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Wealth Distribution and Capital Accumulation under Incomplete Financial Markets and Income Shocks

Supervisors
Ch. Prof. Silvia Faggian
Ch. Prof. Pietro Dino Enrico Dindo

Graduand
Angelica Tiozzo Caenazzo Anzolin
Matriculation Number 882241

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Abstract

The purpose of this thesis is to investigate the effect of uncertainty and incomplete financial markets on interest rates and on the distribution of wealth, namely by studying the impact caused by the introduction of some further form of market uncertainty and the transition of the interest rate and of the wealth distribution curve from a stationary equilibrium to another in financial markets. To this end, this work is primarily based on the analysis of the Aiyagari-Bewley-Huggett heterogeneous agent model of income and wealth distribution in continuous time; its contribution consists in a generalization of the model, in order to investigate the effect of labour income shocks on the interest rates, wealth distribution and risk premium in the case in which agents can self-insure themselves by investing, besides in riskless bonds, also in risky assets. In particular, this extended version is modeled within the framework of Mean Field Games and solved numerically using MATLAB codes.

Keywords: Heterogeneous agents, Interest rate, Wealth distribution, Incomplete markets, Income shocks, Stationary equilibrium, Mean Field Games

1 Introduction

In this work, a quantitative investigation about the introduction of risky investment opportunities within the investable universe of individuals is performed, in particular by assessing the impact of this condition on the risk-free interest rate and on the net worth distribution of individuals by means of a continuous-time general-equilibrium model characterized by an incomplete market wherein a continuum of heterogeneous households allocate their wealth in both unproductive risk-free assets (namely, bonds) and risky assets (in the only form of productive capital) and are subject to uninsured idiosyncratic risk by means of labour income shocks. This structure is founded on the basic implementation of the heterogeneous agent models defined by Achdou et al. (2021), that recasts in continuous time the standard incomplete market models of Huggett (1993), Bewley (1986) and Aiyagari (1994). In particular, the model under analysis can be interpreted as a combination of the parsimonious market model of Huggett, where individuals can only save in unproductive bonds, and the more sophisticated Aiyagari model, where the only asset available for investment is risky and productive capital: this market model includes both bonds and capital in the investable universe of individuals and differs from the set-up of Aiyagari in that it is implemented by a stochastic depreciation on capital.

To be more specific, the main features of the model under consideration are detailed as follows. The economy comprises both heterogeneous individuals and firms. First of all, agents solve an individual optimization problem by optimally choosing their consumption level (thus, defining their consumption and savings policies) and their wealth allocation in risky capital, while the remaining portion of their net worth is invested in riskless assets. They also provide the firms with a labour endowment, which is remunerated at a competitive wage. Then, in every time period, agents are subject to idiosyncratic shocks on their labour income, implying that their heterogeneity results from both their income and their wealth. At the same time, firms borrow capital and use labour for production, undergoing idiosyncratic uncertainty as well, in the form of stochastic depreciation on the value of capital.

The assumption of market incompleteness, namely the inability for households to self-insure against risk (Levine and Zame (2002)), indicates that agents do not have the possibility of writing insurance contracts contingent on the shocks on their labour income (Allais et al. (2020)); in addition to this, it is fundamental to assume also independence between the different sources of uncertainty for individuals, i.e. labor income shocks and risky investments: the presence of some pattern of dependence between these two would allow agents to benefit form a sort of partial insurance. Furthermore, in reaching their optimal allocation, individuals face a borrowing constraint, meaning that they cannot exploit additional loans to fully smooth out their individual income shocks (Allais et al. (2020)). It must also be noted that, consistently with the tradition of relevant literature, the model does not encompass any contribution of social security for income shocks as well, with the result that agents need to figure out how to self insure themselves; in this respect, the only means at their disposal are investments in bond and capital. As a consequence, labor income fluctuations result to be uninsurable for households, except for the possibility to invest in riskless and risky assets.

The general-equilibrium feature of the model implies that the prices or quantities characterizing the economy are endogenously determined. In the specific case of this work, the endogenous dimensions, which are determined by the balance between demand and supply in the market, are the riskless interest rate and the aggregate level of risky capital. Additionally, the wage level for labour and the rental rate of capital, in equilibrium, are exactly the same for all the individuals and firms, respectively, and depend on the aggregate capital level: this means that, although they are exogenous at the individual level, they are endogenously determined in the aggregate.

Aggregate uncertainty or shocks are not covered under this analysis; therefore, given the mere presence of purely idiosyncratic shocks, households face a constant level of interest rate in a steady state. Then, for a given interest rate, optimal individual consumption and saving policies lead to a distribution of agents with different levels of assets, as a consequence of their different histories in terms of labour endowment shocks (Aiyagari (1994)).

This model is encompassed within the broader class of continuous-time incomplete-market heterogeneous-agent models, which are analytically described by a system of two coupled partial differential equations (hereinafter, PDE): first, the so-called Hamilton-Jacobi-Bellman (HJB, for short) Equation characterizes (by means of the so-called *value function*) the optimal choices of an individual agent, who considers the evolution of the distribution as given; secondly, given the optimal policies of the individuals, the so-called Kolmogorov-Forward (KF, shortly) Equation (also called Fokker-Plank Equation) determines the evolution of the distribution. This set-up can be solved in the context of the mathematical theory of Mean Field Games (MFG), which was first introduced by Larsy and Lions (2007) and where the system of coupled HJB and KF Equations is known as the "backward-forward MFG system". Referring to the market model under analysis, the HJB Equation characterizes the optimal consumption and saving policies of individuals, given a stochastic process for the shocks on income and another for the depreciation of capital, whereas the KF Equation characterizes the evolution of the joint distribution of income shocks and wealth. The two equations are coupled because optimal consumption and saving depend on the interest rate which is determined in equilibrium and hence depends on the wealth distribution.

With reference to the numerical resolution, the model is implemented in MATLAB by means of an algorithm based on a finite difference method and a bisection approach. In a nutshell, in the first place the HJB Equation is solved for a given value of the aggregate capital and a given time path of prices (i.e. interest rate); secondly, given the optimal consumption and saving policies of individuals, the KF

Equation is solved for the evolution of the joint distribution of income shocks and wealth. Lastly, the algorithm is iterated and the first two steps are repeated until the equilibrium interest rate and aggregate level of capital are found.

The main contribution of this work is twofold. First, from an economic and financial perspective, it analyzes the consequences of the model under considerations on individuals and on the economy, mainly providing some useful insights on the impact on the interest rate and on the distribution of wealth and inequality. Briefly, the main results in the context of greater uncertainty suggest the following effects: the riskless interest rate is reduced, as well as consumption, whereas savings increase; an increase in the riskiness of capital causes the risk premium of capital (and also its Sharpe Ratio) to increase; inequalities in the wealth distribution are found to be dependent on the output elasticity of capital. In the second place, this work provides the methodological approach with which the HJB and KF Equations are derived, and shows how, respectively, the value function and the joint density of labour income shocks and wealth are solutions to such equations, with arguments that are more than mere heuristics.

This work is organized as follows. Section 2 describes the market model, detailing the mathematical set-up and reasonings. Section 3 illustrates the Dynamic Programming Method in the specific case under analysis, providing a detailed derivation of the HJB Equation, while the KF Equation is derived in Section 4. Section 5 presents the numerical algorithm implemented for the resolution of the model (whereas the exact MATLAB codes are disclosed in the Appendix) and illustrates the results obtained. Section 6 outlines and comments on the results, providing an economic interpretation. Section 7 concludes and gives the reader insights on some possible future work.

2 A Mathematical Description of the Model

Consider a continuos-time infinite-horizon market model. As in Achdou et al. (2021), there exists a continuum of households (with unit mass), who are heterogeneous in terms of their wealth a_t, and of the shocks y_t to which they are subject on their labour income. Individuals have standard preferences over utility flows from future consumption c_t, and discount the future with discount factor $\beta(t) = e^{-\rho t}$, with rate $\rho \geq 0$. The functional of such a problem is indicated as follows[1]:

$$J = \mathbb{E}\left(\int_0^{+\infty} e^{-\rho t} u(c_t) dt\right), \tag{2.1}$$

where the expected value will be later specified. The function u is assumed strictly increasing and strictly concave; it is defined as the Constant Relative Risk Aversion (CRRA) utility function with coefficient of relative risk aversion $\gamma > 0$:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}. \tag{2.2}$$

Differently from Achdou et al. (2021), the economy includes also competitive and risk-neutral firms. At time t, individuals provide the firms with their labour endowment l_t, which is remunerated at the competitive wage w_t. The labour supply is provided inelastically by individuals, meaning that their labour endowment is supplied to the firms entirely, and independently of the level of the wage. Hence, agents do not exhibit other preferences besides work, and individual labour supply can be standardized to a unit: $l_t = 1$[2]. Moreover, the agents are subject to idiosyncratic risk in the form of exogenous shocks, y_t, on their fixed wage. This idiosyncratic exogenous uncertainty can be interpreted as a shock on the agents' idiosyncratic labour endowment. It is assumed that the wage shock follows a two-state Poisson process $y_t \in \{y_1, y_2\}$, with $y_2 > y_1$. The two states can be interpreted as sickness (y_1) and health (y_2): although agents supply their labour inelastically, the ones who get sick can only provide a smaller endowment of labour than those who are healthy. The process y_t jumps from state 1 to state 2 with intensity λ_1 and vice versa with intensity λ_2: in this respect, λ_1 is the recovery rate and λ_2 the sickness rate. However, in the aggregate, since agents have unit mass and sickness and health conditions offset themselves, the values of the switching intensities λ_1 and λ_2 are set such that the aggregate labour supply equals a unit: $L = 1$.

Households choose their level of consumption, c_t, as well as the allocation of their net worth, a_t, between risk-free bonds, b_t, and physical capital, k_t, in order to differentiate investments in riskless and risky

[1] The functional J depends on the consumption function $t \mapsto c_t$ (indicated also with c, when no ambiguity may arise) and will also be made dependent, in Section 3, through the state equation, on the capital holdings function $t \mapsto k_t$ (similarly, k), on the initial instant of time t_0 and on the corresponding initial state of the economy (a, y) in terms of wealth and income shocks. Then, $J = J(c; k; t_0; a; y)$. In the case of (2.1), the initial time is $t_0 = 0$.

[2] More specifically, the rationale behind the inelasticity of labour can be summarized as follows. First, agents do not experience disutility from labour, since their utility function depends only on consumption. Furthermore, when they provide firms with their labour endowment, agents are always remunerated with a positive wage. As a consequence, the optimal choice for individuals it to supply labour inelastically.

assets on the market. Therefore, the following balance sheet constraint holds at time t:

$$a_t = b_t + k_t. \tag{2.3}$$

Bonds evolve with dynamics:

$$db_t = r_t b_t dt, \tag{2.4}$$

where r_t is the endogenous risk-free rate on bonds, to be determined at equilibrium.

The capital value k_t is an Itô process whose variation in time, dk_t, will be specified after a few preliminary considerations.

Firms are risk-neutral. At each instant of time, they produce an output good by means of a standard Cobb-Douglas production technology:

$$Y_t = A k_t^{\alpha} l_t^{1-\alpha}, \tag{2.5}$$

whose inputs are physical capital k_t and labour l_t, and with A the total-factor productivity and $\alpha \in (0, 1)$ the capital share parameter, denoting the relative weight assigned to capital (with respect to labour) in the production process. At the same time, firms bear a cost from borrowing their production inputs from households at rates μ_t (capital rental rate) and w_t (wage), both of which are determined endogenously as a solution of the following (static) maximization problem for instantaneous profits:

$$\max_{k_t, l_t} P(k_t, l_t) \equiv \max_{k_t, l_t} \left\{ Y_t - \underbrace{(k_t \mu_t + l_t w_t)}_{Production\ costs} \right\}, \tag{2.6}$$

where $k_t \mu_t$ is the cost of capital, and $l_t w_t$ the cost of labour, both remunerated to households. The first-order conditions of the problem lead to the following optimal choices for firms in terms of capital rental rate and labour wage:

$$\frac{\partial P(k_t, l_t)}{\partial k_t} = 0 \Leftrightarrow \mu_t^* = \alpha A k_t^{\alpha-1} l_t^{1-\alpha}, \qquad \frac{\partial P(k_t, l_t)}{\partial l_t} = 0 \Leftrightarrow w_t^* = (1-\alpha) A k_t^{\alpha} l_t^{-\alpha}, \tag{2.7}$$

showing that both the two optimal remuneration rates paid by firms depend on the capital, labour and technology employed in the production process, but the capital rental rate is proportional to the relevance attributed to capital with respect to labour, and vice versa for wage. Given the optimal strategies in eq. (2.7), firms' profits as defined in eq. (2.6) result to be always zero (verify that $\frac{\partial Y_t k_t}{\partial k_t} + \frac{\partial Y_t l_t}{\partial l_t} = Y_t$; in other words, all output is spent to remunerate the production factors).

Since firms are competitive in the markets for capital and for labour, they all set the same rental rate on capital and the same labour wage at equilibrium. Therefore, by substituting the aggregate labor supply, L

(which equals 1, as previously explained) and the aggregate capital level (to be determined at equilibrium), K, in the optimal firm choices, one gets that the rental rate is defined as:

$$\mu^* = \mu^*(K) \equiv \alpha A K^{\alpha-1}, \tag{2.8}$$

where $\mu^*(K)$ represents the deterministic component (which is endogenous and depends on the aggregate capital) of the return on capital. Similarly, the equilibrium wage is a function of aggregate capital as well and is obtained as:

$$w^* = w^*(K) \equiv (1-\alpha) A K^{\alpha}. \tag{2.9}$$

As a consequence, α can be interpreted as the share of the firms' profits that are allocated to capital remuneration, whereas the remaining $(1-\alpha)$ is the share set aside to pay for the labour supply.

Finally, in order to assess the evolution in time of capital k_t, it is assumed that firms return capital to households (after borrowing) at rate μ_t^*, and that capital depreciates at the stochastic rate $\delta dt + \sigma dW_t$, as in Waelde (2011). In particular, dW_t is a standard Brownian Motion that indicates a firm-specific (idiosyncratic) shock, representing the risk borne by firms when deciding to borrow capital from individuals. Depreciation shocks are assumed to be independent and idiosyncratic across households. Accordingly, individual households' capital holdings evolve with the following dynamics:

$$dk_t = k_t \left[(\mu_t^* - \delta) dt - \sigma dW_t \right], \tag{2.10}$$

where μ_t^* is given by eq. (2.7). Without loss of generality, it also assumed that $\delta = 0$, so that the expected depreciation for unit of capital stock equals zero:

$$dk_t = k_t \left(\mu_t^* dt - \sigma dW_t \right), \tag{2.11}$$

Rewriting the riskless holding as a function of wealth and capital, $b_t = a_t - k_t$, the dynamic evolution of wealth is then given by:

$$\begin{aligned} da_t = db_t + dk_t &= \left[r_t (a_t - k_t) + y_t w_t^* - c_t \right] dt + k_t \left(\mu_t^* dt - \sigma dW_t \right) \\ &= \left(r_t a_t + y_t w_t^* - c_t + (\mu_t^* - r_t) k_t \right) dt - \sigma k_t dW_t \\ &= r_t a_t dt + y_t w_t^* dt + k_t \left[(\mu_t^* - r_t) dt - \sigma dW_t \right] - c_t dt. \end{aligned} \tag{2.12}$$

It is also assumed that W_t and y_t are independent random variables. The independence assumption between the Brownian Motion W_t that characterizes the uncertainty on the return on capital and the

Poisson process y_t of labour income shocks is fundamental for the construction of the market model. Indeed, as a consequence of independence, the two sources of uncertainty in the model turn out to be uncorrelated: although investing in capital (as well as in bonds) provides individuals with a sort of means of partial-insurance against income uncertainty, the absence of correlation implies that the market is still incomplete, given that agents cannot totally insure themselves against risk.

Altogether, the state of the economy is the joint distribution $x_t = (a_t, y_t)$, while the strategy is the couple $h_t = (c_t, k_t)$[3], and is described by the evolution system:

$$\begin{cases} da_t = r_t a_t dt + y_t w_t^* dt + k_t \left[(\mu_t^* - r_t) dt - \sigma dW_t \right] - c_t dt, & t \in [0, +\infty) \\ dk_t = k_t \left(\mu_t^* dt - \sigma dW_t \right), & t \in [0, +\infty) \\ y_t \in \{y_1, y_2\}, & t \in [0, +\infty) \\ a_0 = a, \; y_0 = y, \end{cases} \tag{2.13}$$

where the μ_t^* and w_t^*, representing the optimal choices taken by firms in terms of capital rental rate and labour wage at time t, are taken as given by households.

Additional constraints on the state and on the controls are required: consumption is naturally non-negative, and the same non-negativity constraint is posed on capital holdings as well, while total wealth must not be lower than a given limit. Altogether:

$$c_t \geq 0, \quad a_t \geq -\underline{a}, \quad k_t \geq 0$$

that translates, as a consequence of the budget constraint, into a restriction on bond holdings, as follows:

$$c_t \geq 0, \quad b_t \geq -k_t - \underline{a}, \quad k_t \geq 0 \tag{2.14}$$

meaning that risky capital cannot be shorted, while the short position on bonds cannot exceed the amount of capital holdings owned, indicating that indebtedness is allowed up to a certain limit in this economy. Hence, the original condition on wealth generates actually a leverage constraint, posing a limitation on the relationship between the amount of bond and capital: at the borrowing constraint, the value of leverage is higher than 1.

In addition to this, it becomes necessary to underline that, in equilibrium, the rates of return on capital

[3] In terms of controls, besides capital holdings, agents optimally choose the labour endowment l_t to provide to firms as well. However, households supply labour inelastically to firms and, as a consequence, their individual labour endowment is standardized to 1, which corresponds to the agents' optimal choice, $l_t = 1$. Hence, l_t is omitted from the control function.

and on bond are determined such that the following relationship is satisfied:

$$\mu^* > r, \tag{2.15}$$

according to which the risky rate of return must be higher than the riskless rate. In other words, it must be convenient for agents to invest in risky capital, since they are assumed to be risk averse. Indeed, the lack of investment in risky capital would cause the absence of future income, given that capital is related by a certain degree of complementarity to labour in the production process: without capital, labour cannot be employed to produce output and, thus, to generate income for agents through their labour wage.

The problem that is addressed is that of choosing the consumption levels c_t and the individual capital holdings k_t so to maximize the overall utility of agents, J, under (2.13) and (2.14), taking as given the evolution of the interest rate r_t, and the evolution of capital at the aggregate level K, together with the following market clearing condition:

$$\sum_{j\in\{1,2\}} \int_{\underline{a}}^{+\infty} a dG_j\left(a,t\right) = \sum_{j\in\{1,2\}} \int_{0}^{+\infty} k\left(a\right) dG_j\left(a,t\right) + B, \tag{2.16}$$

where

- $G_j\left(a,t\right)$ is the joint cumulative distribution function (CDF) of labour income shocks y_j and wealth a, where $\frac{\partial G_j(a,t)}{\partial a} = g_j\left(a,t\right)$, which is the joint density function of these two variables;
- $B \in [0,+\infty)$ is the net fixed bond supply, which is assumed to be set at $B = 0$ indicating that bonds are in zero-net supply.

According to this way of closing the economy following Huggett (1993) (bonds in zero-net supply) and Aiyagari (1994) (total wealth equals total capital), the prices in this economy are the riskless interest rate, r_t, and the deterministic component of the return on capital, μ^*, which are determined by the requirement that, in equilibrium, the total net worth of individuals must equal the aggregate level of capital in the economy, given that the net supply of bonds is zero in the aggregate.

Later on in this work, it will be thoroughly proven that, in a stationary equilibrium, the consumption-saving decision of the individuals (including their allocation choice between risky and riskless assets) and the evolution of the joint distribution of their income shocks and wealth can be expressed by the following couple of HJB and KF differential equations (for further details and clarification, see Subsections 3 and

4, respectively):

$$\rho v_j(a) = \sup_{c,k} \left\{ u(c) + v_j'(a)(ra + y_j w^* - c + (\mu^* - r) k) + \frac{1}{2} v_j''(a) \sigma^2 k^2 + \lambda_j (v_{-j}(a) - v_j(a)) \right\}, \tag{2.17}$$

$$0 = -(ra + y_j w^* - c + (\mu^* - r) k) g_j'(a) + \frac{1}{2} \sigma^2 k^2 g_j''(a) + \lambda_{-j} g_{-j}(a) - \lambda_j g_j(a), \tag{2.18}$$

where $j = 1, 2$, the $-j$ index stands for $3 - j$.

These equations represent the focus of the next Sections, where a more detailed analysis and derivation is presented. In a nutshell, the derivation of the HJB Equation is built upon the definition of the Bellman Equation, which is satisfied by the value function $v_j(a)$, while the KF Equation considers the density of the joint distribution of income shocks and wealth as the density dynamics of a population with some specific birth and death intensities.

Furthermore, the HJB Equation in (2.17) can be rewritten more explicitly, as indicated and proven in Subsection 3.2, by separating the maximization problem with respect to consumption c, and the maximization with respect to capital holding k, and by computing and substituting their maximum values into the equation[4]. In particular, the expression $ra + y_j w^* - c + (\mu^* - r) k$ in (2.17), when determined for the optimal controls c^* and k^*, corresponds to the savings policy function:

$$s_j(a) = ra + y_j w^* - c^* + (\mu^* - r) k^*, \tag{2.19}$$

where the optimal choice of consumption, c^*, will be determined as:

$$c^* = (u')^{-1} (v_j'(a)), \tag{2.20}$$

and the optimal share of risky capital, k^*, will be computed as:

$$k^* = -\frac{\mu^* - r}{\sigma^2} \frac{v_j'(a)}{v_j''(a)}. \tag{2.21}$$

[4] Notice that in (2.20) and (2.21) c^* and k^* are derived assuming that maxima in the Hamiltonians are attained in the interior.

3 The Dynamic Programming Method

The problem stated in Section 2 is solved using the tools of Dynamic Programming, introduced by Bellman in the early 1950s (Bellman (1954)[5], Bellman (1957)). The application of this method to optimal control problems is based on the following idea: by considering a family of optimal control problems that differs in terms of initial times and state variables, relationships among these problems can be established exploiting the HJB Equation, which is a nonlinear second-order PDE (under stochasticity). In case the HJB Equation is solvable (either analytically or numerically), then one can obtain an optimal control by taking the maximizer of the Hamiltonian or generalized Hamiltonian involved in the HJB Equation. More in detail, one would:

(a) establish that the value function of the optimal control problem is a solution (in classical or weakened sense) of the HJB Equation associated to the problem - note that the HJB Equation is the infinitesimal version of the Bellman Equation for the control problem;

(b) obtain a solution of the HJB Equation (if solvable), either analytically or numerically;

(c) obtain a formula for the optimal control by taking the maximizer of the Hamiltonian involved in the HJB Equation; the optimal control is then expressed as a function of the state in real time, i.e. it is an optimal *feedback* control, and such relationship involves (the data of the problem and) the first and second derivatives of the solution of the HJB;

(d) obtain a state equation in feedback form (also known as *closed-loop* equation) by inserting the optimal feedback control into the original state equation, and generate simultaneously an optimal couple, i.e. the couple given by the optimal control and trajectory;

(e) prove that the value function and the solution of the HJB coincide, either because the solution of the HJB is unique or through the so-called verification technique.

Following this approach, one actually finds solutions to the whole family of optimal control problems taken into consideration (with different initial times and state variables), and, in particular, to the original problem to be investigated.

Of this entire procedure, a sketch of the proof for (a) and (c) is performed, while (b) and (d) are addressed numerically.

By applying the Dynamic Programming method to the problem under consideration (with initial time equal to 0), then the problem needs to be immersed into a family of problems with initial time t and initial state $x = (a, y)$ where, for every $\tau \in [t, +\infty)$, the state variable is $x_\tau = (a_\tau, y_\tau)$ and the control

[5] It includes an extensive bibliography of the literature in the area, up to the year 1954.

variable is $h_\tau = (c_\tau, k_\tau)$, and evolution in time is described by means of the following system:

$$
\begin{cases}
da_\tau = \left[r a_\tau + y_\tau w^*_\tau - c_\tau + (\mu^*_\tau - r_\tau) k_\tau \right] d\tau - \sigma k_\tau dW_\tau, & \tau \in [t, +\infty) \\
dk_\tau = k_\tau \left(\mu^*_\tau - \sigma dW_\tau \right), \; k_t = k^0, & \tau \in [t, +\infty) \\
y_\tau \in \{y_1, y_2\}, & \tau \in [t, +\infty) \\
a_t = a, \; y_t = y,
\end{cases}
\tag{3.1}
$$

where the μ^*_τ and w^*_τ, representing the optimal choices taken by firms in terms of capital rental rate and labour wage at time τ, are taken as given by households. The associated constraints on controls and state become, then:

$$c_\tau \geq 0, \quad a_\tau \geq -\underline{a}, \quad k_\tau \geq 0 \quad \forall \tau \geq t$$

that is

$$c_\tau \geq 0, \quad b_\tau \geq -k_\tau - \underline{a}, \quad k_\tau \geq 0 \quad \forall \tau \geq t. \tag{3.2}$$

The overall utility from future consumption, J, is made dependent on the control function $h = (c, k)$ and on the initial state $x = (a, y)$, that is:

$$J(h; t, x) = \mathbb{E} \left(\int_t^{+\infty} e^{-\rho \tau} u(c_\tau) d\tau \right). \tag{3.3}$$

The set of admissible controls at the initial couple $(t, x) = (t, (a, y))$ is the set $\mathcal{H}(t, x)$ of measurable functions (adapted to the filtrations generated by the Brownian Motion W_t and the Poisson process y_t)

$$(c_\tau, k_\tau) : [t, +\infty) \times \Omega \to R^2,$$

satisfying (3.2), and where $a_\tau = a_\tau(t, x)$ represents a solution to the problem (3.1) with initial values (t, x). Moreover, the value function associated to this problem is defined as:

$$v(t, x) = \sup_{h \in \mathcal{H}(t, x)} J\left(h; t, x\right).$$

It is useful to note that the following relationship between the value function at a generic instant of time $t \geq 0$ and the value function at time $t = 0$ exists:

$$v(t, x) = e^{-\rho t} v(0, x) \tag{3.4}$$

since

$$\int_t^{+\infty} e^{-\rho\tau} u(c_\tau) d\tau = e^{-\rho t} \int_0^{+\infty} e^{-\rho\xi} u(c_{\xi+t}) d\xi$$

so that

$$v(t,x) = e^{-\rho t} \sup_{h \in \mathcal{H}(t,x)} \int_0^{+\infty} e^{-\rho\xi} u(c_{\xi+t}) d\xi = e^{-\rho t} \sup_{h(t+\cdot) \in \mathcal{H}(0,x)} \int_0^{+\infty} e^{-\rho\xi} u(c_{\xi+t}) d\xi = e^{-\rho t} v(0,x).$$

As a consequence, to compute the value function $v(t,x)$ for a generic instant of time $t \geq 0$ it is sufficient to know $v(0,x)$, hereinafter indicated by $v(x)$.

The one just described is classified as an optimal control problem for *jump-diffusion* processes, as the stochasticity in the state equations is given jointly by jumps induced by y_τ and by the diffusion represented by the Brownian Motion W_τ. A useful reference for optimal control for jump-diffusion processes is, for instance Oksendal and Sulem (2005), while a more classical reference for merely diffusion processes is Yong and Zhou (1999).

3.1 Bellman Equation

This section contains the derivation of the Bellman Equation associated to the optimal control problem described above, and in which the state equation describes a stochastic process of *jump-diffusion* type. The reader is advised that a mathematically precise treatment of the subject require tools in stochastic control that are beyond the scope of this thesis, and proofs are in fact heuristic.

3.1.1 General equation

Consider the general state variable x, which corresponds to the couple $x_\tau = (a_\tau, y_\tau)$, and the control variable $h_\tau = (c_\tau, k_\tau)$. The following statement then holds.

Lemma 1. *The value function v of the optimal control problem described by (3.1) satisfies the general Bellman Equation*

$$v(t,x) = \sup_{h \in \mathcal{H}(t,x)} \mathbb{E} \left\{ \int_t^\tau e^{-\rho s} u(c_s) ds + v(\tau, x_\tau) \right\}, \tag{3.5}$$

$$v(x) \equiv v(0,x) = \sup_{h \in \mathcal{H}(0,x)} \mathbb{E} \left\{ \int_0^\tau e^{-\rho s} u(c_s) ds + e^{-\rho\tau} v(x_\tau) \right\}. \tag{3.6}$$

Sketch of the Proof. Recalling (3.3), taking a time $\tau \in [t, +\infty)$ one can split the functional J into the sum of two terms:

$$\begin{aligned} J(t,x;h) &= \mathbb{E}\left(\int_t^\tau e^{-\rho s}u(c_s)ds + \int_\tau^{+\infty} e^{-\rho s}u(c_s)ds\right) \\ &= \mathbb{E}\left(\int_t^\tau e^{-\rho s}u(c_s)ds\right) + J(\tau, x_\tau; h(\cdot+\tau)). \end{aligned}$$

Once proven that for all the choices of $h \in \mathcal{H}(0,x)$, one has that $h(\cdot+\tau)$ is admissible at (τ, x_τ), then it is always true that:

$$J(\tau, x_\tau; h(\cdot+\tau)) \leq v(\tau, x_\tau),$$

so that, taking the supremum as $h \in \mathcal{H}(t,x)$ on both sides:

$$v(t,x) \leq \sup_{h\in\mathcal{H}(t,x)} \mathbb{E}\left\{\int_t^\tau e^{-\rho s}u(c_s)ds + v(\tau,x_\tau)\right\}$$

and one has so proven $\leq$ in (3.5). To prove the converse inequality, fix $\epsilon > 0$ and consider an $\epsilon-$optimal control $h_\epsilon^* = (c_\epsilon^*, k_\epsilon^*) \in \mathcal{H}(\tau, x_\tau)$, i.e. maximizing the functional on the right-hand side, short of the value of ϵ:

$$v(\tau, x_\tau) - \epsilon \leq J(\tau, x_\tau; h_\epsilon^*).$$

Then, build a control $\hat{h} = (\hat{c}, \hat{k})$ so that:

$$\widehat{h}(s) = \begin{cases} h(s) & s \in [t,\tau) \\ h_\epsilon^*(s) & s \in [\tau, +\infty) \end{cases}$$

where $h = (c,k) \in \mathcal{H}(t,x)$ is any admissible control at (t,x). Once proven that $\hat{h} \in \mathcal{H}(t,x)$, by definition:

$$\begin{aligned} v(t,x) &\geq J(t,x;(\hat{c},\hat{k})) \\ &= \mathbb{E}\left(\int_t^\tau e^{-\rho s}u(c_s)ds + J(\tau, x_\tau; h_\epsilon^*)\right) \\ &\geq \mathbb{E}\left(\int_t^\tau e^{-\rho s}u(c_s)ds + v(\tau, x_\tau)\right) - \epsilon. \end{aligned}$$

Now taking the supremum as $h \in \mathcal{H}(t,x)$ on both sides, one derives:

$$v(t,x) \geq \sup_{h\in\mathcal{H}(t,x)} \mathbb{E}\left\{\int_t^\tau e^{-\rho s}u(c_s)ds + v(\tau,x_\tau)\right\} - \epsilon$$

and since $\epsilon > 0$ was arbitrarily chosen, the same inequality holds at limits for $\epsilon \to 0^+$, and the proof of (3.5) is complete.

To prove (3.6) one then needs simply to apply (3.4). □

3.1.2 Bellman Equation with a two-state Poisson wage shock process

In the specific case of the problem, since the labour income shock, y_t, is a two-state Poisson process that takes only the values $y_t \in \{y_1, y_2\}$, the two generic income shock states are indicated by y_j and y_{-j}, where j can be either 1 or 2, and the $-j$ index stands for $3-j$. The process y_t jumps from state j to $-j$ with intensity λ_j and vice versa with intensity λ_{-j}, such that individuals with shock on wage y_j receive the same shock with probability $p_j(\tau) = e^{-\lambda_j \tau}$ and switch to state y_{-j} with probability $(1 - p_j(\tau))$.

As before, the control is $h_\tau = (c_\tau, k_\tau)$, while a_τ and y_τ are taken into account separately by means of the following notation for the value function:

$$v_j(a) \equiv v(0, a, y_j) = v(a, y_j).$$

Lemma 2. *The value function v of the optimal control problem described by (3.1) satisfies the following Bellman Equation when $y_t = y_j$*

$$v_j(a) = \sup_{h \in \mathcal{H}(0,x)} \mathbb{E}\left(\int_0^\tau e^{-\rho s} u(c_s)\, ds + e^{-\lambda_j \tau} v_j(a_\tau) + \left(1 - e^{-\lambda_j \tau}\right) v_{-j}(a_\tau)\right). \tag{3.7}$$

Sketch of the Proof. The Bellman Equation (3.6) is rewritten for a fixed j, and taking separately the mean value with respect to the probability spaces for the Brownian Motion , $\mathbb{E}_1$, and the Poisson process, $\mathbb{E}_2$, one gets:

$$\begin{aligned}
v_j(a) &= \sup_{h \in \mathcal{H}(0,x)} \mathbb{E}\left(\int_0^\tau e^{-\rho s} u(c_s)\, ds + e^{-\rho \tau} v(a_\tau, y_\tau)\right) \\
&= \sup_{h \in \mathcal{H}(0,x)} \mathbb{E}_1\left(\int_0^\tau e^{-\rho s} u(c_s)\, ds + e^{-\rho \tau} \mathbb{E}_2(v(a_\tau, y_\tau))\right) \\
&= \sup_{h \in \mathcal{H}(0,x)} \mathbb{E}\left(\int_0^\tau e^{-\rho s} u(c_s)\, ds + e^{-\rho \tau} p_j(\tau) v_j(a_\tau) + e^{-\rho \tau}(1 - p_j(\tau)) v_{-j}(a_\tau)\right) \\
&= \sup_{h \in \mathcal{H}(0,x)} \mathbb{E}\left(\int_0^\tau e^{-\rho s} u(c_s)\, ds + e^{-\rho \tau} e^{-\lambda_j \tau} v_j(a_\tau) + e^{-\rho \tau}\left(1 - e^{-\lambda_j \tau}\right) v_{-j}(a_\tau)\right).
\end{aligned}$$

Summing up, one can write the couple of equations for j and $-j$:

$$\begin{cases}
v_j(a) = \sup_{h \in \mathcal{H}(0,x)} \mathbb{E}\left(\int_0^\tau e^{-\rho s} u(c_s)\, ds + e^{-\lambda_j \tau} v_j(a_\tau) + \left(1 - e^{-\lambda_j \tau}\right) v_{-j}(a_\tau)\right), & y_t = y_j \\
v_{-j}(a) = \sup_{h \in \mathcal{H}(0,x)} \mathbb{E}\left(\int_0^\tau e^{-\rho s} u(c_s)\, ds + e^{-\lambda_{-j} \tau} v_{-j}(a_\tau) + \left(1 - e^{-\lambda_{-j} \tau}\right) v_j(a_\tau)\right), & y_t = y_{-j}
\end{cases}.$$

Given the symmetry of the two equations, the generic equation for j will be considered in the next steps.

□

3.2 Hamilton-Jacobi-Bellman Equation

Now, the HJB Equation in (2.17) is derived heuristically.

Lemma 3. *The Hamilton-Jacobi-Bellman Equation of the optimal control problem described in Section 2 is:*

$$\rho v_j(a) = \sup_{c,k}\left\{u(c) + v_j'(a)\left(ra + y_j w^* - c + (\mu^* - r)k\right) + \frac{1}{2}v_j''(a)\sigma^2 k^2 + \lambda_j\left(v_{-j}(a) - v_j(a)\right)\right\},$$

where $c \geq 0$, and $k \geq -b - \underline{a}$.

Sketch of the Proof. Consider the Bellman Equation:

$$v_j(a) = \sup_{h\in\mathcal{H}(0,x)} \mathbb{E}\left\{\int_0^\tau e^{-\rho s}u(c_s)\,ds + e^{-\rho\tau}\left[e^{-\lambda_j\tau}v_j(a_\tau) + \left(1 - e^{-\lambda_j\tau}\right)v_{-j}(a_\tau)\right]\right\}.$$

Subtract $(1-\rho\tau)\,v_j(a)$ from both sides and divide by τ:

$$\begin{aligned}
\frac{v_j(a) - (1-\rho\tau)v_j(a)}{\tau} &= \sup_{h\in\mathcal{H}(0,x)} \mathbb{E}\left\{\frac{1}{\tau}\int_0^\tau e^{-\rho s}u(c_s)\,ds + \frac{e^{-(\rho+\lambda_j)\tau}v_j(a_\tau) - v_j(a)}{\tau} + \right.\\
&\quad \left. + \frac{e^{-\rho\tau}\left(1-e^{-\lambda_j\tau}\right)}{\tau}v_{-j}(a_\tau) + \frac{\rho\tau}{\tau}v_j(a)\right\}\\
\rho v_j(a) &= \sup_{h\in\mathcal{H}(0,x)}\left\{\underbrace{\mathbb{E}\left(\frac{1}{\tau}\int_0^\tau e^{-\rho s}u(c_s)\,ds\right)}_{(1)} + \underbrace{\mathbb{E}\left(\frac{e^{-(\rho+\lambda_j)\tau}v_j(a_\tau) - v_j(a)}{\tau}\right)}_{(2)}\right\}\\
&\quad + \sup_{h\in\mathcal{H}(0,x)}\left\{\underbrace{\mathbb{E}\left(\frac{e^{-\rho\tau}\left(1-e^{-\lambda_j\tau}\right)}{\tau}v_{-j}(a_\tau) + \rho v_j(a)\right)}_{(3)}\right\}.
\end{aligned}$$

Then, consider the three parts taking the lim as $\tau \to 0$.

$(1) = \mathbb{E}\left(\frac{1}{\tau}\int_0^\tau e^{-\rho s}u(c_s)\,ds\right) \longrightarrow u(c_0)$ as $\tau \to 0$.

$(2) = \mathbb{E}\left(\frac{e^{-\left(\rho+\lambda_j\right)\tau}v_j(a_\tau) - v_j(a)}{\tau}\right) = \mathbb{E}\left(\frac{1}{\tau}\int_0^\tau d\left(e^{-(\rho+\lambda_j)\tau}v_j(a_\tau)\right)dt\right) = (*),$

where $d\left(e^{-(\rho+\lambda_j)\tau}v_j\left(a_\tau\right)\right)$ is the stochastic differential of $e^{-(\rho+\lambda_j)\tau}v_j\left(a_\tau\right)$ which is given by the Itô-Doeblin formula:

$$\begin{aligned} d\left(e^{-(\rho+\lambda_j)\tau}v_j\left(a_\tau\right)\right) &= -\left(\rho+\lambda_j\right)e^{-(\rho+\lambda_j)\tau}v_j\left(a_\tau\right)d\tau + e^{-(\rho+\lambda_j)\tau}v_j'\left(a_\tau\right)da_\tau + \frac{1}{2}e^{-(\rho+\lambda_j)\tau}v_j''\left(a_\tau\right)\left(da_\tau\right)^2 \\ &= e^{-(\rho+\lambda_j)\tau}\left[-\left(\rho+\lambda_j\right)v_j\left(a_\tau\right)d\tau + v_j'\left(a_\tau\right)da_\tau + \frac{1}{2}v_j''\left(a_\tau\right)\left(da_\tau\right)^2\right] \end{aligned}$$

and, since

$$da_\tau = \left(ra_\tau + y_\tau w^* - c_\tau + \left(\mu^* - r\right)k_\tau\right)d\tau + \sigma k_\tau dW_\tau,$$

taking the square

$$\begin{aligned} \left(da_\tau\right)^2 &= \left(ra_\tau + y_\tau w^* - c_\tau + \left(\mu^* - r\right)k_\tau\right)^2 \overbrace{\left(d\tau\right)^2}^{0} + \left(\sigma k_\tau\right)^2 \overbrace{\left(dW_\tau\right)^2}^{d\tau} + \\ &+ 2\left(ra_\tau + y_\tau w^* - c_\tau + \left(\mu^* - r\right)k_\tau\right)\left(\sigma k_\tau\right)\overbrace{d\tau\left(dW_\tau\right)}^{d\tau^{\frac{3}{2}}=0} \\ &= \sigma^2 k_\tau^2 d\tau \end{aligned}$$

the differential becomes

$$\begin{aligned} d\left(e^{-(\rho+\lambda_j)\tau}v_j\left(a_\tau\right)\right) &= e^{-(\rho+\lambda_j)\tau}\left[-\left(\rho+\lambda_j\right)v_j\left(a_\tau\right) + v_j'\left(a_\tau\right)\left(ra_\tau + y_\tau w^* - c_\tau + \left(\mu^* - r\right)k_\tau\right) + \right. \\ &\left. + \frac{1}{2}v_j''\left(a_\tau\right)\sigma^2 k_\tau^2\right]d\tau + e^{-(\rho+\lambda_j)\tau}v_j'\left(a_\tau\right)\sigma k_\tau dW_\tau. \end{aligned}$$

Therefore,

$$\begin{aligned} (*) = \frac{1}{\tau}\mathbb{E}&\left[\int_0^\tau e^{-(\rho+\lambda_j)s}\left[-\left(\rho+\lambda_j\right)v_j\left(a_s\right) + v_j'\left(a_s\right)\left(ra_s + y_s w^* - c_s + \left(\mu^* - r\right)k_s\right) + \frac{1}{2}v_j''\left(a_s\right)\sigma^2 k_s^2\right]ds\right] \\ &+ \frac{1}{\tau}\underbrace{\mathbb{E}\left[\underbrace{\int_0^\tau e^{-(\rho+\lambda_j)\tau}v_j'\left(a_\tau\right)\sigma k_\tau dW_\tau}_{Ito\ integral}\right]}_{=0}. \end{aligned}$$

Taking the lim as $\tau \to 0$ under the expectation:

$$(2) \to -\left(\rho+\lambda_j\right)v_j\left(a\right) + v_j'\left(a\right)\left(ra + y_j w^* - c + \left(\mu^* - r\right)k\right) + \frac{1}{2}v_j''\left(a_s\right)\sigma^2 k^2,$$

where $a_0 = a$, $y_0 = y_j$, $k_0 =: k$ and $c_0 =: c$.

$(3)=\mathbb{E}\left(\frac{e^{-\rho\tau}\left(1-e^{-\lambda_j\tau}\right)}{\tau}v_{-j}\left(a_\tau\right)+\rho v_j\left(a\right)\right)\overset{\tau\to 0}{\longrightarrow}-\left[\frac{d}{d\tau}\left(e^{-\lambda_j\tau}\right)\right]_{|\tau=0}v_{-j}\left(a\right)+\rho v_j\left(a\right)=\lambda_j v_{-j}\left(a\right)+\rho v_j\left(a\right).$

By putting (1), (2) and (3) together, one has proven the equation:

$$\begin{aligned}\rho v_j\left(a\right)&=\sup_{c,k}\left\{u\left(c\right)+v_j'\left(a\right)\left(ra+y_jw^*-c+\left(\mu^*-r\right)k\right)+\right.\\&\left.+\frac{1}{2}v_j''\left(a\right)\sigma^2k^2-\left(\rho+\lambda_j\right)v_j\left(a\right)+\lambda_jv_{-j}\left(a\right)+\rho v_j\left(a\right)\right\}\\&=\sup_{c,k}\left\{u\left(c\right)+v_j'\left(a\right)\left(ra+y_jw^*-c+\left(\mu^*-r\right)k\right)+\frac{1}{2}v_j''\left(a\right)\sigma^2k^2+\lambda_j\left(v_{-j}\left(a\right)-v_j\left(a\right)\right)\right\}.\end{aligned}$$

□

Remark. By computing the maximizers and substituting them, if these values are internal, the equation can be rewritten as follows:

$$\rho v_j\left(a\right)=u^*\left(v_j'\left(a\right)\right)-\frac{\left(\mu^*-r\right)^2}{2\sigma^2}\frac{\left(v_j'\left(a\right)\right)^2}{v_j''\left(a\right)}+v_j'\left(a\right)\left(ra+y_jw^*\right)+\lambda_j\left(v_{-j}\left(a\right)-v_j\left(a\right)\right).\quad(3.8)$$

Indeed, by separating the supremum with respect to c and with respect to k:

$$\begin{aligned}\rho v_j\left(a\right)&=\sup_{c}\left\{u\left(c\right)-cv_j'\left(a\right)\right\}+\sup_{k}\left\{\left(\mu^*-r\right)kv_j'\left(a\right)+\frac{1}{2}v_j''\left(a\right)\sigma^2k^2\right\}\\&+v_j'\left(a\right)\left(ra+y_jw^*\right)+\lambda_j\left(v_{-j}\left(a\right)-v_j\left(a\right)\right)\\&=H_1\left(v_j'\left(a\right)\right)+H_2\left(v_j'\left(a\right),v_j''\left(a\right)\right)+v_j'\left(a\right)\left(ra+y_jw^*\right)+\lambda_j\left(v_{-j}\left(a\right)-v_j\left(a\right)\right)\end{aligned}$$

where it is set:

$$H_1\left(p\right)=\sup_{c\geq 0}\left\{u\left(c\right)-cp\right\},\quad H_2\left(p,q\right)=\sup_{k\geq -b-\underline{a}}\left\{\left(\mu^*-r\right)kp+\frac{1}{2}q\sigma^2k^2\right\}.$$

By imposing first-order conditions, if the maximizers are interior one gets:

$$\frac{\partial}{\partial c}\left[u\left(c\right)-cp\right]=0\Leftrightarrow u'\left(c\right)-p=0\Leftrightarrow c^*=\left(u'\right)^{-1}\left(p\right)$$

implying (2.20), that is:

$$c^*=\left(u'\right)^{-1}\left(v_j'\left(a\right)\right)$$

and meaning that the consumption is such that its marginal utility equals the marginal value of the value function.

Similarly

$$\frac{\partial}{\partial k}[(\mu^* - r)\, kp + \frac{1}{2}qk^2] = 0 \Leftrightarrow k = \frac{(\mu^* - r)\, p}{q\sigma^2}$$

implying (2.21), that is:

$$k^* = -\frac{\mu^* - r}{\sigma^2}\frac{v_j'(a)}{v_j''(a)}.$$

By substituting such expression in the definition of H_1 and H_2, (3.8) is obtained:

$$\rho v_j(a) = u^*\left(v_j'(a)\right) - \frac{(\mu^* - r)^2}{2\sigma^2}\frac{\left(v_j'(a)\right)^2}{v_j''(a)} + v_j'(a)\left(ra + y_j w^*\right) + \lambda_j\left(v_{-j}(a) - v_j(a)\right).$$

4 Kolmogorov-Forward Equation

In this Section, the Kolmogorov-Forward (briefly, KF) Equation is derived, which, associated with the HJB Equation, constitutes a so-called Mean Field Game. The Mean Field Games have been under investigation over the last decade, starting from the pioneering work of Larsy and Lions (2007), and represent one of the most prolific frontiers of research.

In this context, both the transitional (i.e. time-dependent) KF Equation and its stationary version (at equilibrium) are derived. They are, respectively[6]:

$$\frac{\partial g_j(a,t)}{\partial t} = -\left(r_t a_t + y_j w_t^* - c_t + \left(\mu_t^* - r_t\right) k_t\right) \frac{\partial g_j(a,t)}{\partial a} + \frac{1}{2}\sigma_t^2 k_t^2 \frac{\partial g_j(a,t)}{\partial a \partial a} + \lambda_{-j} g_{-j}(a,t) - \lambda_j g_j(a,t), \tag{4.1}$$

and:

$$0 = -\left(ra + y_j w^* - c + \left(\mu^* - r\right) k\right) g_j'(a) + \frac{1}{2}\sigma^2 k^2 g_j''(a) + \lambda_{-j} g_{-j}(a) - \lambda_j g_j(a).$$

To do so, we proceed heuristically as follows. Consider the following equation that describes the density of the joint distribution of income shock y_j and wealth a:

$$dg(a, y_j, t) = dg_j(a,t) = \left[\lambda_{-j} g_{-j}(a,t) - \lambda_j g_j(a,t)\right] dt.$$

This can be interpreted as the density dynamics of a population with birth intensity λ_{-j} and death intensity λ_j, where death means moving from health/sickness to sickness/health and vice versa for birth.

As before, the households' net worth a evolves with dynamics:

$$da_t = \left(r_t a_t + y_j w_t^* - c_t + \left(\mu_t^* - r_t\right) k_t\right) dt + \sigma_t k_t dW_t,$$

where, for the sake of simplicity:

$$\begin{aligned} \eta_t &= r_t a_t + y_j w_t^* - c_t + \left(\mu_t^* - r_t\right) k_t, \\ \beta_t &= \sigma_t k_t, \end{aligned}$$

so that:

$$da_t = \eta_t dt + \beta_t dW_t.$$

[6] Given that the density g is a measure and not a function and that its properties (e.g. continuity or differentiability) are not properly stated, the derivatives of g would need to be adequately interpreted in a mathematical context.

Consider a test function f of class C^2, i.e. with continuous second-order partial derivatives, $f(a, y_j, t) = f_j(a,t)$, and having compact support in $[s, +\infty) \times (\underline{a}, +\infty)$, that is, f is null outside a compact subset of $[s, +\infty) \times (\underline{a}, +\infty)$.Then, the differential $d[f_j(a,t) g_j(a,t)]$ can be computed as follows. Call $F_j(a,t) = f_j(a,t) g_j(a,t)$ and apply the Itô-Doeblin formula with:

$$\begin{aligned}
\frac{\partial F_j(a,t)}{\partial t} &= \frac{\partial f_j(a,t)}{\partial t} g_j(a,t) + f_j(a,t) \frac{\partial g_j(a,t)}{\partial t}, \\
\frac{\partial F_j(a,t)}{\partial a} &= \frac{\partial f_j(a,t)}{\partial a} g_j(a,t) + f_j(a,t) \frac{\partial g_j(a,t)}{\partial a}, \\
\frac{\partial F_j(a,t)}{\partial a \partial a} &= \frac{\partial f_j(a,t)}{\partial a \partial a} g_j(a,t) + f_j(a,t) \frac{\partial g_j(a,t)}{\partial a \partial a} + 2\frac{\partial f_j(a,t)}{\partial a} \frac{\partial g_j(a,t)}{\partial a}.
\end{aligned}$$

The differential becomes:

$$\begin{aligned}
d[f_j(a,t) g_j(a,t)] &= \left[\frac{\partial f_j(a,t)}{\partial t} dt + \frac{\partial f_j(a,t)}{\partial a} da + \frac{1}{2} \frac{\partial f_j(a,t)}{\partial a \partial a} (da)^2\right] g_j(a,t) \\
&+ f_j(a,t) \left[\frac{\partial g_j(a,t)}{\partial t} dt + \frac{\partial g_j(a,t)}{\partial a} da + \frac{1}{2} \frac{\partial g_j(a,t)}{\partial a \partial a} (da)^2\right] \\
&+ \frac{1}{2} 2 \frac{\partial f_j(a,t)}{\partial a} \frac{\partial g_j(a,t)}{\partial a} (da)^2 \\
&= df_j(a,t) g_j(a,t) + f_j(a,t) dg_j(a,t) + \frac{\partial f_j(a,t)}{\partial a} \frac{\partial g_j(a,t)}{\partial a} (da)^2.
\end{aligned}$$

Then, the last term can be rewritten as $df_j(a,t) dg_j(a,t)$ since:

$$\begin{aligned}
df_j(a,t) dg_j(a,t) &= \left[\frac{\partial f_j(a,t)}{\partial t} dt + \frac{\partial f_j(a,t)}{\partial a} da + \frac{1}{2} \frac{\partial f_j(a,t)}{\partial a \partial a} (da)^2\right] \cdot \\
&\cdot \left[\frac{\partial g_j(a,t)}{\partial t} dt + \frac{\partial g_j(a,t)}{\partial a} da + \frac{1}{2} \frac{\partial g_j(a,t)}{\partial a \partial a} (da)^2\right] \\
&= \frac{\partial f_j(a,t)}{\partial t} \frac{\partial g_j(a,t)}{\partial t} (dt)^2 + \frac{\partial f_j(a,t)}{\partial t} \frac{\partial g_j(a,t)}{\partial a} dtda + \frac{\partial f_j(a,t)}{\partial t} \frac{1}{2} \frac{\partial g_j(a,t)}{\partial a \partial a} dt (da)^2 \\
&+ \frac{\partial f_j(a,t)}{\partial a} \frac{\partial g_j(a,t)}{\partial t} dadt + \frac{\partial f_j(a,t)}{\partial a} \frac{\partial g_j(a,t)}{\partial a} (da)^2 + \frac{\partial f_j(a,t)}{\partial a} \frac{1}{2} \frac{\partial g_j(a,t)}{\partial a \partial a} (da)^3 \\
&+ \frac{1}{2} \frac{\partial f_j(a,t)}{\partial a \partial a} \left[\frac{\partial g_j(a,t)}{\partial t} dt (da)^2 + \frac{\partial g_j(a,t)}{\partial a} (da)^3 + \frac{1}{2} \frac{\partial g_j(a,t)}{\partial a \partial a} (da)^4\right] \\
&= \frac{\partial f_j(a,t)}{\partial a} \frac{\partial g_j(a,t)}{\partial a} (da)^2,
\end{aligned}$$

where $(dt)^2 = dtda = dt(da)^2 = (da)^3 = (da)^4 = 0$. Since in the specific case of the problem:

$$dg_j(a,t) = [\lambda_{-j} g_{-j}(a,t) - \lambda_j g_j(a,t)] dt,$$

then

$$
\begin{aligned}
df_j\left(a,t\right)dg_j\left(a,t\right) &= \left[\frac{\partial f_j\left(a,t\right)}{\partial t}dt + \frac{\partial f_j\left(a,t\right)}{\partial a}da + \frac{1}{2}\frac{\partial f_j\left(a,t\right)}{\partial a \partial a}\left(da\right)^2\right]\left[\lambda_{-j}g_{-j}\left(a,t\right) - \lambda_j g_j\left(a,t\right)\right]dt \\
&= 0.
\end{aligned}
$$

By substituting

$$
\begin{aligned}
da_t &= \eta_t dt + \beta_t dW_t, \\
\left(da_t\right)^2 &= \beta_t^2\left(dW_t\right)^2 = \beta_t^2 dt,
\end{aligned}
$$

the differential becomes:

$$
\begin{aligned}
d\left[f_j\left(a,t\right)g_j\left(a,t\right)\right] &= \left[\frac{\partial f_j\left(a,t\right)}{\partial t} + \eta_t\frac{\partial f_j\left(a,t\right)}{\partial a} + \frac{1}{2}\beta_t^2\frac{\partial f_j\left(a,t\right)}{\partial a \partial a}\right]g_j\left(a,t\right)dt \\
&+ \left[\lambda_{-j}g_{-j}\left(a,t\right) - \lambda_j g_j\left(a,t\right)\right]f_j\left(a,t\right)dt \\
&+ \beta_t\frac{\partial f_j\left(a,t\right)}{\partial a}g_j\left(a,t\right)dW_t.
\end{aligned}
$$

Integrating over time and given the properties of the test function over the support:

$$
\begin{aligned}
\int_s^{+\infty} d\left[f_j\left(a,t\right)g_j\left(a,t\right)\right] &= \int_s^{+\infty}\left[\frac{\partial f_j\left(a,t\right)}{\partial t} + \eta_t\frac{\partial f_j\left(a,t\right)}{\partial a} + \frac{1}{2}\beta_t^2\frac{\partial f_j\left(a,t\right)}{\partial a \partial a}\right]g_j\left(a,t\right)dt \\
&+ \int_s^{+\infty}\left[\lambda_{-j}g_{-j}\left(a,t\right) - \lambda_j g_j\left(a,t\right)\right]f_j\left(a,t\right)dt + \int_s^{+\infty}\beta_t\frac{\partial f_j\left(a,t\right)}{\partial a}g_j\left(a,t\right)dW_t
\end{aligned}
$$

$$
\begin{aligned}
\lim_{u\to+\infty}\int_s^{u} d\left[f_j\left(a,t\right)g_j\left(a,t\right)\right] &= \int_s^{+\infty}\left[\frac{\partial f_j\left(a,t\right)}{\partial t} + \eta_t\frac{\partial f_j\left(a,t\right)}{\partial a} + \frac{1}{2}\beta_t^2\frac{\partial f_j\left(a,t\right)}{\partial a \partial a}\right]g_j\left(a,t\right)dt \\
&+ \int_s^{+\infty}\left[\lambda_{-j}g_{-j}\left(a,t\right) - \lambda_j g_j\left(a,t\right)\right]f_j\left(a,t\right)dt + \int_s^{+\infty}\beta_t\frac{\partial f_j\left(a,t\right)}{\partial a}g_j\left(a,t\right)dW_t
\end{aligned}
$$

$$
\begin{aligned}
\lim_{u\to+\infty}\left[f_j\overset{\to 0}{(a,u)}g_j\left(a,u\right) - f_j\overset{=0}{(a,s)}g_j\left(a,s\right)\right] &= \int_s^{+\infty}\left[\frac{\partial f_j\left(a,t\right)}{\partial t} + \eta_t\frac{\partial f_j\left(a,t\right)}{\partial a} + \frac{1}{2}\beta_t^2\frac{\partial f_j\left(a,t\right)}{\partial a \partial a}\right]g_j\left(a,t\right)dt \\
&+ \int_s^{+\infty}\left[\lambda_{-j}g_{-j}\left(a,t\right) - \lambda_j g_j\left(a,t\right)\right]f_j\left(a,t\right)dt \\
&+ \int_s^{+\infty}\beta_t\frac{\partial f_j\left(a,t\right)}{\partial a}g_j\left(a,t\right)dW_t
\end{aligned}
$$

$$0=\int_{s}^{+\infty}\left[\frac{\partial f_j(a,t)}{\partial t}+\eta_t\frac{\partial f_j(a,t)}{\partial a}+\frac{1}{2}\beta_t^2\frac{\partial f_j(a,t)}{\partial a\partial a}\right]g_j(a,t)\,dt$$
$$+\int_{s}^{+\infty}\left[\lambda_{-j}g_{-j}(a,t)-\lambda_j g_j(a,t)\right]f_j(a,t)\,dt+\int_{s}^{+\infty}\beta_t\frac{\partial f_j(a,t)}{\partial a}g_j(a,t)\,dW_t.$$

Integrating over wealth:

$$0=\int_{\underline{a}}^{+\infty}\int_{s}^{+\infty}\left[\frac{\partial f_j(a,t)}{\partial t}+\eta_t\frac{\partial f_j(a,t)}{\partial a}+\frac{1}{2}\beta_t^2\frac{\partial f_j(a,t)}{\partial a\partial a}\right]g_j(a,t)\,dtda$$
$$+\int_{\underline{a}}^{+\infty}\int_{s}^{+\infty}\left[\lambda_{-j}g_{-j}(a,t)-\lambda_j g_j(a,t)\right]f_j(a,t)\,dtda+\int_{\underline{a}}^{+\infty}\int_{s}^{+\infty}\beta_t\frac{\partial f_j(a,t)}{\partial a}g_j(a,t)\,dW_tda,$$

where the last term equals zero because all the Brownian Motions are independent across households over a.

$$0=\int_{\underline{a}}^{+\infty}\int_{s}^{+\infty}\frac{\partial f_j(a,t)}{\partial t}g_j(a,t)\,dtda+\int_{\underline{a}}^{+\infty}\int_{s}^{+\infty}\eta_t\frac{\partial f_j(a,t)}{\partial a}g_j(a,t)\,dtda$$
$$+\frac{1}{2}\int_{\underline{a}}^{+\infty}\int_{s}^{+\infty}\beta_t^2\frac{\partial f_j(a,t)}{\partial a\partial a}g_j(a,t)\,dtda+\int_{\underline{a}}^{+\infty}\int_{s}^{+\infty}\left[\lambda_{-j}g_{-j}(a,t)-\lambda_j g_j(a,t)\right]f_j(a,t)\,dtda.$$

Integrating by parts, piece by piece:

(1) $\int_{\underline{a}}^{+\infty}\left[\int_{s}^{+\infty}\frac{\partial f_j(a,t)}{\partial t}g_j(a,t)\,dt\right]da=\int_{\underline{a}}^{+\infty}\left\{\lim_{u\to+\infty}\int_{s}^{u}\frac{\partial f_j(a,t)}{\partial t}g_j(a,t)\,dt\right\}da$

$=\int_{\underline{a}}^{+\infty}\left\{\lim_{u\to+\infty}\left[\left[f_j(a,t)\,g_j(a,t)\right]_{t=s}^{t=u}-\int_{s}^{u}f_j(a,t)\,\frac{\partial g_j(a,t)}{\partial t}dt\right]\right\}da$

$=\int_{\underline{a}}^{+\infty}\left\{\lim_{u\to+\infty}\left[\left[f_j(a,u)\,g_j(a,u)-f_j(a,s)\,g_j(a,s)\right]-\int_{s}^{u}f_j(a,t)\,\frac{\partial g_j(a,t)}{\partial t}dt\right]\right\}da$

$=\int_{\underline{a}}^{+\infty}\left\{\lim_{u\to+\infty}f_j(a,u)\,g_j(a,u)-f_j(a,s)\,g_j(a,s)-\lim_{u\to+\infty}\int_{s}^{u}f_j(a,t)\,\frac{\partial g_j(a,t)}{\partial t}dt\right\}da$

$=-\int_{\underline{a}}^{+\infty}\int_{s}^{+\infty}f_j(a,t)\,\frac{\partial g_j(a,t)}{\partial t}dtda,$

since $\lim_{u\to+\infty}f_j(a,u)=0$ and $f_j(a,s)=0$ for the boundary conditions on f.

(2) $\int_{\underline{a}}^{+\infty}\left[\int_{s}^{+\infty}\eta_t\frac{\partial f_j(a,t)}{\partial a}g_j(a,t)\,dt\right]da=\int_{s}^{+\infty}\left\{\int_{\underline{a}}^{+\infty}\eta_t\frac{\partial f_j(a,t)}{\partial a}g_j(a,t)\,da\right\}dt$ for the Fubini-Tonelli theorem

$=\int_{s}^{+\infty}\left\{\lim_{x\to+\infty}\int_{\underline{a}}^{x}\eta_t\frac{\partial f_j(a,t)}{\partial a}g_j(a,t)\,da\right\}dt$

$=\int_{s}^{+\infty}\left\{\lim_{x\to+\infty}\left[\left[\eta_t f_j(a,t)\,g_j(a,t)\right]_{a=\underline{a}}^{a=x}-\int_{\underline{a}}^{x}f_j(a,t)\,\frac{\partial g_j(a,t)}{\partial a}da\right]\right\}dt$

$=\int_{s}^{+\infty}\left\{\lim_{x\to+\infty}\left[\left[\eta_t f_j(x,t)\,g_j(x,t)-\eta_t f_j(\underline{a},t)\,g_j(\underline{a},t)\right]-\int_{\underline{a}}^{x}\eta_t f_j(a,t)\,\frac{\partial g_j(a,t)}{\partial a}da\right]\right\}dt$

$=\int_{s}^{+\infty}\left\{\lim_{x\to+\infty}\eta_t f_j(x,t)\,g_j(x,t)-\eta_t f_j(\underline{a},t)\,g_j(\underline{a},t)-\lim_{x\to+\infty}\int_{\underline{a}}^{x}\eta_t f_j(a,t)\,\frac{\partial g_j(a,t)}{\partial a}da\right\}dt$

$=-\int_{s}^{+\infty}\int_{\underline{a}}^{+\infty}\eta_t f_j(a,t)\,\frac{\partial g_j(a,t)}{\partial a}dadt=-\int_{\underline{a}}^{+\infty}\int_{s}^{+\infty}\eta_t f_j(a,t)\,\frac{\partial g_j(a,t)}{\partial a}dtda,$

since $\lim_{x\to+\infty}f_j(x,t)=0$ and $f_j(\underline{a},t)=0$ for the boundary conditions on f.

(3) $\frac{1}{2}\int_{\underline{a}}^{+\infty}\int_{s}^{+\infty}\beta_t^2\frac{\partial f_j(a,t)}{\partial a\partial a}g_j(a,t)\,dtda=\frac{1}{2}\int_{s}^{+\infty}\left\{\int_{\underline{a}}^{+\infty}\beta_t^2\frac{\partial f_j(a,t)}{\partial a\partial a}g_j(a,t)\,da\right\}dt$ for the Fubini-Tonelli theorem

$=\frac{1}{2}\int_{s}^{+\infty}\left\{\lim_{x\to+\infty}\int_{\underline{a}}^{x}\beta_t^2\frac{\partial f_j(a,t)}{\partial a\partial a}g_j(a,t)\,da\right\}dt$

$=\frac{1}{2}\int_{s}^{+\infty}\left\{\lim_{x\to+\infty}\left[\left[\beta_t^2\frac{\partial f_j(a,t)}{\partial a}g_j(a,t)\right]_{a=\underline{a}}^{a=x}-\int_{\underline{a}}^{x}\beta_t^2\frac{\partial f_j(a,t)}{\partial a}\frac{\partial g_j(a,t)}{\partial a}da\right]\right\}dt$

$=\frac{1}{2}\int_{s}^{+\infty}\left\{\lim_{x\to+\infty}\left[\left[\beta_t^2\frac{\partial f_j(x,t)}{\partial x}g_j(x,t)-\beta_t^2\frac{\partial f_j(\underline{a},t)}{\partial \underline{a}}g_j(\underline{a},t)\right]-\int_{\underline{a}}^{x}\beta_t^2\frac{\partial f_j(a,t)}{\partial a}\frac{\partial g_j(a,t)}{\partial a}da\right]\right\}dt$

$=\frac{1}{2}\int_{s}^{+\infty}\left\{\lim_{x\to+\infty}\beta_t^2\frac{\partial f_j(x,t)}{\partial x}g_j(x,t)-\beta_t^2\frac{\partial f_j(\underline{a},t)}{\partial \underline{a}}g_j(\underline{a},t)-\lim_{x\to+\infty}\int_{\underline{a}}^{x}\beta_t^2\frac{\partial f_j(a,t)}{\partial a}\frac{\partial g_j(a,t)}{\partial a}da\right\}dt$

$=-\frac{1}{2}\int_{s}^{+\infty}\left\{\lim_{x\to+\infty}\int_{\underline{a}}^{x}\beta_t^2\frac{\partial f_j(a,t)}{\partial a}\frac{\partial g_j(a,t)}{\partial a}da\right\}dt$

$=-\frac{1}{2}\int_{s}^{+\infty}\left\{\lim_{x\to+\infty}\left[\left[\beta_t^2 f_j(a,t)\frac{\partial g_j(a,t)}{\partial a}\right]_{a=\underline{a}}^{a=x}-\int_{\underline{a}}^{x}\beta_t^2 f_j(a,t)\frac{\partial g_j(a,t)}{\partial a\partial a}da\right]\right\}dt$

$=-\frac{1}{2}\int_{s}^{+\infty}\left\{\lim_{x\to+\infty}\left[\left[\beta_t^2 f_j(x,t)\frac{\partial g_j(x,t)}{\partial x}-\beta_t^2 f_j(\underline{a},t)\frac{\partial g_j(\underline{a},t)}{\partial \underline{a}}\right]-\int_{\underline{a}}^{x}\beta_t^2 f_j(a,t)\frac{\partial g_j(a,t)}{\partial a\partial a}da\right]\right\}dt$

$=-\frac{1}{2}\int_{s}^{+\infty}\left\{\lim_{x\to+\infty}\beta_t^2 f_j(x,t)\frac{\partial g_j(x,t)}{\partial x}-\beta_t^2 f_j(\underline{a},t)\frac{\partial g_j(\underline{a},t)}{\partial \underline{a}}-\lim_{x\to+\infty}\int_{\underline{a}}^{x}\beta_t^2 f_j(a,t)\frac{\partial g_j(a,t)}{\partial a\partial a}da\right\}dt$

$=\frac{1}{2}\int_{s}^{+\infty}\int_{\underline{a}}^{+\infty}\beta_t^2 f_j(a,t)\frac{\partial g_j(a,t)}{\partial a\partial a}dadt=\frac{1}{2}\int_{\underline{a}}^{+\infty}\int_{s}^{+\infty}\beta_t^2 f_j(a,t)\frac{\partial g_j(a,t)}{\partial a\partial a}dtda,$

since $\lim_{x\to+\infty}\frac{\partial f_j(x,t)}{\partial x}=0$, $\frac{\partial f_j(\underline{a},t)}{\partial \underline{a}}=0$, $\lim_{x\to+\infty}f_j(x,t)=0$ and $f_j(\underline{a},t)=0$ for the boundary conditions on f.

Therefore, by putting parts together and rearranging one gets:

$$
\begin{aligned}
0=&-\int_{\underline{a}}^{+\infty}\int_{s}^{+\infty}f_j(a,t)\frac{\partial g_j(a,t)}{\partial t}dtda-\int_{\underline{a}}^{+\infty}\int_{s}^{+\infty}\eta_t f_j(a,t)\frac{\partial g_j(a,t)}{\partial a}dtda\\
&+\frac{1}{2}\int_{\underline{a}}^{+\infty}\int_{s}^{+\infty}\beta_t^2 f_j(a,t)\frac{\partial g_j(a,t)}{\partial a\partial a}dtda+\int_{\underline{a}}^{+\infty}\int_{s}^{+\infty}\left[\lambda_{-j}g_{-j}(a,t)-\lambda_j g_j(a,t)\right]f_j(a,t)\,dtda\\
=&\int_{\underline{a}}^{+\infty}\int_{s}^{+\infty}\left[-\frac{\partial g_j(a,t)}{\partial t}-\eta_t\frac{\partial g_j(a,t)}{\partial a}+\frac{1}{2}\beta_t^2\frac{\partial g_j(a,t)}{\partial a\partial a}+\lambda_{-j}g_{-j}(a,t)-\lambda_j g_j(a,t)\right]f_j(a,t)\,dtda.
\end{aligned}
$$

Since the equality above must hold for any test function f, then, for a well known Lemma, one gets (4.1):

$$
\begin{aligned}
\frac{\partial g_j(a,t)}{\partial t}&=-\eta_t\frac{\partial g_j(a,t)}{\partial a}+\frac{1}{2}\beta_t^2\frac{\partial g_j(a,t)}{\partial a\partial a}+\lambda_{-j}g_{-j}(a,t)-\lambda_j g_j(a,t)\\
&=-\left(r_t a_t+y_j w_t^*-c_t+\left(\mu_t^*-r_t\right)k_t\right)\frac{\partial g_j(a,t)}{\partial a}+\frac{1}{2}\sigma_t^2 k_t^2\frac{\partial g_j(a,t)}{\partial a\partial a}+\lambda_{-j}g_{-j}(a,t)-\lambda_j g_j(a,t).
\end{aligned}
$$

To obtain (2.18), it suffices to observe that, at equilibrium, the density is time-independent, and hence the derivative of g with respect to time is null:

$$
\begin{aligned}
0&=-\eta_t g_j'(a)+\frac{1}{2}\beta_t^2 g_j''(a)+\lambda_{-j}g_{-j}(a)-\lambda_j g_j(a)\\
&=-\left(ra+y_j w^*-c+\left(\mu^*-r\right)k\right)g_j'(a)+\frac{1}{2}\sigma^2k^2 g_j''(a)+\lambda_{-j}g_{-j}(a)-\lambda_j g_j(a).
\end{aligned}
$$

5 Numerical resolution

5.1 Description of the algorithm

The numerical solution to the problem under analysis is found using MATLAB algorithms. Both the basic model as in Achdou et al. (2021), where individuals can only save in unproductive bonds, and its extended version as described in this thesis, introducing also labour supply and investments in risky capital, are solved numerically with the aim of obtaining comparable results and, therefore, to characterize the behavior of the interest rate. Furthermore, exploiting comparative statics within the scope of the model under consideration, it is also possible to investigate how the wealth distribution, the risk-free interest rate, the aggregate capital level and the risky interest rate (limited to its deterministic component) change in relation to variations in the depreciation rate, in the output elasticity of capital and in the stochastic shocks on labour wage.

With regards to the basic model, the algorithm is implemented on the basis of the codes provided in the Benjamin Moll Website for the Huggett economy. By exploiting the available scripts by Moll, the version used for the purpose of this work[7] consists of a combination of the codes to determine the asset supply function and the interest rate in a stationary equilibrium, integrated also in terms of the graphical representation of results. The model with bonds and capital is implemented on the basis of the previous code, taking also advantage of the Moll script for the Aiyagari model with a fat-tailed wealth distribution and two assets. Differently from Moll, the version implemented for the purpose of this thesis does not consider a fat-tailed wealth distribution. A brief explanation of the functioning of the code for the extended model[8] (which derives directly from the numerical resolution of the basic model) is provided below.

Following the online numerical appendix by Achdou et al. (2020)[9] to the reference paper, the algorithm is based on a finite-difference approach and uses a bisection method on the endogenous variables (i.e. the riskless interest rate and the aggregate capital level in the model under analysis). This algorithm - as well as the one for the basic model - uses uniform grids[10], where the discrete grid points are equally spaced from each other. Since in the context of a steady-state equilibrium the HJB and KF Equations constitute a system of coupled equations, the algorithm is required to work iteratively on them. First of all, the values of the riskless interest rate and aggregate capital level are set according to an initial guess, which is then

[7] See *Huggett_bonds.m* in Appendix A.1

[8] See *ExtendedModel_bonds_capital.m* in Appendix A.2

[9] The cited reference is an appendix to the previous version (2020) of the paper by Achdou et al. No later version of this numerical appendix has been published yet, but it is assumed it constitutes a reliable and still updated source for the numerical methods used to solve the model.

[10] By contrast, the code for the Aiyagari model with a fat-tailed distribution by Moll exploits non-uniform grids. The decision not to follow Moll and to adopt, instead, equispaced grids also in the case of the extended version of the model is for the sake of consistency: as mentioned before and specified later (see Subsection 5.2), the codes for the two models should be as analogous as possible in order to be able to compare the results.

updated iteratively. In particular, as a second step, the HJB Equation is solved by approximating the value function, via the finite-difference method, at a finite number of discrete points in the space dimension (namely, wealth). The finite-difference approximation of the HJB Equation at the i-th point in the space dimension, a_i, is the following:

$$\rho v_{i,j}\left(a_i\right) = u\left(c_{i,j}\right) + v'_{i,j}\left(a_i\right)\left(ra_i + y_j w^* - c_{i,j} + \left(\mu^* - r\right)k_{i,j}\right) + \frac{1}{2}v''_{i,j}\left(a_i\right)\sigma^2 k_{i,j}^2 + \lambda_j\left(v_{i,-j}\left(a_i\right) - v_{i,j}\left(a_i\right)\right),$$

where the approximation of the consumption and capital holdings is given by:

$$c_{i,j} = \left(u'\right)^{-1}\left(v'_{i,j}\left(a_i\right)\right),$$
$$k_{i,j} = -\frac{\mu^* - r}{\sigma^2}\frac{v'_{i,j}\left(a_i\right)}{v''_{i,j}\left(a_i\right)}.$$

The so-called “implicit method” is used to update the value function: starting from an initial value, the value function is updated through a functional implicit equation. In addition to this, the so-called “upwind scheme” is employed in order to determine whether a backward or forward difference is more suitable for the approximation: briefly, the forward difference approximation is computed whenever the drift of the state variable (here, savings as the drift of wealth: $s_j\left(a\right) = ra + y_j w^* - c + \left(\mu^* - r\right)k$) is positive, whereas the backward difference approximation is computed whenever this drift is negative. As highlighted by the previous formula, to compute backward or forward savings, also consumption and capital holdings need to be approximated with a backward and forward procedure. Then, the first- and second-order derivatives of the value function can be computed employing the upwind scheme to finally get to the value function approximation. This procedure is iterated as long as the approximated value function at a specific step of the cycle is close enough (according to a given threshold) to the approximation computed at the previous step.

Once the value function is converged to its true approximated value, the KF Equation can be solved, as before, by approximating the wealth distribution through the finite-difference method. The finite-difference approximation of the KF Equation at the i-th point in the space dimension, a_i, is as follows:

$$0 = -\left(ra_i + y_j w^* - c_{i,j} + \left(\mu^* - r\right)k_{i,j}\right)g'_{i,j}\left(a_i\right) + \frac{1}{2}\sigma^2 k_{i,j}^2 g''_{i,j}\left(a\right) + \lambda_{-j} g_{i,-j}\left(a_i\right) - \lambda_j g_{i,j}\left(a_i\right).$$

Differently from the HJB Equation, no iterative procedure is needed to solve this equation, given that it is linear in g_j. As before, an upwind scheme is employed for the computation of the first-order derivative of g.

Given the wealth distribution from the resolution of the KF Equation, the total amount of wealth ($\sum_{j\in\{1,2\}}\int_{\underline{a}}^{+\infty} a dG_j\left(a,t\right)$) and the aggregate level of capital ($\sum_{j\in\{1,2\}}\int_0^{+\infty} k\left(a\right)dG_j\left(a,t\right)$) in the economy, as implied by the model, can be computed. Therefore, exploiting the market clearing condition in (2.16) the riskless interest rate can be update using the bisection algorithm: it is increased whenever the total net worth of individuals is lower than the aggregate level of capital, and decreased whenever the total net worth of individuals is higher than the aggregate level of capital.

Once the updated interest rate is close enough (given a certain threshold) to its previous value, then also the aggregate capital level needs to be updated, according to a similar bisection approach: it is increased whenever the aggregate capital level in the economy implied by the model is higher than its previous value, while it is decreased whenever the aggregate capital level in the economy implied by the model is lower than its previous value. Once the updated level of aggregate capital is close enough (given a certain threshold) to its previous value, then the equilibrium riskless interest rate and aggregate capital level in the economy are found.

5.2 Model parametrization

For the purpose of obtaining comparable results from the two above-mentioned models, the following parameters have been initialized to the same values in the MATLAB scripts. The CRRA coefficient for utility is set to $\gamma = 2.5$, which represents an average value of risk aversion between investors in the real economy, within the range of most commonly accepted measures that lie between 1 and 3, as illustrated in Gandelman and Hernandez-Murillo (2014). The individual discount rate ρ is set equal to 0.05, which is quite low, indicating that individuals consider future consumption almost as relevant as present consumption. The recovery rate (which can be interpreted as the job-finding rate in the basic model, where y is simply the stochastic income process) is assumed to be quite close to 1, namely $\lambda_1 = 0.9$, while the sickness rate (corresponding to the job-destruction rate in the basic model) is considerably lower, $\lambda_2 = 0.075$, indicating that recovering from illness (or finding an employment) has a significantly higher probability than switching from the healthy condition to sickness (or leaving a job). As outlined before (see Section 2), the switching intensities λ_1 and λ_2 are set in a way such that, in the aggregate, the supply of labour L equals a unit. Lastly, individuals are borrowing-constrained through a restriction on their total amount of wealth: $a \geq -\underline{a}$, where $\underline{a} = -2$.

On the other hand, the y process plays different roles in the two models and, therefore, requires to be dealt with accordingly. To solve the basic model, where y represents the two-state stochastic income process, it is set that $y_1 = 0.7$ (unemployment income) and $y_2 = 1.6$ (employment income), whereas in the extension of the model y describes the two-state stochastic multiplicative shock of wage and it is denoted as $y_1 = 0.3$ (sickness state) and $y_2 = 1.1$ (healthy state). It must be underlined that the values of the y's in the two models are chosen in a way such that, after the multiplicative shock is applied in the case of the extended version, the y's result to be of comparable size. Furthermore, the additional parameters of the extended version of the model are set up as follows: the total factor productivity is normalized to the unit, $A = 1$; the output elasticity of capital is specified as $\alpha = \frac{1}{3}$; lastly, the depreciation rate of capital is defined as $\sigma = 20\%$.

5.3 Main results and graphical representation

5.3.1 Model comparison: interest rate, asset supply and optimal policies

In equilibrium, the main result of the numerical resolution for the basic model concerns the risk-free interest rate on bonds, which results to be $r = 3.79\%$.

On the other hand, the numerical algorithm for the extended version of the model provides the following results:

- Riskless interest rate: $r = 3.13\%$,
- Aggregate capital level: $K = 12.23$,
- Deterministic component of the return on capital: $\mu^* = 6.28\%$.

It can be easily observed that the introduction of an additional source of uncertainty, in the form of a stochastic depreciation rate on the capital invested by agents, in the framework of an heterogenous agent model with exogenous idiosyncratic shocks leads to a slight reduction in the interest rate on riskless bonds, which decreases from around 3.8% to approximately 3.1%. Indeed, it is proven that a higher level of uncertainty corresponds to an increase in agents' demand for precautionary saving; this, in turn, leads to lower interest rates, in order to ensure the clearing for the savings market (Amisano and Tristani (2019)).

In addition to this, it can be noted that, in the higher-uncertainty model the aggregate level of assets held (or invested) by individuals, which describes the dimension of the whole economy, is around 12, according to the equilibrium condition of equivalence between net worth and aggregate capital, since the net supply of bonds is imposed to be zero.

Furthermore, these results prove to be consistent with the interest rates inequality illustrated in the theoretical set-up of the model (see eq. (2.15) in Section 2): the riskless interest rate is, indeed, materially lower than the risky return on capital (around half of it), ensuring that households are not disincentivized from choosing riskier investments and continue to contribute to their future income.

In terms of wealth distribution, the following plots illustrate the distributions implied by the benchmark and by the extended model, showing also the different behaviour for the two types of agents in the economy:

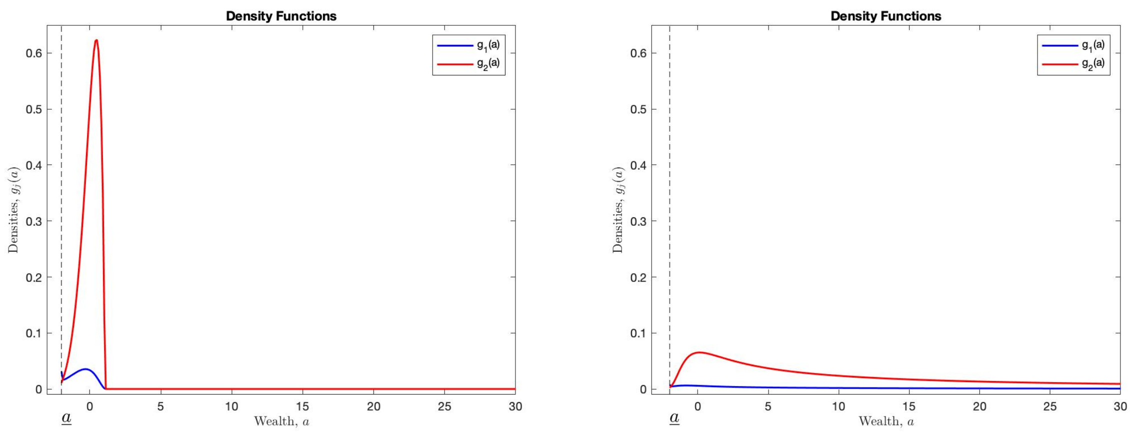

Fig. 5.1: Stationary distribution in the basic model (left) and extended model (right)

It can be seen that, in the basic model, the agents are entirely condensed in the left part of the wealth distribution, namely the part referred to the lowest values of wealth (from the lower limit, -2, to around 1), while from a certain low level of wealth on, the mass of agents reduces drastically to exactly 0. For the extended model, the wealth distribution is significantly different, with households being distributed in a more uniform way: with respect to type-2 agents (that are, individuals with employment income in the basic model and individuals in a healthy condition in the extended model), the majority of the mass is still concentrated in the left tail of the distribution, although some quantity of agents exhibit also higher levels of wealth, quite differently from what occurs for the benchmark model; for type-1 agents (individuals with unemployment income or in a sick condition, depending on the model), the distribution appears almost flat and very close to 0 for every value of wealth (a small mass is only visible in the left tail). It should be noted that the discrepancy between the stationary densities of wealth under the two models may be partly explained also by the fact that the distribution takes into account financial wealth only, which corresponds to total wealth under the benchmark model, whereas in the extended model total wealth comprises labour income as well.

The following graphs, showing the consumption and savings behaviour and the capital and bond holdings of individuals, contain also the illustration of the stationary density functions (as shown in the previous figures) in order to contextualize any comments made with respect to the distribution of agents. Looking at the consumption policies optimally chosen by households in the two models, their functions with respect to wealth are plotted below,:

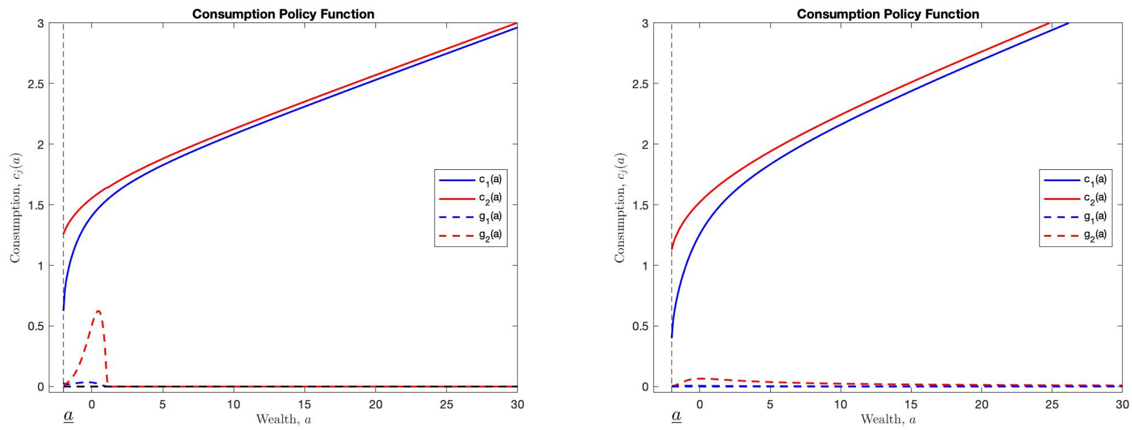

Fig. 5.2: Consumption policy functions in the basic model (left) and extended model (right)

The plots highlight that, in the model with additional uncertainty due to the investment in risky capital, the consumption curve is moderately steeper than in the benchmark model, but consumption level corresponding to low values of wealth is slightly lower with higher uncertainty. For instance, for values of wealth just above the constraint (which is set at -2), the consumption curve for type-1 and type-2 agents in the basic model is at around 0.6 and 1.3, respectively, while in the extended version agents consume below 0.4 and 1.2, respectively.

For what concerns the optimal savings policies determined by households, their functions with respect to wealth in the two models are shown below:

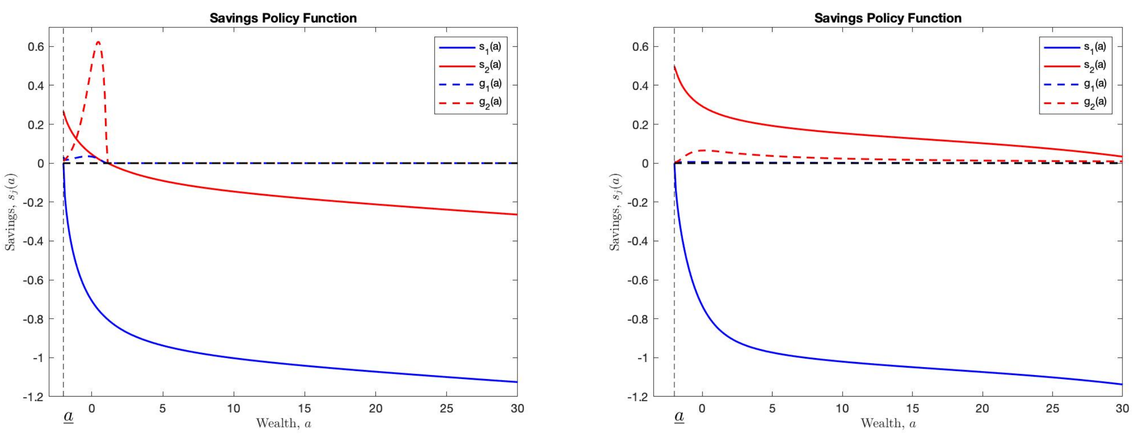

Fig. 5.3: Savings policy functions in the basic model (left) and extended model (right)

It emerges that savings, which are decreasing in wealth, for type-2 households are significantly higher in the model with increased uncertainty, showing positive values for all the wealth levels illustrated in the graph (up to 30); in the basic version of the model, savings for type-2 agents are still positive for wealth levels corresponding to a non-zero mass of individuals, but quite lower than in other model. Differently, for type-1 households savings are negative in all the wealth domain in both models, and decrease very rapidly in wealth; in this case, the savings trend is similar under the two models.

Additionally, as for the extended version of the model, below is a plot of the Lorenz curve implied by the model:

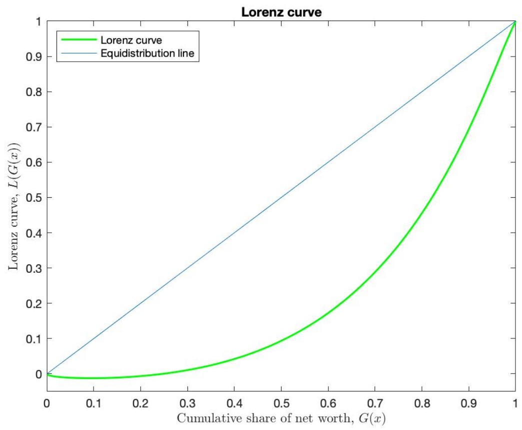

Fig. 5.4: Lorenz curve in the extended model

The Lorenz curve illustrates the distribution of wealth within an economy using cumulative quantities. The area between the line of equidistribution (i.e. the case where wealth is equally distributed in the population) and the true Lorenz curve represents the concentration of wealth in the population. Being below the 45-degree line, the curve indicates that wealth is unequally distributed in the population and presents an high concentration: most of the wealth in the economy is owned by the a small fraction of households. For instance, the richest 30% of households possess more than 70% of total wealth in the economy.

Furthermore, the following plots illustrate the behaviour of capital holdings and bond holdings, respectively, as functions of wealth:

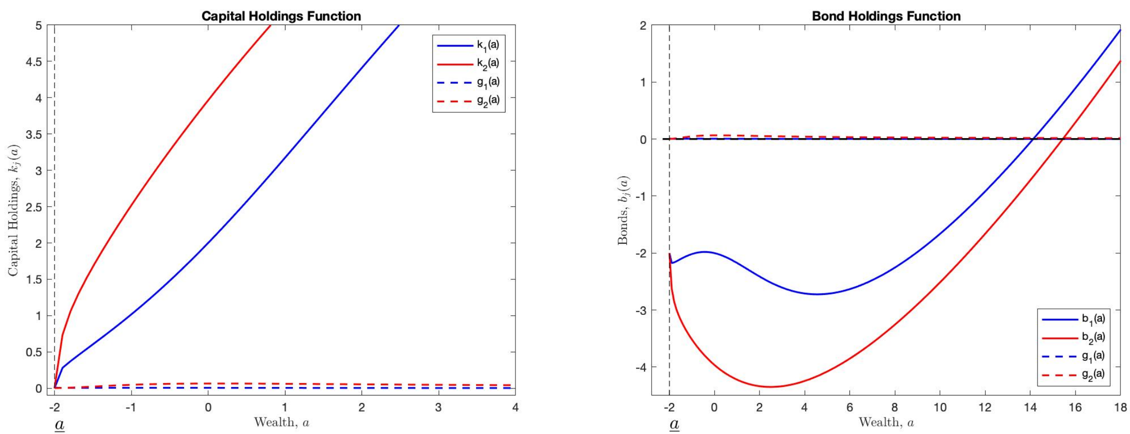

Fig. 5.5: Capital holdings (left) and bond holdings (right) as a function of wealth

In relation to capital, the plot shows that the risky holdings are always positive (due to the constraint posed on capital) and significantly increase, both for type-1 and type-2 agents, as wealth increases. In particular, capital holdings rise much faster for the agents of the second type, for whom the function of capital with respect to wealth is also clearly concave; this evidence is not so explicit for the other category of agents, whose function seems, instead, to be convex in wealth.

Looking at the second plot, one can observe that bond holdings are initially negative and diminishing in wealth, decreasing more for type-2 agents and reaching a low at below -4. From values of wealth approximately in the range $(3, 5)$, the holdings of riskless bonds start to increase exponentially and become positive for both types of agents, so that bond holdings are convex functions in wealth.

5.3.2 Comparative statics: changes in wealth distribution, interest rate, aggregate capital and capital rental rate

Exploiting some comparative statics, it is possible to investigate the changes in the wealth distribution, in the riskless interest rate and in the capital rental rate that are caused by changes in the depreciation rate of capital, in the output elasticity of capital and in the stochastic shocks on labour wage.

First, by considering the depreciation rate of capital, σ, is must be highlighted that the MATLAB code is very sensitive to changes in its value. Indeed, convergence for the level of aggregate capital is not always guaranteed for different values of the depreciation. Ceteris paribus, small changes in the depreciation rate of capital lead to the following trends in the other parameters:

Depreciation rate, σ	**Interest rate, r**	**Aggregate capital, K**	**Deterministic capital rental rate, μ^***
10%	4.40%	15.97	5.26%
15%	3.65%	14.47	5.61%
17%	3.40%	13.61	5.85%
20%	3.13%	12.23	6.28%

It emerges that halving the depreciation rate from 20% to 10% implies the following: the riskless interest rate increases from above 3% to around 4.5%; the aggregate capital level is also increased from approximately 12 to almost 16; by contrast, the deterministic rental rate on capital shows a slight decrease, reaching around 5% from an initial value of more than 6%. Furthermore, the density functions of wealth for the different values of the depreciation rate are illustrated in the following plots:

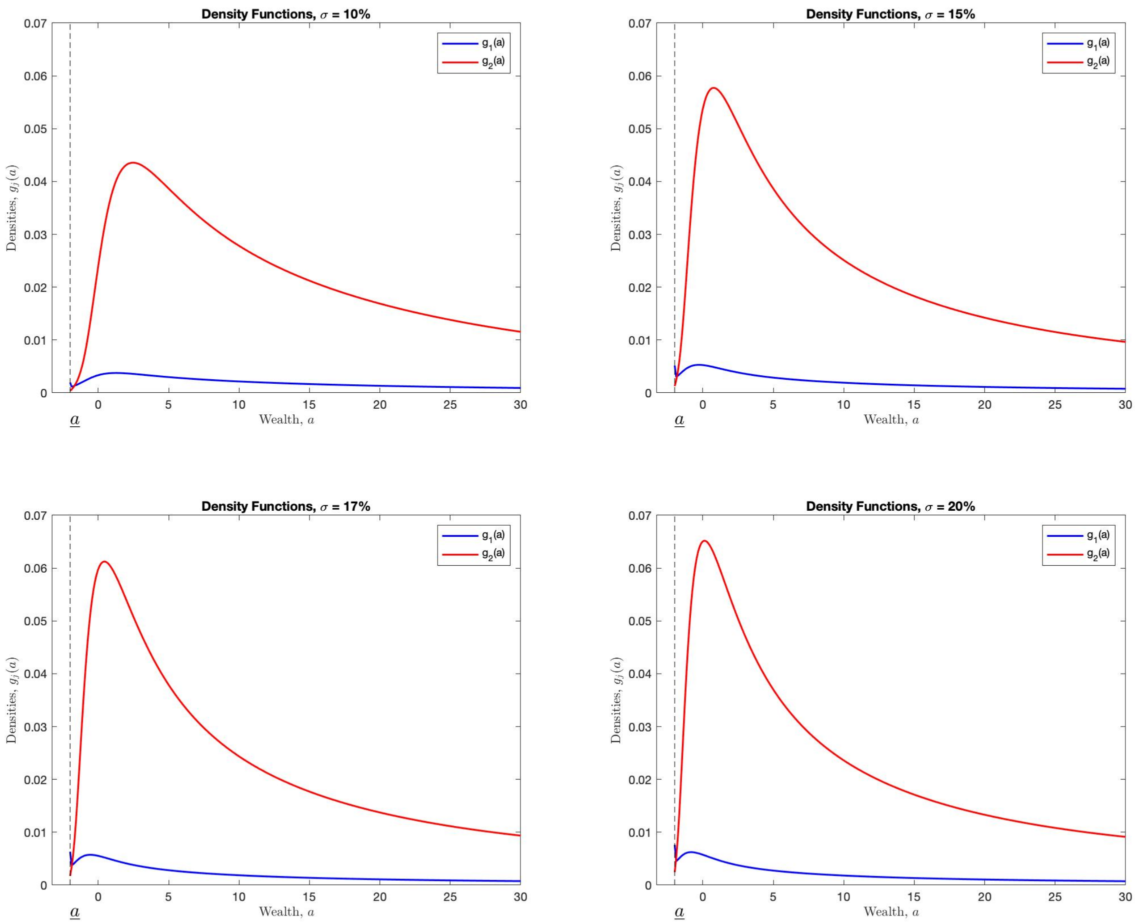

Fig. 5.6: Stationary density functions for different values of the depreciation rate

As the depreciation rate on capital increases, the percentage of households that exhibit the lowest amount of net worth (wealth from the minimum value, -2, until 5, for instance) increases and the distribution shows an increasingly higher peak. This effect is clearly evident for type-2 agents, while it is much more smoothed for type-1 agents.

Then, allowing the output elasticity of capital, α, to vary between 0 and 1, all the rest being equal, one can observe the following trends in the parameters under analysis:

Output elasticity of capital, α	**Interest rate, r**	**Aggregate capital, K**	**Deterministic capital rental rate, μ^***
1/4	2.82%	7.35	5.60%
1/3	3.13%	12.23	6.28%
0.4	2.95%	18.59	6.93%

It can be noted that an increase in the output elasticity of capital from 25% to 40% leads to the following evidences: the aggregate capital level almost triple, from approximately 7 to almost 19; also the deterministic rental rate on capital shows a moderate increase, reaching around 7% from an initial value of 5.6%; on the other hand, the riskless interest rate does not display a clear trend, first increasing and then decreasing. Like for the depreciation rate, also in this case the MATLAB code is very sensitive to changes in the output elasticity of capital. Furthermore, the density functions of wealth for the different values of the output elasticity of capital are illustrated in the following plots:

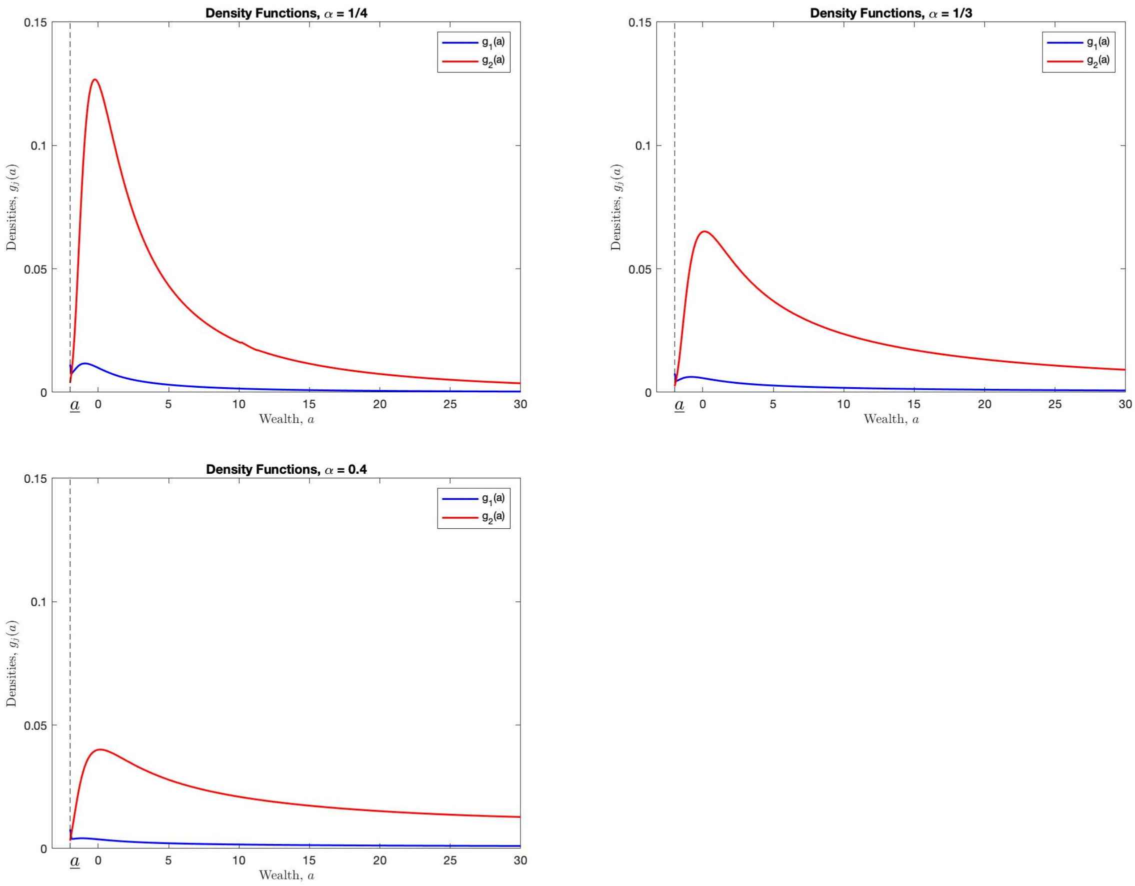

Fig. 5.7: Stationary density functions for different values of the output elasticity of capital

The graphs show that, as the output elasticity of capital increases, the percentage of households that exhibit the lowest amount of net worth decreases exponentially. This is clearly observable in the last plot, corresponding to $\alpha = 0.4$: the distribution of the density function is considerably smoothed with respect to the other plots, where the peak in the left-hand side of the function indicates that the majority of the individuals owns a limited amount of wealth. As before, the effect is much more tangible for type-2 agents than for type-1 individuals, whose density function is smoother.

Lastly, focusing the attention on changes on the stochastic shocks on labour wage that households are subject to, y_j with $j = 1, 2$, all the rest being equal, the following trends in the parameters under analysis are observed:

Shocks on wage, y_j	**Interest rate, r**	**Aggregate capital, K**	**Deterministic capital rental rate, μ^***
$y_1 = 0.5$, $y_2 = 0.7$	4.71%	10.21	7.08%
$y_1 = 0.9$, $y_2 = 1.1$	4.66%	10.78	6.83%
$y_1 = 0.3$, $y_2 = 1.1$	3.13%	12.23	6.28%
$y_1 = 0.3$, $y_2 = 1.5$	2.79%	13.59	5.85%

It is noted that an increase in the gap between the two states of the stochastic shock on labour wage (meaning that the two values are more extreme compared to one another) with respect to the basic case of $y_1 = 0.3$ and $y_2 = 1.1$ implemented in the model (look, for example, at the case for $y_1 = 0.3$ and $y_2 = 1.5$) leads to a slight increase in the aggregate capital level, whereas it causes a small decrease in the riskless interest rate and in the capital rental rate; accordingly, a decrease in the gap (see the case for $y_1 = 0.9$ and $y_2 = 1.1$ and the one for $y_1 = 0.5$ and $y_2 = 0.7$) causes a decrease in the aggregate capital level, while the riskless interest rate and the capital rentale rate are subject to a small increase. Furthermore, the plots below graphically show the density functions of wealth for the different values of the stochastic shocks on labour wage:

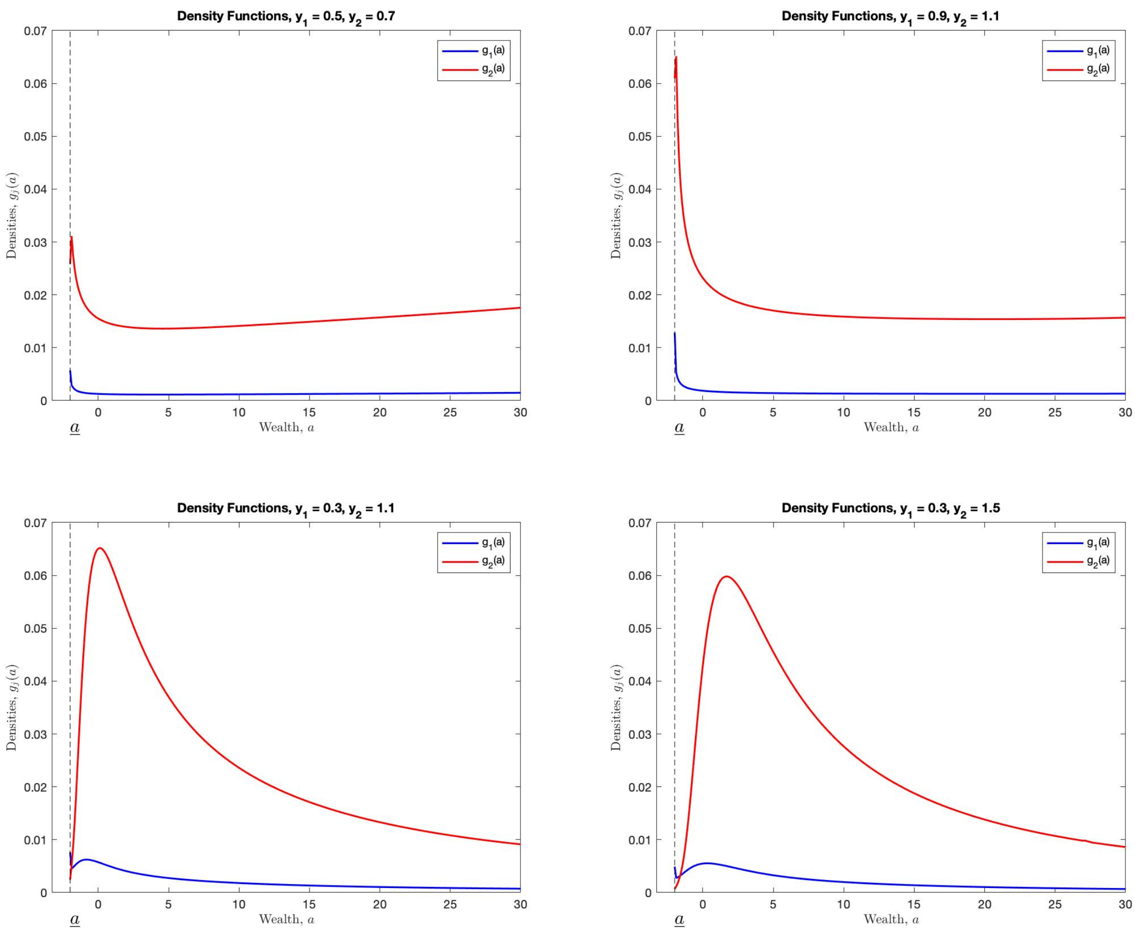

Fig. 5.8: Stationary density functions for different values of the shocks on labour wage

It is evident from the graphs that the probability density functions become much more homogeneous as the divergence between the two states of the stochastic shock on labour wage increases.

As a final remark, again, all the simulations performed in this Subsection fulfill the condition on the interest rates that must be verified in equilibrium, according to which the return on risky capital must be greater than the return on riskless bonds.

6 Economic interpretation of the results

The comparison of the results obtained by the numerical resolution of the basic market model à la Huggett and its implementation to incorporate risky investment (see Sub-subsection 5.3.1) offers several interesting insights from an economic point of view.

Above all, regarding the consumption policies under both models, as it should be natural, agents of the second type (namely, employed or in a healthy condition, depending on the model considered) consume more than other agents, given that they receive an higher income from labour, after accounting for the effect of the stochastic shocks. In addition to this, comparing the two scenarios, the analysis shows that consumption is slightly reduced under uncertainty. As one would expect, increased uncertainty typically depress consumption. It has been widely noted that households are, in fact, incentivized to smooth consumption as a consequence of future income uncertainty and have a precautionary motive to accumulate assets. Indeed, as it will be better detailed in the next paragraph, precautionary saving as a response to changing income uncertainty plays an important role in explaining the observed changes in both consumption and savings, as proven by Hahm and Steigerwald (1999).

In terms of savings, the optimal policies adopted by households underline the relevance of the precautionary saving motive, which consists essentially in the positive extra saving arising, by way of example, from the fact that future income (here, labour income) is stochastic, and thus random, rather than deterministic and known beforehand (Leland (1968)). Indeed, for agents of the second type, i.e. the ones receiving an higher income from labour (either because they are employed or in a healthy condition, depending on the model considered), the savings are higher with increased uncertainty than in the basic model, for given levels of individual wealth. This is in accordance with the inclination of households to save more in times where uncertainty about future earnings is higher. However, although it is true for both agent types, as well as for both models, that the precautionary motive for saving has a greater impact on lower wealth levels, for type-1 agents savings show a similar trend under the two models. This outcome could be seen in light of the fact that there is yet no consensus in literature about the intensity and the impact of the rationale for saving on individuals (Lugilde et al. (2017)). On the other hand, it is also true that there is evidence, in literature, that a large share of saving and wealth accumulation is not attributable to earnings (here, labour wage) uncertainty exclusively, although it being the most addressed rationale underlying precautionary savings (Guiso et al. (1992)). In particular, it is argued that the precautionary motive alone cannot justify the wealth levels of the richest, as outlined in Lusardi (1998).

For what concerns capital holdings, which correspond to the risky component of the households' portfolios, what emerges from the previous simulations is that households increase their investment in risky and productive capital as their wealth rises. Indeed, considering that agents exhibit a certain degree of risk aversion by assumption, they are inclined to invest more in risky assets as they become wealthier, since this choice entails an element of uncertainty while providing the possibility to earn larger returns. Furthermore, the fact that the function of capital holdings for agents of the second type is concave in wealth indicates that households' propensity to invest in risky capital slows down for very large values of wealth, which may be caused by an higher risk aversion.

By considering bonds, since the holdings result to be negative for low wealth values, it emerges that the poorest borrow money instead of investing in riskless securities. This observed evolution can be supported by a reasoning, comparable to the previous one, based on the risk aversion of individuals. The initial negative holdings of bonds, suggest that low-wealth households are prone to indebt themselves - and increase their indebtedness - at a risk-free interest rate in order to improve their economic condition. This is even more tangible for agents of the second type, who may be assumed to wish to exploit riskless borrowing to increase their wealth more than type-1 agents. After reaching a certain level of wealth, individuals cease to take on more debt and start to invest in risk-free bonds, whose holdings steeply increase as wealth rises, in a similar fashion for both agent categories. In other words, it may be said that, as households reach a satisfactory wealth level, they start to invest exploiting a risk-free interest rate.

Below some further considerations on the comparative statics analysis performed in Sub-subsection 5.3.2.

First, changes in the depreciation rate of capital significantly affect the other dimensions investigated. An increase in the depreciation rate can be translated into an increase in the riskiness of capital; in turn, this causes a reduction in the riskless interest rate on bonds (for the savings market clearing, as explained in the previous Sub-subsection, 5.3.1), while the rental rate on capital increases. As a consequence, the risk premium of risky capital with respect to riskless bonds is increased. This may be seen in terms of the Sharpe Ratio of capital. To this purpose, below are reported the Sharpe Ratio values computed for the different values of the depreciation rate and, thus, capital rental rate and risk-free rate:

Depreciation rate, σ	**Risk premium,** $\mu^* - r$	**Sharpe Ratio,** $\frac{\mu^* - r}{\sigma}$
10%	0.86%	8.60%
15%	1.96%	13.07%
17%	2.45%	14.41%
20%	3.15%	15.75%

It can be observed that the Sharpe Ratio increases with depreciation, meaning that the risk-adjusted performance of capital improves as its riskiness increases. Therefore, capital holdings constitute profitable investments, even accounting for their riskiness.

With regards to the output elasticity of capital, an increase in its value implies an increase in the relevance of capital within the production process, translating into a more capital-intensive production process. This can be clearly observed in terms of the increase in the rental rate of capital and in the higher aggregate level of capital in the economy. Furthermore, the greater importance of capital can be interpreted in terms of the trend for automation in the production process: as capital becomes more and more used to produce output, the inequality between households is more evident, given that the poorest (type-1 agents) cannot increase their investments in capital as they wish, and capital is the most-profitable good in the economy for high levels of its output elasticity. This inequality in the income distribution is clear in the increased distance between the densities of the two categories of individuals (as shown in the last plot of figure 5.7).

Lastly, it is worth mentioning the double effect of the depreciation rate and the rental rate on capital on the risk-free interest rate. Since the interest rate on bonds depends on both of these two dimensions, it has been empirically observed that a change in one of them does not always provide straightforward and unequivocal results with respect to the interest rate, given that, for instance, changes in the depreciation rate cause changes in the rental rate as well. Indeed, the final impact on the risk-free interest rate depends on how these two effects counterbalance one another and on which of them dominates on the other.

7 Conclusion

The analysis performed in this work is aimed at investigating the impact that the introduction of additional market uncertainty causes on the riskless interest rate on bond and on the distribution of wealth (thus, inequality) in the economy. In particular, the heterogeneous households, besides being subject to stochastic shocks on their labour income, are given the possibility to invest in risky and productive capital, which is subject to stochastic depreciation shocks.

As outlined in the introduction, the twofold contribution of this work materializes, first, in the analysis of the presented in model in order to provide some useful insights in terms of the impact on the interest rate and on the distribution of wealth, and, secondly, in the provision of a methodological approach to derive and deal with the system of coupled PDE that describes the economy (namely, the HJB and KF Equations).

The main evidences arisen from this work are reported below. First, as one would expect, the risk-free interest rate decreases under greater uncertainty. The rational behind this evidence can be found in the precautionary motive, according to which increased uncertainty leads to a growing demand for precautionary saving, which in turn lowers interest rates, in order to ensure the clearing of the market (Amisano and Tristani (2019)). In terms of the distribution of wealth, it is noted that, in the context of higher uncertainty, the stationary density becomes more uniformly-distributed than in the benchmark case. This result must be seen in light of the fact that the models deal with financial wealth only: while it corresponds to total wealth under the benchmark model, the same is not true for the extended model, where total wealth comprises labour income as well. Additionally, through the analysis of the Lorenz curve implied by the model under analysis, it is evident that this type of economy implies significant inequality in the distribution of wealth. With reference to the consumption and savings policies adopted by households in the two scenarios, greater uncertainty implicates that consumption is depressed, whereas savings are increased. Again, an important contribution in explaining these evidences is provided by the above-mentioned precautionary motive: as a consequence of future income uncertainty, households are inclined to smooth consumption and have a precautionary motive to accumulate assets. Therefore, it appears clear that the precautionary motive plays a relevant role in clarifying these dynamics, although it is not sufficient to justify the large accumulation of wealth and the high wealth holdings of the very rich, as commonly agreed in literature (Lusardi (1998)). Focusing on the depreciation rate of capital, the analysis shows that an increase in the depreciation, which indicates that the riskiness of capital is higher, causes a rise in the risk premium of risky capital with respect to riskless bonds, since the risk-free interest rate decreases, while the return on capital increases. Another interesting evidence is related to the the output elasticity of capital, indicating the relative importance of capital with respect to labour in the production process. Positive variations in its value have a positive impact on both the return on capital and the individual capital holdings, leading to a higher level of capital in the aggregate; this is proven to increase wealth inequalities in the economy. As a last remark, it should be noted that, under the present set-up, the effect on the riskless interest rate caused by the increased uncertainty is not always straightforward, due to the fact that it takes into account variations in both the depreciation rate and the rental rate on capital, which affect each other.

By way of conclusion, the model under analysis presents some clear limitations, which lead the way to further work. First, the stochastic process defined for the labour income shocks is a basic two-state Poisson process, which could be implemented to consider a more realistic outlook. Then, individuals could be assumed to have preferences besides work, so that their labour supply is not inelastic with respect to wage. In addition, the model does not take into account inflation, according to which no asset could be actually considered risk-free.

A Appendix

This appendix provides the full MATLAB codes implemented and used in order to solve numerically the two models: for the basic model entailing only bonds, the code corresponds mostly to the script by Moll (Benjamin Moll Website) for the steady-state equilibrium in a Huggett economy, where integrations are made for what concerns the set-up of the model and the layout of the results; the code for the extended version of the model is based on the previously-mentioned code, but implemented to include capital investments.

A.1 Basic model MATLAB code

Huggett_bonds.m

```
%% HUGGETT MODEL - INVESTMENT IN BONDS
% uses implicit upwind method for HJB
clear all; clc; close all;
tic;
%% SET UP
gamma = 2.5; %CRRA utility with parameter gamma
rho = .05; %discount rate
y1 = .7; %income of type 1 agent
y2 = 1.6; %income of type 2 agent
y = [y1,y2]; %income (state variable)
la1 = .9; %lambda_1, switching intensity from state 1 to state 2
la2 = .075; %lambda_2, switching intensity from state 2 to state 1
la = [la1,la2]; %lambda
I = 500; %number of discrete points in the space dimension (a_i with i=1,...,I) used to approximate the value function (v_i_j=v_j(a_i))
%wealth a (state variable)
amin = -2; %borrowing constraint
amax = 50; %borrowing upper limit
a = linspace(amin,amax,I)'; %generation of a column vector of I points, with space between them of (amax-amin)/(I-1)
da = (amax-amin)/(I-1); %space between points (delta_a)
aa = [a,a]; %matrix with 2 identical columns of wealth values
yy = ones(I,1)*y; %matrix with 2 columns: the first contains I times y1, the second I times y2
maxit = 100; %maximum number of iterations for value function
crit = 10^(-6); %critical value to evaluate distance of value function at (n+1) from value function at n
```

```
Delta = 1000; %step size of the implicit method used for the HJB
dVf = zeros(I,2); %set up of a matrix of forward differences of value function
dVb = zeros(I,2); %set up of a matrix of backward differences of value function
dV0 = zeros(I,2); %set up of a matrix of differences of value function at steady state
c = zeros(I,2); %set up of a matrix of consumption values (control variable)
%speye(I) creates a sparse IxI identity matrix (1s in the main diagonal, 0s elsewhere)
Aswitch = [-speye(I)*la(1),speye(I)*la(1);speye(I)*la(2),-speye(I)*la(2)];
r0 = .01; %initial guess
Ir = 100; %maximum number of iterations for interest rate
rmin = .01; rmax = .05;
crit_S = 10^(-5); %critical value to evaluate the interest rate
%% RESOLUTION
%Initial guesses
%For interest rate
r = r0;
%For value function: a natural initial guess is the value function of "staying put", which is equal to u(c)/rho
v0(:,1) = (y(1) + r*a).^(1-gamma)/(1-gamma)/rho;
v0(:,2) = (y(2) + r*a).^(1-gamma)/(1-gamma)/rho;
for ir=1:Ir %iterate for the interest rate
r_r(ir) = r; %save the interest rates
rmin_r(ir) = rmin;
rmax_r(ir) = rmax;
if ir>1
v0 = V_r(:,:,ir-1); %update the initial value function (from step 2 of the for cycle)
end
%% HJB EQUATION
v = v0;
for n=1:maxit %iterate for the value function
V = v;
V_n(:,:,n)=V;
%forward difference approximation of first derivative
dVf(1:I-1,:) = (V(2:I,:)-V(1:I-1,:))/da;
dVf(I,:) = (y + r.*amax).^(-gamma); %will never be used, but impose state constraint a<=amax just in case: dVf=u'(c(amax))
%backward difference approximation of first derivative
dVb(2:I,:) = (V(2:I,:)-V(1:I-1,:))/da;
```

```
dVb(1,:) = (y + r.*amin).^(-gamma); %state constraint boundary condition: dVb=u'(c(amin))

I_concave = dVb > dVf; %indicator whether value function is concave (problems arise if this is not the case) %this condition corresponds to ssb > ssf

%consumption and savings with forward difference

cf = max(dVf,10^(-10)).^(-1/gamma);

ssf = yy + r.*aa - cf;

%consumption and savings with backward difference

cb = max(dVb,10^(-10)).^(-1/gamma);

ssb = yy + r.*aa - cb;

%consumption and derivative of value function at steady state

c0 = yy + r.*aa;

dV0 = max(c0,10^(-10)).^(-gamma); %u'(c)

%dV_upwind makes a choice of forward or backward differences based on the sign of the drift

If = ssf > 0; %positive drift --> forward difference

Ib = ssb < 0; %negative drift --> backward difference

I0 = (1-If-Ib); %at steady state

dV_Upwind = dVf.*If + dVb.*Ib + dV0.*I0; %include the third term for taking into account cases where sf <= 0 <= sb (we set savings=0)

c = max(dV_Upwind,10^(-10)).^(-1/gamma);

u = c.^(1-gamma)/(1-gamma);

%Construct matrix

X = - min(ssb,0)/da;

Y = - max(ssf,0)/da + min(ssb,0)/da; %lambda is subtracted later, when Aswitch is added

Z = max(ssf,0)/da;

%create the part of matrix A referred to values for j=1;

%spdiags(a,b,c,d) creates a cxd matrix and puts in diagonal b the values of a (the main diagonal is indicated by 0)

A1 = spdiags(Y(:,1),0,I,I)+spdiags(X(2:I,1),-1,I,I)+spdiags([0;Z(1:I-1,1)],1,I,I);

%create the part of matrix A referred to values for j=2

A2 = spdiags(Y(:,2),0,I,I)+spdiags(X(2:I,2),-1,I,I)+spdiags([0;Z(1:I-1,2)],1,I,I);

A = [A1,sparse(I,I);sparse(I,I),A2] + Aswitch; %2I x 2I matrix called Poisson transition matrix or Intensity matrix

if max(abs(sum(A,2)))>10^(-9) %sum(A,2) is a column vector containing the sum of each row of A

disp('Improper Transition Matrix')

break

end

B = (1/Delta + rho)*speye(2*I) - A;

u_stacked = [u(:,1);u(:,2)]; %column vector of 2xI elements (first the ones for j=1, then for j=2)
```

```
V_stacked = [V(:,1);V(:,2)]; %similarly but for the value function
b = u_stacked + V_stacked/Delta;
V_stacked = B\b;
%Solve system of equations
V = [V_stacked(1:I),V_stacked(I+1:2*I)]; %re-create a matrix with 2 different columns for j=1 and j=2
Vchange = V - v;
v = V;
dist(n) = max(max(abs(Vchange))); %first compute the max of the 2 columns for j=1 and for j=2, then compute the max of the 2 max
if dist(n)<crit %convergence criterion
disp('Value Function Converged, Iteration = ')
disp(n)
break %stop the for cycle of the value function (find the converged value function for each value of the interest rate)
end
end
toc;
%% FOKKER-PLANCK EQUATION
AT = A';
b = zeros(2*I,1); %since ATg must be equal to 0
%need to fix one value, otherwise matrix is singular
i_fix = 1; b(i_fix)=.1;
row = [zeros(1,i_fix-1),1,zeros(1,2*I-i_fix)];
AT(i_fix,:) = row; %substitute the first row of AT with the values in row
%Solve linear system
gg = AT\b;
g_sum = gg'*ones(2*I,1)*da;
gg = gg./g_sum;
g = [gg(1:I),gg(I+1:2*I)];
check1 = g(:,1)'*ones(I,1)*da;
check2 = g(:,2)'*ones(I,1)*da;
g_r(:,:,ir) = g;
adot(:,:,ir) = yy + r.*aa - c; %first derivative of a, which corresponds to saving
V_r(:,:,ir) = V;
dV_r(:,:,ir) = dV_Upwind;
c_r(:,:,ir) = c;
S(ir) = g(:,1)'*a*da + g(:,2)'*a*da; %asset supply
```

```
%% UPDATE INTEREST RATE
if S(ir)>crit_S
disp('Excess Supply')
rmax = r;
r = .5*(r+rmin); %decrease the interest rate
elseif S(ir)<-crit_S
disp('Excess Demand')
rmin = r;
r = .5*(r+rmax); %increase the interest rate
elseif abs(S(ir))<crit_S
disp('Equilibrium Found, Interest rate =')
disp(r)
disp('Asset supply =')
disp(S(ir))
break %stop the for cycle of the interest rate
end
end
amax1 = 0.6;
amin1 = amin-0.03;
r_S = [r_r' S']; %create a matrix where each row contains the interest rate and the corresponding asset
supply
r_S = sortrows(r_S,1); %order the matrix according to the increasing interest rate
%% GRAPHICAL REPRESENTATION OF THE RESULTS
%Consumption Policy Function
figure(1)
set(gca,'FontSize',16)
plot(a,c_r(:,1,ir),'b',a,c_r(:,2,ir),'r',a,g_r(:,1,ir),'b--',a,g_r(:,2,ir),'r--',a,zeros(1,I),'k--','LineWidth',1.5);
text(amin,-.235,'$\underline{a}$','FontSize',16,'interpreter','latex')
line([amin amin], [min(min(adot(:,:,ir)))-0.05 amax],'Color','Black','LineStyle','--')
legend('c_1(a)','c_2(a)','g_1(a)','g_2(a)','Location','East')
xlabel('Wealth, $a$','interpreter','latex')
ylabel('Consumption, $c_j(a)$','interpreter','latex')
xlim([amin1-1 amax-20]) ylim([-.05 3])
title('Consumption Policy Function')
saveas(gcf,'Consumption policy function_bonds.jpg')
%Savings Policy Function
figure(2)
```

```
set(gca,'FontSize',16)
plot(a,adot(:,1,ir),'b',a,adot(:,2,ir),'r',a,g_r(:,1,ir),'b--',a,g_r(:,2,ir),'r--',linspace(amin1,amax,I),zeros(1,I),'k--','LineWidth',1.5);
text(amin,-1.3,'$\underline{a}$','FontSize',16,'interpreter','latex')
line([amin amin], [min(min(adot(:,:,ir)))-0.05 amax],'Color','Black','LineStyle','--')
legend('s_1(a)','s_2(a)','g_1(a)','g_2(a)','Location','NorthEast')
xlabel('Wealth, $a$','interpreter','latex')
ylabel('Savings, $s_j(a)$','interpreter','latex')
xlim([amin1-1 amax-20])
ylim([-1.2 0.7])
title('Savings Policy Function')
saveas(gcf,'Savings policy function_bonds.jpg')
%Density Functions
figure(3)
set(gca,'FontSize',16)
plot(a,g_r(:,1,ir),'b',a,g_r(:,2,ir),'r','LineWidth',1.5);
text(amin,-.05,'$\underline{a}$','FontSize',16,'interpreter','latex')
line([amin amin], [-0.05 0.8],'Color','Black','LineStyle','--')
legend('g_1(a)','g_2(a)')
xlabel('Wealth, $a$','interpreter','latex')
ylabel('Densities, $g_j(a)$','interpreter','latex')
xlim([amin1-1 amax-20])
ylim([-0.01 0.65])
title('Density Functions')
saveas(gcf,'Density functions_bonds.jpg')
```

A.2 Extended model MATLAB code

ExtendedModel_bonds_capital.m

```
%% EXTENDED MODEL - INVESTMENT IN BONDS AND CAPITAL
% uses implicit upwind method for HJB
clear all; clc; close all;
tic;
%% SET UP
gamma = 2.5; %CRRA utility with parameter gamma
rho = .05; %discount rate
y1 = .3; %state 1 shock on wage (sickness)
```

y2 = 1.1; %state 2 shock on wage (health)

y = [y1,y2]; %idiosyncratic shock on wage

la1 = .9; %lambda_1, switching intensity from state 1 to state 2

la2 = .075; %lambda_2, switching intensity from state 2 to state 1

la = [la1,la2]; %lambda

updk = .2; % Sensitivity parameter <---- important for the convergence of K

%Cobb-Douglas production function

Aprod = 1; % total factor productivity (normalized at 1)

al = 1/3; % alpha (output elasticity of capital)

I = 500; %number of discrete points in the space dimension (a_i with i=1,...,I) used to approximate the value function (v_i_j=v_j(a_i))

%total wealth (state variable)

amin = -2; %borrowing constraint

amax = 50; %maximum wealth

a = linspace(amin,amax,I)'; %generation of a column vector of I points, with space between them of (amax-amin)/(I-1)

da = (amax-amin)/(I-1); %space between points (delta_a)

aa = [a,a]; %matrix with 2 identical columns of wealth values

yy = ones(I,1)*y; %matrix with 2 columns: the first contains I times y1, the second I times y2

maxit = 100; %maximum number of iterations for value function

crit = 10^(-6); %critical value to evaluate distance of value function at (n+1) from value function at n

Delta = 1000; %step size of the implicit method used for the HJB

dVf = zeros(I,2); %set up of a matrix of forward differences of value function

dVb = zeros(I,2); %set up of a matrix of backward differences of value function

dV0 = zeros(I,2); %set up of a matrix of differences of value function at steady state

dV2 = zeros(I,2); %set up of a matrix of approximated differences of value function second derivative

c = zeros(I,2); %set up of a matrix of consumption values (first control variable)

k = zeros(I,2); %set up of a matrix of capital values (second control variable)

%speye(I) creates a sparse IxI identity matrix (1s in the main diagonal, 0s elsewhere)

Aswitch = [-speye(I)*la(1),speye(I)*la(1);speye(I)*la(2),-speye(I)*la(2)];

%Interest rate (first endogenous variable)

r0 = .01; %initial guess for interest rate

Ir = 100; %maximum number of iterations for interest rate

rmin = -.01;

rmax = .05; %must be lower than or equal to mu by hypothesis

crit_S = 10^(-4); %critical value to evaluate the interest rate

%Aggregate capital (second endogenous variable)

```
K0 = 15; %initial guess for aggregate capital
Kr = 10; %maximum number of iterations for capital
Kmin = 13;
Kmax = 14;
crit_K = 10^(-3); %critical value to evaluate the aggregate capital level
test = zeros(1,Kr);
sig = .20;
sig2 = sig^2;
%% RESOLUTION
%Initial guesses
%For aggregate capital
K = K0;
%For interest rate
r = r0;
%Compute the initial values of the deterministic part of the return on capital and wage
mu = al * Aprod * K^(al-1);
wbar = (1-al) * Aprod * K^al;
%For value function: a natural initial guess is the value function of "staying put", which is equal to u(c)/rho
v0(:,1) = (wbar*y(1) + r*a + (mu-r)^2/(gamma*sig2)*a).^(1-gamma)/(1-gamma)/rho; %due to the state constraint
v0(:,2) = (wbar*y(2) + r*a + (mu-r)^2/(gamma*sig2)*a).^(1-gamma)/(1-gamma)/rho; %due to the state constraint
for kr=1:Kr %iterate for aggregate capital
K_K(kr) = K; %save the aggregate capital level
Kmin_K(kr) = Kmin;
Kmax_K(kr) = Kmax;
if kr>1
mu = al * Aprod * K^(al-1); %update the mu (from step 2 of the for cycle)
wbar = (1-al) * Aprod * K^al; %update the wage (from step 2 of the for cycle)
end
for ir=1:Ir %iterate for the interest rate
r_r(ir) = r; %save the interest rates
rmin_r(ir) = rmin;
rmax_r(ir) = rmax;
if ir>1
v0 = V_r(:,:,ir-1); %update the initial value function (from step 2 of the for cycle)
```

```
end
%% HJB EQUATION
v = v0;
for n=1:maxit %iterate for the value function
V = v;
V_n(:,:,n)=V;
%forward difference approximation of first derivative
dVf(1:I-1,:) = (V(2:I,:)-V(1:I-1,:))/da;
dVf(I,:) = (wbar*y + r.*amax + (mu-r)^2/(gamma*sig2)*amax).^(-gamma); %will never be used, but impose state constraint a<=amax just in case: dVf=u'(c(amax))
%backward difference approximation of first derivative
dVb(2:I,:) = (V(2:I,:)-V(1:I-1,:))/da;
dVb(1,:) = (wbar*y + r.*amin).^(-gamma); %state constraint boundary condition: dVb=u'(c(amin))
I_concave = dVb > dVf; %indicator whether value function is concave (problems arise if this is not the case), this condition corresponds to ssb > ssf
%second derivative approximation (backward and forward only differs at amax)
dV2b(2:I-1,:) = dVf(2:I-1,:)/da - dVb(2:I-1,:)/da;
dV2f(2:I-1,:) = dVf(2:I-1,:)/da - dVb(2:I-1,:)/da;
dV2b(I,:) = -gamma*dVb(I,:)/amax; %boundary condition
dV2f(I,:) = -gamma*dVf(I,:)/amax; %boundary condition
%consumption, capital share and savings with forward difference
cf = max(dVf,10^(-10)).^(-1/gamma);
kf = max(- dVf./dV2f.*(mu-r)/(sig2),0); %optimal choice of capital
ssf = wbar*yy + r.*aa - cf + (mu-r).*kf;
%consumption, capital share and savings with backward difference
cb = max(dVb,10^(-10)).^(-1/gamma);
kb = max(- dVb./dV2b.*(mu-r)/(sig2),0); %optimal choice of capital
ssb = wbar*yy + r.*aa - cb + (mu-r).*kb;
%capital share, consumption and derivative of value function at steady state
k0 = (kb + kf)/2; %very simple but seems to work well and it is fast
c0 = wbar*yy + r.*aa + (mu-r).*k0;
dV0 = max(c0,10^(-10)).^(-gamma); %u'(c)
%dV_upwind makes a choice of forward or backward differences based on the sign of the drift
If = ssf > 0; %positive drift --> forward difference
Ib = ssb < 0; %negative drift --> backward difference
I0 = (1-If-Ib); %at steady state
dV_Upwind = dVf.*If + dVb.*Ib + dV0.*I0; %include the third term for taking into account cases where sf <= 0 <= sb (set savings=0)
```

```
c = max(dV_Upwind,10^(-10)).^(-1/gamma);
u = c.^(1-gamma)/(1-gamma);
k = max(-dV_Upwind./dV2b.*(mu-r)/sig2,0);
%Construct matrix
X = -Ib.*ssb/da + sig2/2.*k.^2/da^2;
Y = -If.*ssf/da + Ib.*ssb/da - sig2.*k.^2/da^2; %lambda is subtracted later, when Aswitch is added
Z = If.*ssf/da + sig2/2.*k.^2/da^2;
%at the upper boundary a=amax
eps = -amax*(mu-r)^2/(2*gamma*sig2);
X(I,:) = -min(ssb(I,:),0)/da - eps/da;
Y(I,:) = -max(ssf(I,:),0)/da + min(ssb(I,:),0)/da + eps/da; Z(I,:) = max(ssf(I,:),0)/da;
%create the part of matrix A referred to values for j=1;
%spdiags(a,b,c,d) creates a cxd matrix and puts in diagonal b the values of a (the main diagonal is indi-
cated by 0)
A1 = spdiags(Y(:,1),0,I,I)+spdiags(X(2:I,1),-1,I,I)+spdiags([0;Z(1:I-1,1)],1,I,I);
%create the part of matrix A referred to values for j=2
A2 = spdiags(Y(:,2),0,I,I)+spdiags(X(2:I,2),-1,I,I)+spdiags([0;Z(1:I-1,2)],1,I,I);
%at the upper boundary a=amax
A1(I,I) = Y(I,1) + Z(I,1);
A2(I,I) = Y(I,2) + Z(I,2);
A = [A1,sparse(I,I);sparse(I,I),A2] + Aswitch; %2I x 2I matrix called Poisson transition matrix or Inten-
sity matrix
if max(abs(sum(A,2)))>10^(-9) %sum(A,2) is a column vector containing the sum of each row of A
disp('Improper Transition Matrix')
break
end
B = (1/Delta + rho)*speye(2*I) - A;
u_stacked = [u(:,1);u(:,2)]; %column vector of 2xI elements (first the ones for j=1, then for j=2)
V_stacked = [V(:,1);V(:,2)]; %similarly but for the value function
b = u_stacked + V_stacked/Delta;
V_stacked = B\b;
%Solve system of equations
V = [V_stacked(1:I),V_stacked(I+1:2*I)]; %re-create a matrix with 2 different columns for j=1 and j=2
Vchange = V - v;
v = V;
dist(n) = max(max(abs(Vchange))); %first compute the max of the 2 columns for j=1 and for j=2, then
compute the max of the 2 max
```

```
if dist(n)<crit %convergence criterion
disp('Value Function Converged, Iteration = ')
disp(n)
break %stop the for cycle of the value function (find the converged value function for each value of the interest rate)
end
end
toc;
%% FOKKER-PLANCK EQUATION
%Recompute transition matrix with reflecting barrier at amax
X = -min(ssb,0)/da + sig2/2.*k.^2/da^2;
Y = -max(ssf,0)/da + min(ssb,0)/da - sig2.*k.^2/da^2; %lambda is subtracted later, when Aswitch is added
Z = max(ssf,0)/da + sig2/2.*k.^2/da^2;
%at the upper boundary a=amax
%X(I,:) = -min(ssb(I,:),0)/da + sig2/2.*st(I,:).^2/da^2; %unuseful because it is the same formula used for the other rows
Y(I,:) = min(ssb(I,:),0)/da - sig2/2.*k(I,:).^2/da^2; %lambda is subtracted later, when Aswitch is added
Z(I,:) = 0;
A1=spdiags(Y(:,1),0,I,I)+spdiags(X(2:I,1),-1,I,I)+spdiags([0;Z(1:I-1,1)],1,I,I);
A2=spdiags(Y(:,2),0,I,I)+spdiags(X(2:I,2),-1,I,I)+spdiags([0;Z(1:I-1,2)],1,I,I);
A1(I,I) = Y(I,1) + Z(I,1);
A2(I,I) = Y(I,2) + Z(I,2);
A = [A1,sparse(I,I);sparse(I,I),A2] + Aswitch;
AT = A';
b = zeros(2*I,1); %since ATg must be equal to 0
%need to fix one value, otherwise matrix is singular
i_fix = 1;
b(i_fix)=.1;
row = [zeros(1,i_fix-1),1,zeros(1,2*I-i_fix)];
AT(i_fix,:) = row; %substitute the first row of AT with the values in row
%Solve linear system
gg = AT\b;
g_sum = gg'*ones(2*I,1)*da;
gg = gg./g_sum;
g = [gg(1:I),gg(I+1:2*I)];
check1 = g(:,1)'*ones(I,1)*da;
```

```
check2 = g(:,2)'*ones(I,1)*da;
g_r(:,:,ir) = g;
adot(:,:,ir) = wbar*yy + r.*aa - c + (mu-r)*k; %first derivative of a, which corresponds to saving
V_r(:,:,ir) = V;
dV_r(:,:,ir) = dV_Upwind;
c_r(:,:,ir) = c;
k_r(:,:,ir) = k;
S(ir) = g(:,1)'*a*da + g(:,2)'*a*da; %total amount of wealth
K = g(:,1)'*k(:,1)*da + g(:,2)'*k(:,2)*da; %aggregate level of capital
%% UPDATE INTEREST RATE
%market clearing condition:
%total amount of wealth equals total amount of capital
if (S(ir)-K)>crit_S
disp('Excess Supply')
rmax = r;
r = .5*(r+rmin); %decrease the interest rate
elseif (S(ir)-K)<-crit_S
disp('Excess Demand')
rmin = r;
r = .5*(r+rmax); %increase the interest rate
elseif abs(S(ir)-K)<crit_S
disp('Equilibrium Found, Interest rate =')
disp(r)
break %stop the for cycle of the interest rate
end
end
%% UPDATE AGGREGATE CAPITAL LEVEL
test(kr) = K-K_K(kr);
if (K-K_K(kr))<-crit_K
disp('Shortage of investment in capital')
K = K_K(kr)-updk*abs(K-K_K(kr)); %decrease the aggregate capital
elseif (K-K_K(kr))>crit_K
disp('Excess investment in capital')
K = K_K(kr)+updk*abs(K-K_K(kr)); %increase the aggregate capital
elseif abs(K-K_K(kr))<crit_K
disp('Equilibrium Found, Aggregate capital level =')
```

```
disp(K)
break %stop the for cycle of capital
end
end
%% GRAPHICAL REPRESENTATION OF THE RESULTS
%Consumption Policy Function
figure(1)
set(gca,'FontSize',16)
plot(a,c_r(:,1,ir),'b',a,c_r(:,2,ir),'r',a,g(:,1),'b--',a,g(:,2),'r--',a,zeros(1,I),'b--','LineWidth',1.5);
text(amin,-.235,'$\underline{a}$','FontSize',16,'interpreter','latex')
line([amin amin], [min(min(adot(:,:,ir)))-0.05 amax],'Color','Black','LineStyle','--')
legend('c_1(a)','c_2(a)','g_1(a)','g_2(a)','Location','East')
xlabel('Wealth, $a$','interpreter','latex')
ylabel('Consumption, $c_j(a)$','interpreter','latex')
xlim([amin-1.3 amax-20])
ylim([-.05 3])
title('Consumption Policy Function')
saveas(gcf,'Consumption policy function_bonds_capital.jpg')
%Savings Policy Function
amax1 = 8;
amin1 = amin-.5;
figure(2)
set(gca,'FontSize',16)
plot(a,adot(:,1,ir),'b',a,adot(:,2,ir),'r',a,g(:,1),'b--',a,g(:,2),'r--',linspace(amin,amax,I),zeros(1,I),'k--','LineWidth',1.5);
text(amin,-1.3,'$\underline{a}$','FontSize',16,'interpreter','latex')
line([amin amin], [min(min(adot(:,:,ir)))-1 max(max(adot(:,:,ir)))+1],'Color','Black','LineStyle','--')
xlabel('Wealth, $a$','interpreter','latex')
ylabel('Savings, $s_j(a)$','interpreter','latex')
xlim([amin1-1 amax-20])
ylim([-1.2 0.7])
legend('s_1(a)','s_2(a)','g_1(a)','g_2(a)','Location','NorthEast')
title('Savings Policy Function')
saveas(gcf,'Savings policy function_bonds_capital.jpg')
%Capital holdings
amax1 = 6;
amin1 = amin-.3;
```

```
figure(3)
set(gca,'FontSize',16)
h1 = plot(a,k(:,1),'b',a,k(:,2),'r',a,g(:,1),'b--',a,g(:,2),'r--','LineWidth',1.5);
text(amin,-0.5,'$\underline{a}$','FontSize',16,'interpreter','latex')
line([amin amin], [0 max(max(k))+1],'Color','Black','LineStyle','--')
xlabel('Wealth, $a$','interpreter','latex')
ylabel('Capital Holdings, $k_j(a)$','interpreter','latex')
xlim([amin-0.1 4]) ylim([-0.1 5])
legend(h1,'k_1(a)','k_2(a)','g_1(a)','g_2(a)','Location','NorthEast')
title('Capital Holdings Function')
saveas(gcf,'Capital holdings_bonds_capital.jpg')
%Bond holdings
figure(4)
set(gca,'FontSize',16)
h1 = plot(a,a-k(:,1),'b',a,a-k(:,2),'r',a,g(:,1),'b--',a,g(:,2),'r--',linspace(amin1,amax,I),zeros(1,I),'k--','LineWidth',1.5);
text(amin,-5,'$\underline{a}$','FontSize',16,'interpreter','latex')
line([amin amin], [min(min(a-k))-1 max(max(a-k))+1],'Color','Black','LineStyle','--')
xlabel('Wealth, $a$','interpreter','latex')
ylabel('Bonds, $b_j(a)$','interpreter','latex')
xlim([amin1-0.5 18])
ylim([-4.5 2])
legend(h1,'b_1(a)','b_2(a)','g_1(a)','g_2(a)','Location','SouthEast')
title('Bond Holdings Function')
saveas(gcf,'Bond holdings_bonds_capital.jpg')
%Density Functions
figure(5)
set(gca,'FontSize',16)
h1 = plot(a,g(:,1),'b',a,g(:,2),'r','LineWidth',1.5);
text(amin,-0.05,'$\underline{a}$','FontSize',16,'interpreter','latex')
line([amin amin], [-0.01 0.65],'Color','Black','LineStyle','--')
xlabel('Wealth, $a$','interpreter','latex')
ylabel('Densities, $g_j(a)$','interpreter','latex')
xlim([amin-1.3 amax-20])
ylim([-0.01 0.65])
legend(h1,'g_1(a)','g_2(a)')
title('Density Functions')
```

```
saveas(gcf,'Density functions_bonds_capital.jpg')
%Lorenz curve
%Computes y-axis to plot the Lorenz curve
wealth=aa.*g*da; %computes aggregate wealth (net worth) for each type
cum_wealth=cumsum(wealth,1); %computes cumulative wealth for each type
cum_wealth_agg=cum_wealth(:,1)+cum_wealth(:,1); %computes aggregate cumulative wealth
Lorenz=cum_wealth_agg./cum_wealth_agg(end); %computes Lorenz curve
%Computes x-axis to plot the Lorenz curve (cumulative density)
G=cumsum(g,1).*da; G_cum=G(:,1)+G(:,2);
figure(6)
set(gca,'FontSize',16)
plot(G_cum,Lorenz,'g','Linewidth',1.5)
xlabel('Cumulative share of net worth, $G(x)$','interpreter','latex')
ylabel('Lorenz curve, $L(G(x))$','interpreter','latex')
title('Lorenz curve')
xlim([0 1])
ylim([-0.05 1])
refline(1,0)
legend('Lorenz curve','Equidistribution line','Location','NorthWest')
saveas(gcf,'Lorenz curve_bonds_capital.jpg')
```

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