1$
- $\alpha_1 + \alpha_2 = 1$.

To maximize the function, it is necessary to compute the first derivative of the function with respect to the price q, which I have supposed to be q in the state $\sigma = 1$ and $1-q$ in state $\sigma = 2$. It follows that:

$$\frac{d\mathbf{(47)}}{q} = 0 \rightarrow (\pi^F log\frac{\alpha_1^F}{q} + (1-\pi)^F log\frac{\alpha_2^F}{1-q}) = 0.$$

Below, I report the mathematical steps that provide me with the link between α and π:

$$(\pi^F log\frac{\alpha_1^F}{q} + (1-\pi)^F log\frac{\alpha_2^F}{1-q}) = 0;$$

$$-\frac{\pi^F}{\alpha_1^F} + \frac{(1-\pi)^F}{1-\alpha_1^F} = 0;$$

$$\frac{(1-\pi)^F}{1-\alpha_1^F} = \frac{\pi^F}{\alpha_1^F};$$

$$\alpha_1^F(1-\pi)^F = \pi^F(1-\alpha_1^F);$$

$$\alpha_1^F - \alpha_1^F\pi^F = \pi^F - \alpha_1^F\pi^F;$$

In the end, I will have that:

$$\begin{cases} \alpha_1^F = \pi^F \\ \alpha_2^F = (1-\pi)^F \end{cases} . \ \mathbf{(48)}$$

Looking to the system above I notice that the value of α_1^F which maximize the utility function of an informed traders is given by the value of the probability π^F. Moreover, I have to fix α_2^F equal to $(1-\pi)^F$.

4.4 The maximization of the ex-ante utility function for a "speculator"

Afer having treated the maximization of the utility function of the "fundamentalists", it is time to move the attention on the one of the uninformed investors, which has told at the beginning are able to understand only that π is a realization of the random variable Π, whose support is δ.

I need to set up a three-equation system, in particular putting together the mathematical statements **(42)**, **(43)** and **(44)**.

Below, I represent the system which is essential to maximize the utility function for the "speculators":

$$\begin{cases} w_t^S(\sigma^t) = \dfrac{\alpha_{(t-1),\sigma_t}^S}{q_{(t-1),\sigma_t}} w_{(t-1)}^S(\sigma^{t-1}) \\ \alpha_{t,\sigma_{t+1}}^S = \begin{cases} \dfrac{1}{2}, E(\Pi) = \dfrac{1}{2}, k = 0 \\ \hat{\Pi}_k(q), k > 0 \end{cases} \\ \dfrac{\alpha_{t,\sigma_t}^F \phi_{t-1}^F}{q_{t,\sigma_t}} + \dfrac{\alpha_{t,\sigma_t}^S \phi_{t-1}^S}{q_{t,\sigma_t}} = 1 \end{cases} \qquad \textbf{(49)}.$$

At this point, I have to analyze the single equations presented within the system.

Starting from $w_t^S(\sigma^t) = \dfrac{\alpha_{(t-1),\sigma_t}^S}{q_{(t-1),\sigma_t}} w_{(t-1)}^S(\sigma^{t-1})$ it is possible to compare between them the wealth dynamics and the total wealth, expressing the equation just mentioned as a function of ϕ_t^i.

Firstly, I must divide the term at the left of the equal of the equation cited above for the sum of the wealth of the two different categories of traders in time t in the state of the world σ_t, $w_t^F(\sigma_t) + w_t^S(\sigma_t)$.

Morevoer, I have to divide the term at the right of the equal for $w_{t-1}^F(\sigma_{t-1}) + w_{t-1}^S(\sigma_{t-1})$.

Going into more detail, as demonstrated in the previous section, I am able to state that $w_t^F(\sigma_t) + w_t^S(\sigma_t) = w_{t-1}^F(\sigma_{t-1}) + w_{t-1}^S(\sigma_{t-1})$.

After making these premises, I can write the following:

$$\frac{w_t^i(\sigma^t)}{w_t^F(\sigma_t) + w_t^S(\sigma_t)} = \frac{\alpha_{(t-1),\sigma_t}^i}{q_{(t-1),\sigma_t}} \frac{w_{(t-1)(\sigma^{t-1})}^i}{w_{t-1}^F(\sigma_{t-1}) + w_{t-1}^S(\sigma_{t-1})}.$$

To make things easier, in this case I consider traders of type "i", where $i = F, S$.

It also known that $\dfrac{w_t^i(\sigma^t)}{w_t^F(\sigma_t)+w_t^S(\sigma_t)} = \phi_t^i(\sigma_t)$ and $\dfrac{w_{t-1}^i(\sigma^{t-1})}{w_{t-1}^F(\sigma_t)+w_{t-1}^S(\sigma_t)} = \phi_{t-1}^i(\sigma_{t-1})$.

This discussion is useful because it allows me to report the following mathematical statement:

$$w_t^S(\sigma^t) = \frac{\alpha_{(t-1),\sigma_t}^S}{q_{(t-1),\sigma_t}} w_{(t-1)}^S(\sigma^{t-1}) \text{ equal to } \phi_t^i(\sigma_t) = \frac{\alpha_{(t-1),\sigma_t}^i}{q_{(t-1),\sigma_t}} \phi_{t-1}^i(\sigma_{t-1}).$$

To maximize the utility function of a "speculator", now I will use into the system **(49)** the equation just reported:

$$\phi_t^i(\sigma_t) = \frac{\alpha_{(t-1),\sigma_t}^i}{q_{(t-1),\sigma_t}} \phi_{t-1}^i(\sigma_{t-1}). \ \mathbf{(50)}$$

I can notice that for both the informed and the uninformed traders the marginal wealth in t in σ_t depends on the marginal wealth calculated in the previous period with respect to T and on the relationship between $\alpha_{(t-1),\sigma_t}^i$ and $q_{(t-1),\sigma_t}$.

Moreover, I can take into consideration the equation which represents the market equilibrium **(46)** and express it as a function of q_{t-1}, σ_t.

Following simple mathematical steps, I obtain:

$$\frac{\alpha_{(t-1),\sigma_t}^F \phi_{t-1}^F(\sigma_{t-1})}{q_{(t-1),\sigma_t}} + \frac{\alpha_{(t-1),\sigma_t}^S \phi_{t-1}^S(\sigma_{t-1})}{q_{(t-1),\sigma_t}} = 1;$$

$$\frac{\alpha_{(t-1),\sigma_t}^F \phi_{t-1}^F(\sigma_{t-1}) + \alpha_{(t-1),\sigma_t}^S \phi_{t-1}^S(\sigma_{t-1})}{q_{(t-1),\sigma_t}} = 1;$$

finally, I have:

$$q_{(t-1),\sigma_t} = \alpha_{(t-1),\sigma_t}^F \phi_{t-1}^F(\sigma_{t-1}) + \alpha_{(t-1),\sigma_t}^S \phi_{t-1}^S(\sigma_{t-1}). \ \mathbf{(51)}$$

This equation shows that the price q in $t-1$ for each state of the world σ_t depends on the α at time $t-1$ for each category of investors and on the marginal wealth whose values are calculated in time $t-1$.

I will devise a new system into which I insert this specific mathematical statement representing the so-called market equilibrium.

The last variable that I have to treat is α.

I will consider $\alpha_{(t-1),\sigma_t}$ for both categories of traders and in particular for the uninformed traders I contemplate only the case in which $k > 0$ (I want to underline that for $k = 0$ we have that $\alpha_{(t-1),\sigma_t}^S = \dfrac{1}{2}$.)

From the discussion reported in the previous sections of this chapter, it is possible to show that:

$$\begin{cases} \alpha_{(t-1),\sigma_t}^F = \pi_{\sigma_t}^F \\ \alpha_{(t-1),\sigma_t}^S = \hat{\Pi}_{k,\sigma_t}^S(q_{t-1}), k > 0 \end{cases} \quad \mathbf{(52)}.$$

It is known that $\pi_{\sigma_t}^F$ is given because the "fundamentalists" know the probability of the realization of the random shocks, while the amount $\hat{\Pi}_{k,\sigma_t}^S(q_{t-1})$ is unknown and depends on the value of q_{t-1}.

The aim is to find out the value of $\hat{\Pi}_{k,\sigma_t}^S$ with a level-k reasoning $k > 0$.

Now, I have all the elements to redefine the system **(49)** with the presence of other variables which allows me to maximize the ex-ante utility function for an uninformed trader.
Putting together equation **(50)** and **(51)** and the system just reported **(52)** it is possible to obtain:

$$\begin{cases} \phi_t^i(\sigma_t) = \dfrac{\alpha^i_{(t-1),\sigma_t}}{q_{(t-1),\sigma_t}}\phi^i_{t-1}(\sigma_{t-1}) \\ \alpha^F_{t-1,\sigma_t} = \pi^F_{\sigma_t} \\ \alpha^S_{(t-1),\sigma_t} = \hat{\Pi}^S_{k,\sigma_t}(q_{t-1}) \\ q_{(t-1),\sigma_t} = \alpha^F_{(t-1),\sigma_t}\phi^F_{t-1}(\sigma_{t-1}) + \alpha^S_{(t-1),\sigma_t}\phi^S_{t-1}(\sigma_{t-1}) \end{cases} \quad \textbf{(53)}.$$

This last system will be important in the fifth and last chapter when I will create some codes on Matlab to represent the differences in wealt and security prices for the two categories of traders considered in the model.

4.5 The function of price $Q_k(\pi)$ at level K

Turning back to the section ”Learning of the speculators” presented in the second chapter, I want to take into consideration the equation which represents $Q_0(\pi)$, obtained putting into $\hat{\Pi}_0(q)$ the value $\dfrac{1}{2} = E(\Pi)$.
The mathematical statement just cited is the following:

$$Q_0(\pi) = \frac{2\pi\Omega + \omega}{2(1-\pi)\Omega + \omega}.$$

The reason I have reconsidered this equation is that I want to obtain the inverse function $\pi(Q_1(\pi)$, which will result useful later.
Below, I present the mathematical steps to obtain the variable concerned. To simplify things, I use price q to refer to $Q_0(\pi)$:

$$q = \frac{2\pi\Omega + \omega}{2(1-\pi)\Omega + \omega};$$

Multiplying both terms by the denominator it is possible to obtain:

$$q(2(1-\pi)\Omega + \omega) = 2\pi\Omega + \omega;$$

$$2q\Omega - 2q\pi\Omega + \omega q = 2\pi\Omega + \omega;$$

Keeping in mind that the aim is to obtain π, so I move the terms that contain π to the left of the equal:

$$2\pi\Omega + 2q\pi\Omega = 2q\Omega + \omega q - \omega.$$

Isolating π, I obtain:

$$\pi = \frac{2q\Omega + \omega(q-1)}{2\Omega(q+1)};$$

Separating the two terms in the numerator, I finally get:

$$\pi = \frac{q}{q+1} + \frac{\omega(q-1)}{2\Omega(q+1)}. \quad \mathbf{(54)}$$

It is necessary to keep in mind this last mathematical statement that I will use later.

Referring to section 2.5 "Learning of the speculators", I take into consideration equation **(11)** which is constructed using the demands of the two classes of investors, $Y^F(Q_k(\pi);\pi)$ and $Y_k^S(Q_k(\pi))$, and the pricing function Q_k. In this specific case I want to discover the value of $Q_1(\pi)$.
The first step is to substitute in the equation **(11)** $Q_k(\pi)$ with $Q_1(\pi)$, knowing that that the value of $Q_0(\pi)$ has been already calculated ($Q_0(\pi) = \frac{2\pi\Omega + \omega}{2(1-\pi)\Omega + \omega}$):

$$\frac{\pi}{Q_1(\pi)}(Q_k(\pi)\omega_1^F + \omega_2^F) - \omega_1^F = \omega_1^S - \frac{\hat{\Pi}_1(Q_1(\pi))}{Q_1(\pi)}(Q_1(\pi)\omega_1^S + \omega_2^S).$$

Simplifying things and considering the assumption that $\omega_1^F = \omega_2^F = \Omega > \omega = \omega_1^S = \omega_2^S$, I can rewrite the equation above as:

$$[\pi(1 + \frac{1}{Q_1(\pi)}) - 1]\Omega = [1 - \hat{\Pi}_1(Q_0(\pi))(1 + \frac{1}{Q_1(\pi)})]\omega.$$

In this moment, I substitute equation **(54)** into $\hat{\Pi}_1(Q_0(\pi))$ to express $Q_1(\pi)$ as a function of ω and Ω.
It is possible to obtain the following mathematical statement:

$$[\pi(1 + \frac{1}{Q_1(\pi)}) - 1]\Omega = [1 - (\frac{Q_1(\pi)}{Q_1(\pi)+1} + \frac{\omega(Q_1(\pi)-1)}{2\Omega(Q_1(\pi)+1)})(1 + \frac{1}{Q_1(\pi)})]\omega. \quad \mathbf{(55)}$$

To facilitate the mathematical steps, I substitute $Q_1(\pi)$ with q as made previously.

$$[\pi(1 + \frac{1}{q}) - 1]\Omega = [1 - (\frac{q}{q+1} + \frac{\omega(q-1)}{2\Omega(q+1)})(1 + \frac{1}{q})]\omega.$$

Below, I present the mathematical steps necessary to derive the price q:

$$\pi\Omega + \frac{\pi\Omega}{q} - \Omega = [1 - (\frac{2\Omega q + \omega q - \omega}{2\Omega(1+q)})(1 + \frac{1}{q})]\omega,$$

multiplying the terms within the round brackets together, I obtain:

$$\pi\Omega + \frac{\pi\Omega}{q} - \Omega = [1 - (\frac{2\Omega q + \omega q - \omega}{2\Omega(1+q)}) - (\frac{2\Omega q + \omega q - \omega}{2\Omega q(1+q)})]\omega;$$

$$\pi\Omega + \frac{\pi\Omega}{q} - \Omega = [\frac{2\Omega q(1+q) - 2\Omega q^2 - \omega q^2 + \omega q - 2\Omega q - \omega q + \omega}{2\Omega q(1+q)}]\omega$$

Eliminating the terms with opposite sign at the numerator, I get:

$$\pi\Omega + \frac{\pi\Omega}{q} - \Omega = [\frac{\omega(1-q^2)}{2\Omega q(1+q)}]\omega;$$

$$\pi\Omega + \frac{\pi\Omega}{q} - \Omega = [\frac{\omega(1-q)(1+q)}{2\Omega q(1+q)}]\omega.$$

Simplifying the terms (1 + q) and applying the common denominator, it is possible to write:

$$2\Omega^2\pi q + 2\pi\Omega^2 - 2\Omega^2 q = \omega^2 - q\omega^2;$$

isolating the term q, I obtain:

$$q = \frac{2\Omega^2\pi - \omega^2}{2\Omega^2 - \omega^2 - 2\Omega^2\pi};$$

$$q = \frac{2\Omega^2\pi - \omega^2}{2\Omega^2(1-\pi) - \omega^2}.$$

The final step is to substitute q with $Q_1(\pi)$ getting:

$$Q_1(\pi) = \frac{2\Omega^2\pi - \omega^2}{2\Omega^2(1-\pi) - \omega^2}. \ \mathbf{(56)}$$

Going into more detail, I consider equation **(13)** presented in the second chapter:

$$Q_k(\pi) = \frac{2\pi\Omega^{k+1} + (-1)^k\omega^{k+1}}{2(1-\pi)\Omega^{k+1} + (-1)^k\omega^{k+1}},$$

and rewrite equation **(56)** in the same way using mathematical induction, it is possible to get:

$$Q_1(\pi) = \frac{2\Omega^{k+1}\pi - \omega^{k+1}}{2(1-\pi)\Omega^{k+1} - \omega^{k+1}}. \ \mathbf{(57)}$$

I want to underline that in equation **(56)**, I have considered the case in which $k = 1$.

It is possible to notice that substituting $k = 1$ into equation **(13)** I will obtain equation **(56)**.

In the same way, to calculate $Q_2(\pi)$, I must derive π from equation **(56)** putting it into equation **(11)** instead of $\hat{\Pi}_2(Q_2(\pi))$, deriving in a second moment $Q_2(\pi)$.

The same procedure can be repeated for all k.

Below, I present the general case for any value assumed by k.

To prove that equation **(13)** is true for all the values of k, I start trying to extract from the same mathematical expression the variable π.

I present the mathematical steps to obtain it. As made previously, I use the letter q instead of $Q_k(\pi)$.

$$q(2(1-\pi)\Omega^{k+1} + (-1)^k\omega^{k+1}) = 2\pi\Omega^{k+1} + (-1)^k\omega^{k+1};$$

$$2q\Omega^{k+1} - 2q\pi\Omega^{k+1} + q(-1)^k\omega^{k+1} = 2\pi\Omega^{k+1} + (-1)^k\omega^{k+1},$$

bringing the terms containing the price q to the left of the equal, I obtain:

$$\pi(2\Omega^{k+1} + 2q\Omega^{k+1}) = 2q\Omega^{k+1} + q(-1)^k\omega^{k+1} - (-1)^k\omega^{k+1}.$$

Isolating the variable I am taking into consideration, I get:

$$\pi = \frac{2q\Omega^{k+1} + q(-1)^k\omega^{k+1} - (-1)^k\omega^{k+1}}{2\Omega^{k+1}(q+1)}.$$

In the end, it is possible to write the function which represents the value of the probability:

$$\pi = \frac{q}{q+1} + \frac{(-1)^k\omega^{k+1}(q-1)}{2\Omega^{k+1}(q+1)};$$

$$\pi = \frac{Q_k(\pi)}{Q_k(\pi)+1} + \frac{(-1)^k \omega^{k+1}(Q_k(\pi)-1)}{2\Omega^{k+1}(Q_k(\pi)+1)}. \quad \textbf{(58)}$$

At this point, I have to substitute the mathematical statement just found into the variable $\hat{\Pi}_k(Q_k(\pi))$ contained into the generic equation **(12)**.

To simplifly the mathematical steps, I use q instead of $Q_k(\pi)$:

$$[\pi(1+\frac{1}{q})-1]\Omega = [1-(\frac{q}{q+1}+\frac{(-1)^k\omega^{k+1}(q-1)}{2\Omega^{k+1}(q+1)})(1+\frac{1}{q})]\omega;$$

$$\pi\Omega + \frac{\pi\Omega}{q} - \Omega = [1-(\frac{2\Omega^{k+1}q+(-1)^k\omega^{k+1}(q-1)}{2\Omega^{k+1}(q+1)})(\frac{q+1}{q})]\omega,$$

multiplying the terms into the parenthesis, I get:

$$\pi\Omega + \frac{\pi\Omega}{q} - \Omega = [1-\frac{2\Omega^{k+1}q+(-1)^k\omega^{k+1}(q-1)}{2\Omega^{k+1}q}]\omega.$$

Applying the common denominator at the right of the equal:

$$\pi\Omega + \frac{\pi\Omega}{q} - \Omega = [\frac{2\Omega^{k+1}q-2\Omega^{k+1}q-(-1)^k\omega^{k+1}(q-1)}{2\Omega^{k+1}q}]\omega.$$

$$\pi\Omega + \frac{\pi\Omega}{q} - \Omega = [\frac{-(-1)^k\omega^{k+1}(q-1)}{2\Omega^{k+1}q}]\omega,$$

and implementing the common denominator to the entire mathematical expression, I will obtain:

$$2\Omega^{k+2}\pi q + 2\Omega^{k+2}\pi - 2\Omega^{k+2}q = -(-1)^k\omega^{k+2}(q-1);$$

$$2\Omega^{k+2}\pi q + 2\Omega^{k+2}\pi - 2\Omega^{k+2}q = -(-1)^k\omega^{k+2}q + (-1)^k\omega^{k+2}.$$

Bringing the terms containing the price q to the left of the equal, it is possible to write:

$$-2\Omega^{k+2}\pi q + 2\Omega^{k+2}q - (-1)^k\omega^{k+2}q = -(-1)^k\omega^{k+2} + 2\Omega^{k+2}\pi.$$

Finally, isolating the term q I get:

$$q = \frac{+2\Omega^{k+2}\pi - (-1)^k\omega^{k+2}}{2\Omega^{k+2}(1-\pi)-(-1)^k\omega^{k+2}};$$

$$Q_{k+1}(\pi) = \frac{+2\Omega^{k+2}\pi - (-1)^k\omega^{k+2}}{2\Omega^{k+2}(1-\pi)-(-1)^k\omega^{k+2}}. \quad \textbf{(59)}$$

I have demonstrated that equation **(13)** and **(59)** are completely equal if compared for every value assumed by k.

4.6 $Q_k(\pi)$ represented as a function of the marginal wealth

An alternative way to represent the price q is using a mathematical expression represented as a function of α and of the marginal wealth ϕ^S and ϕ^F.
First of all, to show this I must take into consideration equation **(32)** treated in chapter 3 in the section "The relationship between the price q, α and the marginal wealth".
Starting from the equation just cited above, I want to obtain the value of π:

$$q_0 = \alpha_1^F \phi^F + \alpha_1^S \phi^S,$$

knowing that $\alpha_1^F = \pi$ and $\alpha_1^S = \frac{1}{2}$, in the case in which $k = 0$, I get:

$$q_0 = \pi\phi^F + \frac{1}{2}\phi^S;$$

$$\pi = \frac{q}{\phi^F} - \frac{\phi^S}{2\phi^F}. \quad \textbf{(60)}$$

At this point, I can express the equation just cited that represents q_0 as a function of $Q_k(\pi)$ and $\hat{\Pi}_k(Q_k(\pi))$ for any generic value assumed by k. Substituting these variables into the equation I get:

$$Q_k(\pi) = \pi\phi^F + \hat{\Pi}_k(Q_k(\pi))\phi^S. \quad \textbf{(61)}$$

Going into more detail and replicating what I have previously done for the calculation of $Q_k(\pi)$ as a function of Ω and ω, I have to substitute the mathematical expression **(60)** into the variable $\hat{\Pi}_k(Q_k(\pi))$ contained in the equation above obtaining:

$$Q_k(\pi) = \pi\phi^F + (\frac{Q_k(\pi)}{\phi^F} - \frac{\phi^S}{2\phi^F})\phi^S.$$

It can be noticed that instead of $\frac{1}{2}$, which is the value corresponding to $\hat{\Pi}_0(Q_0(\pi))$, it is possible to write $\frac{q}{\phi^F} - \frac{\phi^S}{2\phi^F}$, which embodies the value of $\hat{\Pi}_k(Q_k(\pi))$ for a generic k.
Now, I must derive the function representing $Q_k(\pi)$.
Using q instead of $Q_k(\pi)$ I get:

$$q = \pi\phi^F + (\frac{2q - \phi^S}{2\phi^F})\phi^S,$$

applying the common denominator, I obtain:

$$2\phi^F q = 2\phi^{2F}\pi + 2\phi^S q - \phi^{2S};$$

$$2\phi^F q - 2\phi^S q = 2\phi^{2F}\pi - \phi^{2S}.$$

Isolating the variable I am interested in, I get:

$$q = \frac{2\phi^{2F}\pi - \phi^{2S}}{2\phi^F - 2\phi^S};$$

$$Q_1(\pi) = \frac{2\phi^{2F}\pi - \phi^{2S}}{2(\phi^F - \phi^S)}. \quad \textbf{(62)}$$

Equation **(62)** represents the formula which expresses the value of $Q_1(\pi)$ as a function of the marginal wealth of both the informed traders (F) and the uninformed ones (S).
If I want to be more detailed, I can derive from the last mathematical statement the value of π which will be useful to get the function that represents $Q_2(\pi)$.
Below, I show the mathematical steps to obtain the function of π starting from equation **(62)**:

$$2q(\phi^F - \phi^S) = 2\phi^{2F}\pi - \phi^{2S};$$

$$2\phi^{2F}\pi = 2q\phi^F + \phi^{2S} - 2q\phi^S,$$

isolating the term π, I get:

$$\pi = \frac{2q\phi^F}{2\phi^{2F}} + \frac{\phi^S(\phi^S - 2q)}{2\phi^{2F}}.$$

Finally, simplifying I will have that the mathematical statement which represents $\hat{\Pi}_2(Q_2(\pi))$ is:

$$\hat{\Pi}_2(Q_2(\pi)) = \frac{Q_2(\pi)}{\phi^F} + \frac{\phi^S(\phi^S - 2Q_2(\pi))}{2\phi^{2F}}. \textbf{ (63)}$$

The aim of this discussion is to find the function representing $Q_2(\pi)$. As made before for the calculation of $Q_2(\pi)$, I must substitute the equation just found into equation **(61)** in place of $\hat{\Pi}_k(Q_k(\pi))$.
Doing this, I obtain:

$$Q_2(\pi) = \pi\phi^F + (\frac{Q_2(\pi)}{\phi^F} + \frac{\phi^S(\phi^S - 2Q_2(\pi))}{2\phi^{2F}})\phi^S.$$

Below, I show all the mathematical steps to derive the function representing $Q_2(\pi)$):

$$q = \pi\phi^F + (\frac{2\phi^F q + \phi^{2S} - 2\phi^S q}{2\phi^{2F}})\phi^S,$$

applying the common denominator, I get:

$$2\phi^{2F}q = 2\pi\phi^{3F} + 2\phi^F\phi^S q + \phi^{3S} - 2\phi^{2S}q;$$

$$q(2\phi^{2F} + 2\phi^{2S} - 2\phi^F\phi^S) = 2\pi\phi^{3F} + \phi^{3S}.$$

Isolating the term q, I obtain:

$$q = \frac{2\pi\phi^{3F} + \phi^{3S}}{2(\phi^{2F} + \phi^{2S} - \phi^F\phi^S)},$$

$$Q_2(\pi) = \frac{2\pi\phi^{3F} + \phi^{3S}}{2(\phi^{2F} + \phi^{2S} - \phi^F\phi^S)}. \textbf{ (64)}$$

At this point, I can make another step calculating, in the same way as made for $Q_1(\pi)$ and $Q_2(\pi)$, $Q_3(\pi)$. This will result useful to find the equation which will be used to compute $Q_k(\pi)$ regardless of the value assumed by k.
Starting from equation **(64)**, I derive the function that represents π:

$$2\phi^{2F}q + 2\phi^{2S}q - 2\phi^S\phi^F q = 2\pi\phi^{3F} + \phi^{3S};$$

$$2\pi\phi^{3F} = 2\phi^{2F}q + 2\phi^{2S}q - 2\phi^S\phi^F q - \phi^{3S}.$$

Closing off π, I get:

$$\hat{\Pi}_3(Q_3(\pi)) = \frac{Q_3(\pi)(2\phi^{2F} + 2\phi^{2S} - 2\phi^S\phi^F) - \phi^{3S}}{2\phi^{3F}}. \quad \textbf{(65)}$$

The final step is to gather the mathematical expression which embodies the value of $Q_3(\pi)$. All mathematical steps are reported below; as made previously, I begin from equation **(61)**:

$$q = \pi\phi^F + \frac{(2\phi^{2F}q + 2\phi^{2S}q - 2\phi^S\phi^F q - \phi^{3S})\phi^S}{2\phi^{3F}},$$

$$2\phi^{3F}q = 2\pi\phi^{4F} + 2\phi^{2F}\phi^S q + 2\phi^{3S}q - 2\phi^{2S}\phi^F q - \phi^{4S};$$

bringing the terms containing the price q to the left of the equal, I get:

$$q(2\phi^{3F} - 2\phi^{3S} - 2\phi^{2F}\phi^S + 2\phi^{2S}\phi^F) = 2\pi\phi^{4F} - \phi^{4S};$$

$$q = \frac{2\pi\phi^{4F} - \phi^{4S}}{2(\phi^{3F} - \phi^{3S} - \phi^F\phi^S(\phi^F - \phi^S))}.$$

Finally, I obtain mathematical expression of Q_3 as a function of π:

$$Q_3(\pi) = \frac{2\pi\phi^{4F} - \phi^{4S}}{2(\phi^{3F} - \phi^{3S} - \phi^F\phi^S(\phi^F - \phi^S))}. \quad \textbf{(66)}$$

Now, the last step is to obtain a generic equation for $Q_k(\pi)$.
I can get the mathematical expression for $Q_k(\pi)$ expressing the equation **(13)** as a function of the marginal wealth of both the informed investors (ϕ_F) and the uninformed ones (ϕ_S).
Starting from equation **(13)**, I must divide both the numerator and the denominator, at the left of the equal, for $(\omega + \Omega)^{k+1}$ obtaining:

$$Q_k(\pi) = \frac{\dfrac{2\pi\Omega^{k+1}}{(\Omega+\omega)^{k+1}} + \dfrac{(-1)^k\omega^{k+1}}{(\Omega+\omega)^{k+1}}}{\dfrac{2(1-\pi)\Omega^{k+1}}{(\Omega+\omega)^{k+1}} + \dfrac{(-1)^k\omega^{k+1}}{(\Omega+\omega)^{k+1}}}.$$

As discussed in the previous chapter, I know that:

- $\dfrac{\Omega^{k+1}}{(\Omega+\omega)^{k+1}} = \phi^F$;
- $\dfrac{\omega^{k+1}}{(\Omega+\omega)^{k+1}} = \phi^S$.

Knowing this, I can rewrite the equation above in a more restricted way:

$$Q_k(\pi) = \frac{2\pi(\phi^F)^{k+1} + (-1)^k(\phi^S)^{k+1}}{2(1-\pi)(\phi^F)^{k+1} + (-1)^k(\phi^S)^{k+1}}. \quad \textbf{(67)}$$

The following step is to divide the equation just found for $(1 + Q_k(\pi))$:

$$\frac{Q_k(\pi)}{1+Q_k(\pi)} = \frac{\dfrac{2\pi(\phi^F)^{k+1} + (-1)^k(\phi^S)^{k+1}}{2(1-\pi)(\phi^F)^{k+1} + (-1)^k(\phi^S)^{k+1}}}{\dfrac{2(1-\pi)(\phi^F)^{k+1} + (-1)^k(\phi^S)^{k+1} + 2\pi(\phi^F)^{k+1} + (-1)^k(\phi^S)^{k+1}}{2(1-\pi)(\phi^F)^{k+1} + (-1)^k(\phi^S)^{k+1}}};$$

eliminating the terms with opposite sign and simplifying the two equal denominators, I get:

$$Q_k(\pi) = \frac{2\pi(\phi^F)^{k+1} + (-1)^k(\phi^S)^{k+1}}{2(\phi^F)^{k+1} + 2(-1)^k(\phi^S)^{k+1}}.$$

Finally, I obtain:

$$Q_k(\pi) = \frac{2\pi(\phi^F)^{k+1} + (-1)^k(\phi^S)^{k+1}}{2[(\phi^F)^{k+1} + (-1)^k(\phi^S)^{k+1}]}. \quad \textbf{(68)}$$

This is the general equation for the calculation of the price $Q_k(\pi)$ for any generic value of k as a function of the marginal wealth ϕ^F and ϕ^S.

So, I can write the mathematical expression for a generic price Q_k both as a function of Ω and ω (equation **(13)**) and expressed with the marginal wealth ϕ (equation **(68)**).

4.7 The accuracy of the general equation for the calculation of $Q_k(\pi)$

To prove that this last mathematical expression is true, it is possible to study the behaviour with $k = 1$, $k = 2$, $k = 3$, ...

Starting from $k = 1$, if I will obtain equation **(62)** doing this, I will prove the accuracy of the mathematical statement **(68)**.

$$Q_1(\pi) = \frac{2\pi\phi^{2F} - \phi^{2S}}{2[\phi^{2F} - \phi^{2S}]};$$

using the properties of exponents, it is possible to write that:

$$Q_1(\pi) = \frac{2\pi\phi^{2F} - \phi^{2S}}{2[(\phi^F - \phi^S)(\phi^F + \phi^S)]},$$

and taking into consideration equation **(33)**, it is known that $\phi^F + \phi^S = 1$.

So, as shown below, I have demonstrated that I have obtained equation **(62)**:

$$Q_1(\pi) = \frac{2\pi\phi^{2F} - \phi^{2S}}{2[(\phi^F - \phi^S)]}.$$

Further proof of the reliability of the general equation is given by inserting into it $k = 2$ and comparing the same with the mathematical statement **(64)**. I get:

$$Q_2(\pi) = \frac{2\pi\phi^{3F} + \phi^{3S}}{2[\phi^{3F} + \phi^{3S}]}.$$

Also in this case, as I have made for $k = 1$, I can adopt the properties of exponents getting:

$$Q_2(\pi) = \frac{2\pi\phi^{3F} + \phi^{3S}}{2[(\phi^F + (\phi)^S)(\phi^{2F} + (\phi)^{2S} - \phi^F\phi^S)]},$$

knowing that $\phi^F + (\phi)^S = 1$, I obtain:

$$Q_2(\pi) = \frac{2\pi\phi^{3F} + \phi^{3S}}{2[\phi^{2F} + \phi^{2S} - \phi^F\phi^S]},$$

which is exactly equal to equation **(64)**.

One last test to prove accuracy of the general equation is to insert within it $k = 3$ and demonstrate that I get equation **(68)**, which is the mathematical expression which embodies the value of $Q_3(\pi)$. Below, I show the mathematical steps to demonstrate this:

$$Q_3(\pi) = \frac{2\pi\phi^{4F} - \phi^{4S}}{2[\phi^{4F} - \phi^{4S}]};$$

$$Q_3(\pi) = \frac{2\pi\phi^{4F} - \phi^{4S}}{2[(\phi^{2F} - \phi^{2S})(\phi^{2F} + \phi^{2S})]};$$

always adopting the properties of exponents, I obtain:

$$Q_3(\pi) = \frac{2\pi\phi^{4F} - \phi^{4S}}{2[(\phi^F + \phi^S)(\phi^F - \phi^S)(\phi^{2F} + \phi^{2S})]};$$

knowing that $\phi^F + \phi^S = 1$:

$$Q_3(\pi) = \frac{2\pi\phi^{4F} - \phi^{4S}}{2[(\phi^F - \phi^S)(\phi^{2F} + \phi^{2S})]};$$

$$Q_3(\pi) = \frac{2\pi\phi^{4F} - \phi^{4S}}{2[\phi^{3F} - \phi^{3S} - (\phi^F\phi^S)(\phi^F - \phi^S)]},$$

which exactly equal to equation **(66)**.

I have demonstrated that the general equation for $Q_k(\pi)$ for any generic value of k represented as a function of the marginal wealth of both traders is true for $k = 1$, $k = 2$ and $k = 3$.

It is possible to repeat the same procedure for all k, and it is possible to notice that it is the most reliable mathematical statement.

Chapter 5

The long run survival of "speculators" and level-k reasoning

5.1 The variables used in the code

In the previous chapters, I dealt with the formulas representing the variables that are useful within the model, in particular the price q of the securities, the α function, the marginal wealth of the two types of investors (ϕ^F and ϕ^S) and the probability associated to the traders.

Then, I have created a code in Matlab with the aim to understand how much the "speculators" are able to reduce the gap of information with the "fundamentalists".

Going into more detail, as reported in the appendix, I have realized a simulation with two distinct states, respectively $\sigma = 1$ and $\sigma = 2$, and with a number "n" of iterations between the informed and the uninformed traders. Moreover, "pi" is the real probability and the voice "seed" expresses a random number generator.

In the code I present the formulas representing the value of $q(t, 1)$ and $q(t, 2)$, which are respectively the prices of the securities of the informed traders and the uninformed ones.

To represent the function of q for both investors I have taken into account the last equation of the system **(53)**, having the following mathematical expressions for the two categories of traders:

- $q(t, 1) = w(t, 1)a1(t, 1) + w(t, 2)a2(t, 1)$, for the informed traders; **(69)**
- $q(t, 2) = w(t, 1)a1(t, 2) + w(t, 2)a2(t, 2)$, for the uninformed traders. **(70)**

Turning back to the representation of the marginal wealth of the two types of traders, which I have presented as $w(t + 1, 1)$ for the categories of the "fundamentalists" and as $w(t + 1, 2)$ for the "speculators", I have taken into consideration the first equation of the system **(53)** treated in

section 4.4 "The maximization of the ex-ante utility function for a "speculator"".
So, the marginal wealth for the informed traders considered at time $t+1$ is the following:

$$w(t+1,1) = \frac{a1(t,1)}{q(t,1)}w(t,1).$$

On the other hand, the marginal wealth for the uninformed traders is given by:

$$w(t+1,2) = \frac{a2(t,2)}{q(t,1)}w(t,2).$$

In the code I rewrite the functions which represent investor's wealth substituting into $q(t,1)$ and $q(t,2)$ the equations cited above.
The final representation of $w(t+1,1)$ and $w(t+1,2)$ is:

$$w(t+1,1) = \frac{a1(t,1)}{w(t,1)a1(t,1)+w(t,2)a2(t,1)}w(t,1) \ \mathbf{(71)},$$

for the category of the fundamentalists, and

$$w(t+1,2) = \frac{a2(t,1)}{(w(t,1)a1(t,1)+w(t,2)a2(t,1)}w(t,2) \ \mathbf{(72)},$$

which represents "speculators"' wealth.
Moreover, I have included in the code the mathematical expressions for the function alpha of both the informed traders and the uninformed ones. Knowing that the informed trader is the one who comprehend the market system, differently from the uninformed, I can write the system reported below to represent the function *alpha* for this trader:

$$\begin{cases} a1(t,1) = pi1; \\ a1(t,2) = 1 - pi1; \end{cases} \qquad \mathbf{(73)}.$$

It can be noticed that the value of the function α for the "fundamentalist" can be fixed equal to the real probability pi1.
Differently from the analysis just reported, I must consider the equation representing the function α for the "speculator" in a different way.
As discussed in the previous chapters, it is clear that the uninformed trader discerns information from the other investor and for this reason, I have to take into account the third equation of the system **(53)** presented in the fourth chapter, in particular:

$$\alpha^S_{(t-1),\sigma_t} = \hat{\Pi}^S_{k,\sigma_t}(q_{t-1}).$$

Considering the case in which $k=1$, I have taken into consideration equation **(60)**:

$$\pi = \frac{q}{\phi^F} - \frac{\phi^S}{2\phi^F};$$

the next step is to substitute into the variable q presented at the numerator of the equation above equation **(62)** ($Q_1(\pi)$), finally obtaining the function which embodies the value of $a2(t,1)$:

$$a2(t,1) = \frac{2(w(t,1)w(t,1))pi1 - (w(t,2)w(t,2)}{2(w(t,1)-w(t,2)}\frac{1}{w(t,1)} - \frac{w(t,2)}{2w(t,1)} \ \mathbf{(74)}.$$

On the other hand, knowing that $a2(t,1) + a2(t,2) = 1$ it is clear that:

$$a2(t,2) = 1 - a2(t,1). \ \textbf{(75)}$$

To give more stability to the model I have created in Matlab, I have decided to impose two constraints to the value assumed by $a2(t,2)$, in order to avoid excessive fluctuations in the value of the variable, due to the number of high iterations n considered during the construction of the model.

In particular, I have observed some fluctuations of the value of α corresponding to high values of n in the model with $k = 2$.

Being more precise, I impose that the value of $a2(t,2)$ and naturally also of $a2(t,2)$ cannot be higher than $0,99$ and lower than $0,01$.

The discussion just reported for the function which embodies the value of $a2(t,1)$ is applied to the model constructed considering $k = 1$.

For the construction of the model with $k = 2$ and, in particular, to define the value of the function α I consider firstly equation **(63)**:

$$\hat{\Pi}_2(Q_2(\pi)) = \frac{Q_2(\pi)}{\phi^F} + \frac{\phi^S(\phi^S - 2Q_2(\pi))}{2\phi^{2F}}.$$

The second step, as made previously, is to substitute the value of $Q_2(\pi)$ with equation **(64)** discussed in the section "$Q_k(\pi)$ represented as a function of the marginal wealth", having finally the subsequent relation:

$$a2(t,1) = \hat{\Pi}_2(Q_2(\pi)) = \frac{2(w(t,1)^3)pi1 + (w(t,2)^3}{2(w(t,1)^2 + w(t,2)^2 - (w(t,1)(w(t,2)} \frac{1}{w(t,1)} + \frac{w(t,2)^2}{2w(t,1)^2} - \frac{(w(t,2))(2(w(t,1)^3)pi1 + (w(t,2)^3)}{2(w(t,1)^2 + w(t,2)^2 - (w(t,1)(w(t,2)} \frac{1}{w(t,1)^2} \ \textbf{(76)}.$$

As in the code with $k = 1$, also in this case I take into account the constraints on the values of $a2(t,1)$ and $a2(t,2)$.

After having reported the functions which I have taken into account to create the code, I can show in the next section the results of some examples I have made with the code in Matlab, by repeatedly varying the values of n, $w1$, $w2$, $pi1$ and pi to understand how the value of both the price and the wealth of the two different types of investors change, and then comparing them with each other.

One of the main objectives of this study is to understand if the "speculators", starting from a lower knowledge of the market system, can learn and to discern information from the "fundamentalist" looking to the prices $q(t)$ used by the traders who have full knowledge of the market. I try to investigate if the uninformed traders are able to learn faster as the value of k increases.

5.2 The application of the code with function with k = 1

In this section, I try to explain the aim and the utility of the code created in Matlab, highlighting how investors' wealth and securities' prices vary changing the values assigned to the different variables taken into consideration.
Taking into consideration that fundamentalists have full knowledge of the market system, in every simulation I have attributed the same value both to the real probability pi and to the probability $pi1$ of the informed trader.
I have made different attempts with different values of the variables, both for the model with $k = 1$ and $k = 2$, looking at the behaviour of $w1$, $w2$, $q1$ and $q2$ represented in some graphs. The functions I have gradually changed are investors' wealth and the real probability $pi = pi1$.
Of course, infinite attempts could be made, continuing to vary the number of iterations between the two distinct types of traders, to see in each specific situation how the trend of the curves in the graphs varies and which state σ is the best solution for the two types of investors.
In this paper, to simplify things and make them less full-bodied , I have decided to represent, for both the two functions analyzed in Matlab, only some graphs to give an idea of the behavior of the variables in the two models, similarly to the graphs presented in [5].
Firstly, taking as a point of reference the code constructed with the value of k equal to 1, I have decided to fix the variable of investors' wealth and of the real probability of the realization in the future of the state of the world σ, varying in the three graphs reported only the value of the number of iterations n to see how the behaviour of the traders changes during a certain period of time.
Going into more detail, I have considered the following specific values for this analysis:

- $pi = pi1 = 0.3$
- $w1 = 0.7$
- $w2 = 0.3$
- $seed = 10$

Of course, the value of the variables just cited can be changed infinitesimal times getting always a recurring behavior of the output functions (investors' wealth and prices of securities.)
To carry out a detailed analysis and be able to obtain satisfactory and understandable results, I consider three pairs of graphs, the first with $n = 10$, the second with $n = 100$ and the third with $n = 1000$.
Each pair of graphs consists of one representing the wealth of the two investors at different times t and the other representing the price of the securities for the two distinct traders.
I show the graphs starting from the one obtained by inserting in the code the smallest value of n to see how the graphs evolve with the increase of n.

In Figure 5.1 I report the first pair of graphs constructed considering $n = 10$. As it can be

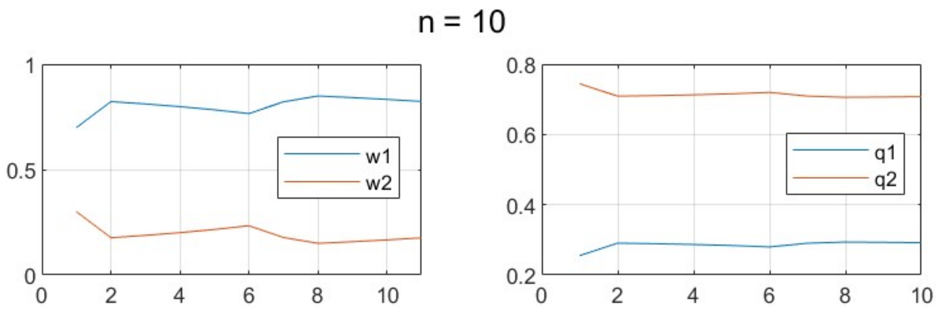

Figure 5.1: Function with K = 1, n = 10

noticed observing the trend of $w1$ and $w2$, it is clear that $w1 + w2 = 1$, which is in line with the equation **(33)**, reported in the section 3.5 "The relationship between the price q, α and the marginal wealth".

Putting now the attention on the stock prices trend, it can be observed that the price curve referring to "fundamentalists" tends to the value of 0.3. On the other hand the price curve referring to "speculators" tends to the value of 0.7.

Going on in the discussion and considering the case in which $n = 100$, it is clear, looking to Figure 5.2 presented below, that as the value of the number of iterations increases $w1$ tends to 1 and $w2$ tends to 0.

If uninformed trader's wealth tends to 0, the risk for the trader is to go out of the market.

Paying attention to the chart that represents the prices of the stocks, it is possible to see a tendency more and more emphasized of $q1$ towards 0.3 and of $q2$ towards 0.7. It is important to

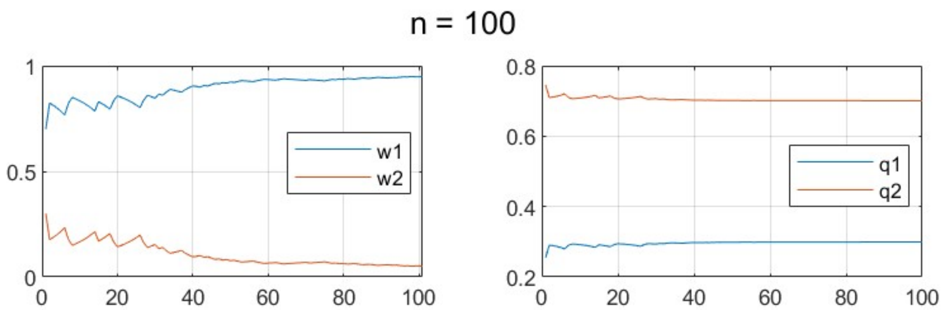

Figure 5.2: Function with k = 1, n = 100

underline that the main aim of the code created is to understand if the uninformed trader, who has heterogeneous beliefs if compaired to the informed ones, is able to learn and understand the market, observing the price fixed by the "fundamentalist".

As demonstrated from Figure 5.3, as the number of iterations increases the "speculators" are able to learn more and more market information, getting closer and closer to the knowledge of reality. This is proved by the trend of the prices which tend closer and closer to 0.7 for the uninformed traders and to 0.3 for the informed ones.

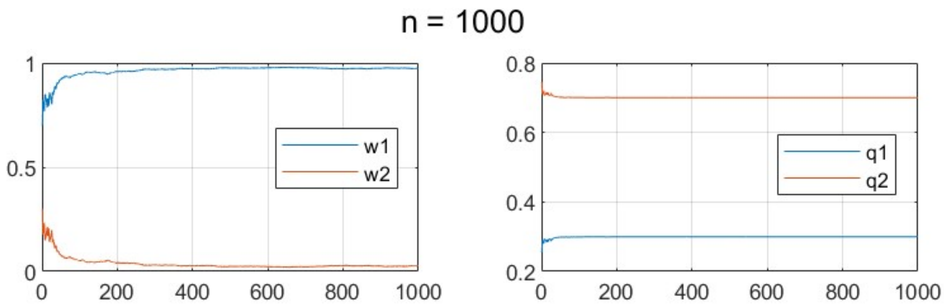

Figure 5.3: Function with k = 1, n = 1000

5.3 The application of the code with function with k = 2

In this section, I want to focus my attention on the analysis of the function with $k = 2$.
As in the model previously analyzed, I choose to fix some values for investors' wealth and for the real probability pi.
For simplicity, I have chosen the same values used in the code with the function with $k = 1$.
In particular: $pi = pi1 = 0.3$, $w1 = 0.7$, $w2 = 0.3$ and $seed = 10$.
It seems clear looking to Figure 5.4 that differently from Figure 5.1 the price curve seems to be more fluctuating both for the informed trader and for the uninformed one.

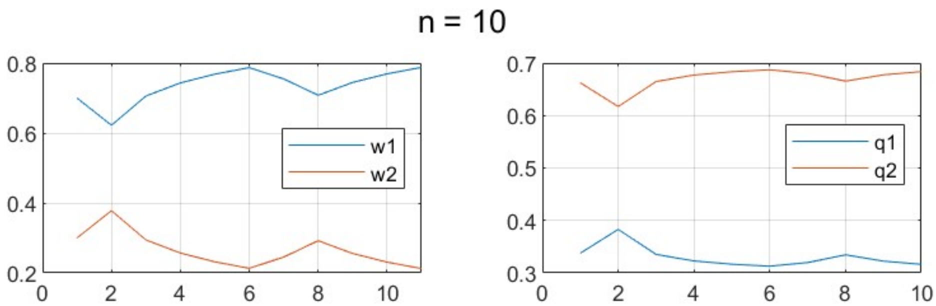

Figure 5.4: Function with k = 2, n = 10

This kind of trend is confirmed even more by the graph constructed considering as number of iterations $n = 100$, as represented in Figure 5.5, where in particular in the range of n between 40 and 60, an irregular trend can be observed in the curves of the two prices, which tend both towards the value of q equal to 0.5.
This behaviour suggests that the uninformed trader is not able to discern information and moves away from the knowledge of reality.

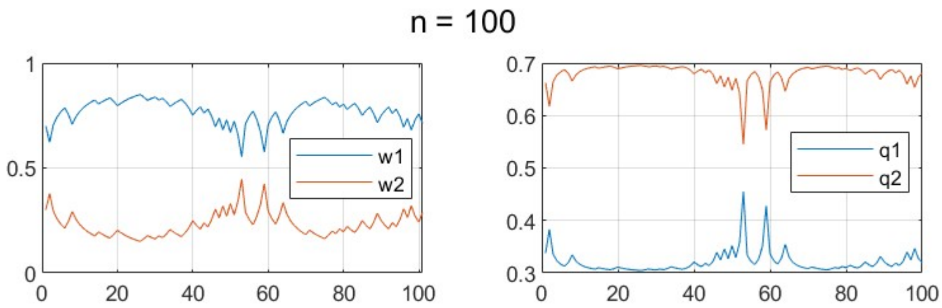

Figure 5.5: Function with k = 2, n = 100

At the end, looking at the graph constructed in Figure 5.6 considering $n = 1000$, it is possible to notice a further irregular behavior in correspondence of a value of n little more than 100.

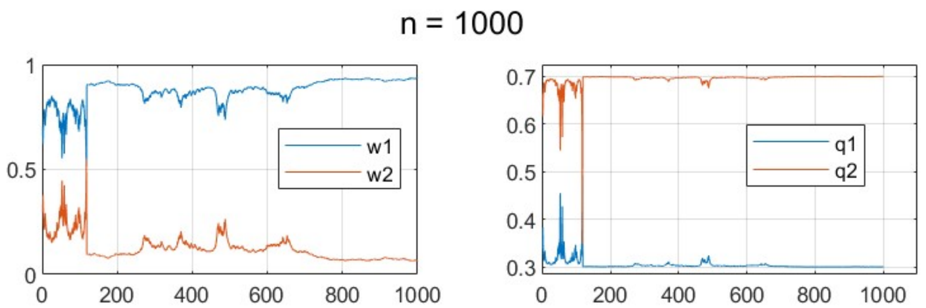

Figure 5.6: Function with K =2, n = 1000

Then, the trend resumes regularly, therefore it can be said that even with $k = 2$ "speculators" slowly manage to learn information by observing the price fixed by "fundamentalists", managing to know the market more and more.
Unlike the graphs with $k = 1$, the speculator's wealth with $k = 2$ tends to zero less quickly. For this reason the investor manages to survive longer, remaining competitive in the market.

5.4 How the ability of "speculators" to learn over time varies

It is important to underline that one of the main purposes of making the two codes in Matlab is to understand how "speculators" behave over time, how much they can learn by observing the prices of "fundamentalists" and how quickly they can actually get closer to the knowledge of the markets and the truth.
After analyzing with specific graphs the differences between the riches of the different types of

investors and the prices of the stocks in the model with $k = 1$ and in the model with $k = 2$, I want to continue and conclude the discussion by creating two more codes in Matlab, for the values of $k = 1$ and $k = 2$.
With these codes, I try to analyze the situation with a more general perspective and not limited to the three graphs constructed using the three different values of the number of iterations.
The goal is to understand if uninformed traders over time can understand more quickly the information coming from informed traders, who of course know the reality but can still have a distorted view of the market.
In this particular case, "speculators" who rely on the prices of stocks set by "fundamentalists" would learn incorrectly, arriving at a distorted view of reality.
Turning back to the two new codes I have just cited, I present the differences that I have introduced with respect to the codes analyzed in sections 5.2 and 5.3.
I have constructed the new codes in the same way as the ones presented in the appendix below, adding some elements which are necessary to provide a more general and broad view of the variables taken into account.
Going into more detail, the variables considered as output of the single function are no more $w, q, s, a1, a2$, as in the code presented in the appendix, but y, pi and $pi1$.
In particular, y is defined as a matrix $y = \begin{bmatrix} N & N \end{bmatrix}$ and $pi = pi1 = \begin{bmatrix} N & 1 \end{bmatrix}$.
Since i and j go from 1 to N, I have defined the real probability and the probability of the informed trader as follow:

- $pi(i, 1) = i/(N + 1);$
- $pi1(j, 1) = j/(N + 1)$.

After having reported the expressions representing stock prices, investor's wealth and the function α, having regard to a period of time t that is included between $n999/1000+1$ and n, I have defined the value of $y(i, j)$ which is equal to:

$$y(i, j) = y(i, j) + (w(t, 2)1000)/n. \quad \textbf{(77)}$$

A further discussion must be made on the variables that I have considered as an input to obtain satisfactory results in the codes.
As discussed before, I maintain the value of the *seed* constant and equal to 10, as the value of traders' wealth, which are respectively equal to 0.7 for the informed traders and to 0.3 for the uninformed ones.
Then, I have fixed N equal to 9 in order to obtain a matrix $\begin{bmatrix} 9 & 9 \end{bmatrix}$, and I consider a high number of iterations equal to 10000.
Running the codes, I have obtained two tables representing the values of $y(i, j)$ in different instants of time.
Starting from the discussion of the code in which I take into consideration $k = 1$, it is possible to

see by looking at the table underlying a characteristic element.
In fact, regardless of the row considered, in the column corresponding to the value equal to 0.5 it is possible to notice a recurrence of 0.3.
As analyzed in the previous chapters, if informed traders use a probability corresponding to 0.5, then uninformed traders will also have an equal probability.
Looking at the table, it is possible that the values in the boxes near the main diagonal are larger than the outer values, which are closer to zero faster. This behavior is given by the fact that the more I approach the diagonal, the more the uninformed traders approach the knowledge of the truth.
It should be kept in mind that the starting point of the discussion of this work is that the two different types of investors considered in the model have heterogeneous beliefs about market knowledge.
The purpose of uninformed traders is to capture as much information as possible from informed traders who have a wider knowledge.

		j								
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
i	0.1	0.01	0.08	0.05	0.02	0.3	4.8e-04	1.7e-04	7.6e-05	2.8e-05
	0.2	2,2e-04	0.01	0.05	0.07	0.3	6e-04	2.1e-04	8.9e-05	3.2e-05
	0.3	1.1e-04	5.1e-04	0.02	0.06	0.3	7.9e-04	2.6e-04	1.1e-04	3.7e-05
	0.4	7.4e-05	2.6e-04	9.9e-04	0.02	0.3	0.001	3.5e-04	1.3e-04	4.5e-05
	0.5	5.5e-05	1.7e-04	5.02e-04	0.002	0.3	0.002	5.4e-04	1.8e-04	5.7e-05
	0.6	4.4e-05	1.3e-04	3.4e-04	0.001	0.3	0.02	0.001	2.7e-04	7.7e-05
	0.7	3.7e-05	1.04e-04	2.5e-04	7.7e-04	0.3	0.06	0.03	5.75e-04	1.2e-04
	0.8	3.2e-05	8.7e-05	2.1e-04	5.9e-04	0.3	0.14	0.11	0.02	2.4e-04
	0.9	2.8e-05	7.6e-05	1.7e-04	4.7e-04	0.3	0.04	0.02	0.05	0.01

Table 5.1: Representation of the values of $y(i, j)$ in correspondence of $k = 1$.

But it is important to keep in mind that "fundamentalists" do not always have a correct vision of reality and truth and in this case learning quickly for "speculators" is definitely not the best solution to make up for the lack of information. In this specific case for the "speculator" it would be better to learn more slowly.
On the contrary, if the information transmitted by the "fundamentalist" is in line with the truth, the faster the other investor learns the better.
Moreover, comparing the tables, both consisting of nine rows and nine columns obtained using the last two codes made in Matlab, it is possible to notice a big difference.

		j								
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
i	0.1	0.032	0.009	0.006	0.005	0.004	0.003	0.0026	0.002	0.0013
	0.2	0.05	0.04	0.009	0.006	0.005	0.004	0.003	0.002	0.0014
	0.3	0.25	0.03	0.04	0.008	0.006	0.004	0.003	0.002	0.0015
	0.4	0.03	0.06	0.07	0.03	0.008	0.005	0.004	0.003	0.002
	0.5	0.02	0.07	0.06	0.13	0.03	0.007	0.005	0.003	0.002
	0.6	0.05	0.05	0.02	0.02	0.1	0.04	0.007	0.004	0.002
	0.7	1	0.02	0.02	0.1	0.04	0.05	0.04	0.005	0.003
	0.8	1	1	1	0.02	0.09	0.08	0.08	0.03	0.004
	0.9	1	1	1	1	1	1	0.02	0.03	0.025

Table 5.2: Representation of the values of $y(i, j)$ in correspondence of $k = 2$.

Going specifically, in correspondence of the boxes having the same value for row and column (es: 0.1 - 0.1, 0.2 - 0.2, ...), in the table obtained by inserting in the code the expressions for $k = 1$ the values are in all cases lower than those obtained in the other table.
In these specific cases, the values for $k = 2$ tend to go to zero slowly if compared to the ones gained with the expressions for $k = 1$.
To conclude the discussion, it is important to point out that not always at $k = 2$ traders who have less information about the market can learn faster if compared to the same traders at level $k = 1$.
To support the above statement, it is possible to observe that by comparing some cells in both tables, values for $k = 2$ are not always higher than values for $k = 1$. Going into more detail, it is possible to locate larger values for $k = 1$ in correspondence of cells located in the upper left and lower right near the diagonal (for example: considering the cell as (i;j) some examples are: $(0.1; 0.3)$, $(0.1; 0.4)$, $(0.8; 0.7)$, $(0.8; 0.6)$).

Conclusion

As mentioned in the introduction, I have articulated the work in five chapters. After giving a general introduction to the main topics in the first chapter, based on papers by various economists, I have analyzed a static ecomomic model with complete markets in the second chapter.
In the chapter just mentioned, I focus in particular on the figures of informed traders and uninformed traders. I have analyzed the demand functions and some variables, as wealth, securities prices, fot both the category of investors. The main aim is to try to understand if the "speculators", who have heterogeneous beliefs with respect to the "fundamentalist", are able to fill the lack of information.
Proceeding with the third chapter, I have tried to expand my analysis, introducing further variables, such as alpha and marginal wealth, considering the market selection with a level-k optimization. The aim is to deal with a more dynamic model, highlighting in particular the relationships that bind together the various variables presented within the elaborated.
Moreover, after analyzing the wealth of investors at time $t = 0$ and $t = 1$, I have applied the final wealth to all T states of the world.
In the fourth chapter I have realized a detailed analysis of the ex-ante utility function of both informed traders and uninformed ones.
Thanks to these maximizations, I have obtained the formulas that I then come back useful in the fifth and last chapter, using them within the codes I have created in Matlab.
Going into more detail, I have achieved the expressions representing respectively the investors' wealth, the α function, the price of the stocks and the probability that a particular σ state will be realized in the future.
Finally, in the last chapter, thanks to Matlab, I have realized some codes using some expressions I have treated in the previous chapters.
I have decided to analyze the figures of the two different types of traders, in particular of their wealth and prices, with the aid of some charts and two tables.
After dealing with the main topics of the various chapters,it is important to highlight that the main aim of the model analyzed is to study and understand if the uninformed trader is able to discern information of the market system, taking as a point of reference the stock price of an informed trader.

It may seem obvious that by going forward with time and increasing the value of k, "speculators" are able to learn information faster, filling their gap with the other category of investors and reaching the truth in no time.

Going into more detail and looking to the graphs and the two tables reported in the fifth and last chapter, I have demonstrated that this belief is not always true.

In fact, it is always important to pay attention to the information given and transmitted by the "fundamentalist". The information transmitted by informed traders is not always reliable.

In this case, the faster the "speculator" learns and the faster he moves away from the truth, reducing in time his general wealth.

Therefore, in specific situations it is convenient for the "speculators" to analyze in detail the information given by the choices made by the "fundamentalists", so as to remain as long as possible within the market and increase their wealth over time.

In the end, it is of primary importance to underline that in general, thanks to the analyzed model, it is possible to conclude that uninformed traders can quickly fill the lack of information they have with respect to informed traders.

This allows them to increase their wealth and become competitive within the market.

Appendix

Function with $k = 1$.

```
function [w,q,s,a1,a2]=simprice_example1_2final(n,pi,w1,w2,pi1,seed)

% CODE GENERATED CONSIDERING THE VALUE OF K = 1

% simultation with two states sigma = 1 and sigma = 2
% n is the number of iterations
% w1 is the initial wealth of the informed trader
% w2 is the initial wealth of the uninformed trader

% a1 is the function alpha associated to the infomed trader
% a2 is the function alpha associated to the uninformed trader
% pi1 is the probability associated to the infomed trader
% pi2 is the probability associated to the uninformed trader
% pi is the real probability


format short e

% parameters

 rand("seed", seed);

% 2 agents (informed trader and the uninformed one), 2 states



w=zeros(n+1,2);
q=zeros(n,2);
s=zeros(n+1,1);
```

```
a1=zeros(n,2);
a2=zeros(n,2);
chance=zeros(n+1,1);



%initial conditions

w(1,:)=[w1,w2];
w(2,:)=[w1,w2];



%loading the stochastic process

for t=1:n
chance(t,1)=rand(1,1);
if (chance(t,1) >= 0) && (chance(t,1) <= pi)
  s(t+1,1)=1;
 else
    s(t+1,1)=2;
end
end


%iteration of 2 agents

q(1,1)=0.5;
q(1,2)=0.5;
%a1(1,1)=pi1;




for t=1:n

%finding the equilibrium price, past prices

a1(t,1)=pi1;
a1(t,2)=1-pi1;
```

```



% controllo
%q(t,1)=q(t-1,1);
%q(t,2)=q(t-1,2);

%q(t,1)=(2*(w(t,1)*w(t,1))*pi1 - (w(t,2)*w(t,2)))/(2*(w(t,1) - w(t,2)))
    ;
%q(t,2)=1 - q(t,1);

a2(t,1)=(((2*(w(t,1)*w(t,1))*pi1 - (w(t,2)*w(t,2)))/(2*(w(t,1) - w(t,2)
    )))/w(t,1)) - (w(t,2)/(2*w(t,1)));

 if       a2(t,1) < 0.01
          a2(t,1) = 0.01;
 elseif   a2(t,1) > 0.99
          a2(t,1) =0.99;

end

a2(t,2)=1 - a2(t,1);


q(t,1)=w(t,1)*a1(t,1)+w(t,2)*a2(t,1);
q(t,2)=w(t,1)*a1(t,2)+w(t,2)*a2(t,2);

 if (s(t+1,1)==1)
   w(t+1,1)= (a1(t,1)/(w(t,1)*a1(t,1)+w(t,2)*a2(t,1)))*w(t,1);
w(t+1,2)= (a2(t,1)/(w(t,1)*a1(t,1)+w(t,2)*a2(t,1)))*w(t,2);
%w(t+1,2)= 1 - w(t+1,1);

  else
    w(t+1,1) = (a1(t,2))/(w(t,1)*a1(t,2)+w(t,2)*a2(t,2))*w(t,1);
w(t+1,2) = ((a2(t,2))/(w(t,1)*a1(t,2)+w(t,2)*a2(t,2)))*w(t,2);
%w(t+1,2)= 1 - w(t+1,1);
end
end
```

Function with $k = 2$.

```
function [w,q,s,a1,a2]=simprice_example1_2final2q(n,pi,w1,w2,pi1,seed)

% CODE GENERATED CONSIDERING THE VALUE OF K = 2

% simulation with two states sigma = 1 and sigma = 2
% n is the number of iterations
% w1 is the initial wealth of the informed trader
% w2 is the initial wealth of the uninformed trader
% a1 is the function alpha associated to the infomed trader
% a2 is the function alpha associated to the uninformed trader
% pi1 is the probability associated to the infomed trader
% pi2 is the probability associated to the uninformed trader
% pi is the real probability

format short e

% parameters

rand("seed", seed);

% 2 agents (informed trader and the uninformed one), 2 states


w=zeros(n+1,2);
q=zeros(n,2);
s=zeros(n+1,1);
a1=zeros(n,2);
a2=zeros(n,2);
chance=zeros(n+1,1);


%initial conditions

w(1,:)=[w1,w2];
w(2,:)=[w1,w2];
```

```
%loading the stochastic process

for t=1:n
chance(t,1)=rand(1,1);
if (chance(t,1) >= 0) && (chance(t,1) <= pi)
  s(t+1,1)=1;
 else
   s(t+1,1)=2;
end
end



%iteration of 2 agents

q(1,1)=0.5;
q(1,2)=0.5;
%a1(1,1)=pi1;



for t=1:n

%finding the equilibrium price, past prices

a1(t,1)=pi1;
a1(t,2)=1-pi1;



% controllo
%q(t,1)=q(t-1,1);
%q(t,2)=q(t-1,2);

%q(t,1)=(2*(w(t,1)^3)*pi1 + (w(t,2)^3))/(2*(w(t,1)^2 + w(t,2)^2 - (w(t
    ,1)*(w(t,2)))));
%q(t,2)=(2*(w(t,1))^2*(1 - 2*(w(t,1)*pi1)) + (w(t,2)^2*(2 - w(t,2))) -
```

```
        2*w(t,1)*w(t,2)) /(2*(w(t,1)^2 + w(t,2)^2 -w(t,1)*w(t,2)));

a2(t,1)=((2*(w(t,1)^3)*pi1 + (w(t,2)^3))/(2*(w(t,1)^2 + w(t,2)^2 - (w(t
        ,1)*(w(t,2)))))/w(t,1) + (w(t,2)*(w(t,2) - 2*((2*(w(t,1)^3)*pi1 +
         (w(t,2)^3))/(2*(w(t,1)^2 + w(t,2)^2 - (w(t,1)*(w(t,2))))))))/2*w(
        t,1)*w(t,1);

if          a2(t,1) < 0.01
            a2(t,1) = 0.01;
elseif      a2(t,1) > 0.99
            a2(t,1)  =0.99;

end

a2(t,2)=1 - a2(t,1);


q(t,1)=w(t,1)*a1(t,1)+w(t,2)*a2(t,1);
q(t,2)=w(t,1)*a1(t,2)+w(t,2)*a2(t,2);

if (s(t+1,1)==1)
  w(t+1,1)= (a1(t,1)/(w(t,1)*a1(t,1)+w(t,2)*a2(t,1)))*w(t,1);
w(t+1,2)= (a2(t,1)/(w(t,1)*a1(t,1)+w(t,2)*a2(t,1)))*w(t,2);
%w(t+1,2)= 1 - w(t+1,1);


 else
   w(t+1,1) = ((a1(t,2))/(w(t,1)*a1(t,2)+w(t,2)*a2(t,2)))*w(t,1);
w(t+1,2) = ((a2(t,2))/(w(t,1)*a1(t,2)+w(t,2)*a2(t,2)))*w(t,2);
%w(t+1,2)= 1 - w(t+1,1);
end
end
```

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