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Managing a Black and Litterman Portfolio's Rebalancing with Gold for an Italian Retail Investor

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1 Introduction

Markowitz, in the 1950s, innovated the financial sector of the economy with his studies on the problem of finding an optimal financial portfolio. He was among the first to give a theoretical architecture to the problem, using well-known statistical concepts to describe financial assets. In particular, he defined the risk of a financial asset, through its variance and the relationships between the assets through their covariance. Therefore, an asset could be defined using the mean-variance in terms of returns-risks and placed in relation to the other assets in the portfolio using correlation.

By solving a mathematical maximization problem, the Markowitz model allows finding the composition, in terms of asset weights, of the optimal portfolio for an investor.

With the concept of correlation, he could also theorize the effects of the diversification in reducing portfolio risk.

Starting from this model, many scholars have tried to find a solution to portfolio optimisation.

With the Capital Asset Pricing Model (CAPM), Sharpe took another step forward in terms of diversification. He defined the beta of a stock, which, in his model, is the only source of risk of a stock, from the observation that "that when the market is bullish, the prices of most of the shares go up so that there should be a positive correlation with a common factor ". This theoretical solution made it possible to divide the risk into systematic risk and idiosyncratic risk; while the former remains within the portfolio, the latter can be reduced thanks to diversification.

The initial paragraph of the first chapter of this thesis is therefore dedicated to these two models to retrace the birth of the idea of looking for an optimal portfolio and the contribution that diversification, a fundamental world in finance, has given to this field.

The first paragraph ends with analysing the historical problems related to the so-called Modern Portfolio Theory, the portfolio theory originated by Markowitz. These are well-known problems, which have limited the practical diffusion of the proposed models. As analysed within the thesis, although perfect from a theoretical point of view, and fundamental for opening the research field, these models present some problems. The optimal portfolios constructed with these models are often unstable in the composition during time and strongly influenced to errors in estimating the parameters and moreover they don't include the possibility for the investor to plug in her own views, all these problems will be analyzed, more in details, in the fourth paragraph

In the second paragraph is presented the Black and Litterman model. Black and Litterman, starting from the 90s, gave their contribution to the solution of these problems with their model. Taking cues from the models previously described, they created their model, which manages to combine greater stability in the composition of the final portfolio by including the investor's personal views on the portfolio's assets. The Black - Litterman model uses the reverse optimisation technique of Sharpe's CAPM model to generate the so-called equilibrium returns and, after combining them with the investor's views, inserts them into a Markowitz-like maximisation problem.

As noted, therefore, diversification is essential in reducing the risk of a portfolio. The second chapter then closes with a summary on diversification and an analysis of which instruments or assets may be most suitable for obtaining it. The focus is mainly based on commodities, on their correlation with equities. Finally, the case of gold is reported, with an analysis of the literature that mentions it as a possible safe asset. "The safe asset (or safe heaven) is an asset that is negative correlated with other assets, specifically during a market crisis", and for this reason, the possibility of using it within a portfolio is investigated.

The third chapter is functional to the passage from theory to practice. The possibility of creating a portfolio for an Italian retail investor is then analysed, trying to improve its fiscal management. Some of the typical long-term investment instruments for retail investors, such as mutual funds and ETFs, are then analysed, exploited for their transparency and intrinsic diversification[1]. However, their tax limits are mentioned, in compensating capital losses and capital gains; these specific limits are attributed to the specific Italian legislation on the subject. To overcome this problem, is proposed to use in the portfolio an ETC, exchange-traded commodities, an asset that can be fiscally compensated with the ETFs. The ETC proposed in the thesis replicates the trend of the price of gold.

[1] Since they have many securities within them.

The thesis's objective is to propose an effective portfolio and a procedure for an Italian retail investor who can exploit the tax benefits in a long-term investment. To achieve this objective, the Black and Litterman model is used, with the advantages that we have seen that it can bring, giving views that can be inserted on the assets.

As stated before, an ETC on gold will also be included within the portfolio. The portfolio will be periodically rebalanced, and doing this, capital loss and capital gains will be compensated fiscally, thus trying to obtain a better result than a portfolio without rebalancing. The rebalancing will be performed according to the views that were initially given on the assets.

In the last chapter, the procedure used, the parameters, and the data of the model will be explained, and finally, the results will be shown.

To complete the analysis and to be able to make comparisons, two other portfolios were created. The first without the rebalancing process, the second without the rebalancing *and* without adding gold, both the portfolios are been capitalized in the same time interval. We expect to see if the rebalancing process is effective in terms of both absolute and risk-related performance increases from the first portfolio. From the second, we also want to see how helpful gold is in diversification and reducing volatility.

2 Asset Allocation and Diversification

2.1 Modern Portfolio Theory

When dealing with a financial portfolio, the Markowitz model is undoubtedly the starting point. Specifically, Markowitz created the portfolio theory from a scientific point of view with his work. Although innovative from a theoretical point of view, the proposed model soon showed some limitations for its practical application. New proposals for models for the optimisation of a portfolio are still being made. The ultimate goal of this thesis is to work with the Black Litterman model. However, without presenting some of the models preceding it, from which Black and Litterman draw ideas and procedures, the thesis would be incomplete.

The chapter begins by illustrating the Markowitz model, focusing on the theorisation of diversification as a source of risk reduction. Markowitz had given a particular role to correlation, which explains the comovements of the assets' prices.

In particular, from the Markowitz model, the Black and Litterman model’s takes the optimisation process to determine the final composition of the optimal portfolio.

The third paragraph is dedicated to Sharpe's models, the Single Index and the CAPM, his attempts to overcome some limitations of the Markowitz model. The CAPM proposes a different method of calculating the risk of an asset, introducing the important concept of beta as the only source of risk in a portfolio, born from observing the correlation between assets. Sharpe introduced the so-called *reverse optimisation*, a procedure that will be taken up by Black and Litterman for the calculation of the returns to be included in their portfolio.

The fourth paragraph, which is the critical elements of the Modern Portfolio Theory, represents the limits that Black and Litterman have tried to overcome with their model to obtain a portfolio optimisation model that could perform well even in practical applications.

Therefore, the models previously presented will be beneficial for understanding the Black and Litterman model used within this thesis.

2.1.1 Markowitz's model

This paragraph illustrates the Markowitz model, with particular attention to cases where correlation reduces risk.

The general idea that diversification can lower the risks is well known for a long time ago. However, only in recent years, it has been formalised. In 1952 Harry Markowitz gave his fundamental contributions with his revolutionary work *Portfolio Selection*[2]; this work began a pillar for the entire financial community. The basis were few simple assumptions:

- Investors want to maximise the utility of their final wealth;
- Investors are risk-averse: they always prefer outcomes with low uncertainty to outcomes with high uncertainty, even if the latter's predicted outcome is equal to or higher in utility than the more certain outcome;
- The investment horizon is one-period;
- There are no transaction costs or taxes, and the securities are perfectly divisible (frictionless market);
- All investors have access to the same information and the same expectation on the assets;
- Investors are price takers: a single investor is too small to influence market prices;
- Investors can borrow unlimited amounts of capital at the risk-free rate;
- The returns of the assets are normally distributed.

So, the *Modern Portfolio Theor*y was created, and the author introduced two milestones in his article[3].

Firstly, Markowitz used the standard deviation as the measure of risk of a financial asset. The standard deviation is a statistical indicator that indicates the average distance of the realisation of a random variable from its mean. According to Markowitz, every security can be described by the expected value of its return, generally estimated by the mean of the past returns, and its risk, generally estimated by the variance of those past

[2] (Markowitz, 1952)

[3] "The main innovation introduced by Markowitz was to measure the risk of a portfolio via the joint (multivariate) distribution of returns of all asset. Multivariate distributions are characterized by the statistical (marginal) properties of all component random variables and by their dependence structure. Markowitz described the former by the first two moments of the univariate distributions – the asset – returns and the later via the linear (Pearson) correlation coefficient between each pair of random returns." (Szego, 2005).

returns. In this way, the scholar introduced the so-called *mean-variance* approach, which defines every financial asset by its expected value and risk and gives the possibility to compare two different securities at a time. To do that, he used the mean-variance dominance criterion.

Let X_1 and X_2 be two securities, X_1 is not preferred to X_2 for the mean-variance criterion:

$$E(X_1) \geq E(X_{2)}$$

$$and$$

$$Var(X_1) \leq Var(X_2)$$

Notice that if at least one of these two inequalities is satisfied in the strong form, then X_1 is preferred than (or dominates) X_2. Nevertheless, this relation has some limits; in some cases, the strong sense requirement can be not satisfied. In those cases, the mean-variance approach is not able to recognise a practical preference. Figure 1 represents three assets in a mean-variance plain. Asset "2" is preferred to asset "3" because it has a higher return with lower risk. However, we can't say anything about the preference between "1" and "2" as the second has a greater expected return but also greater volatility; it is a situation with no solutions. This approach cannot always produce a preference between those two assets.

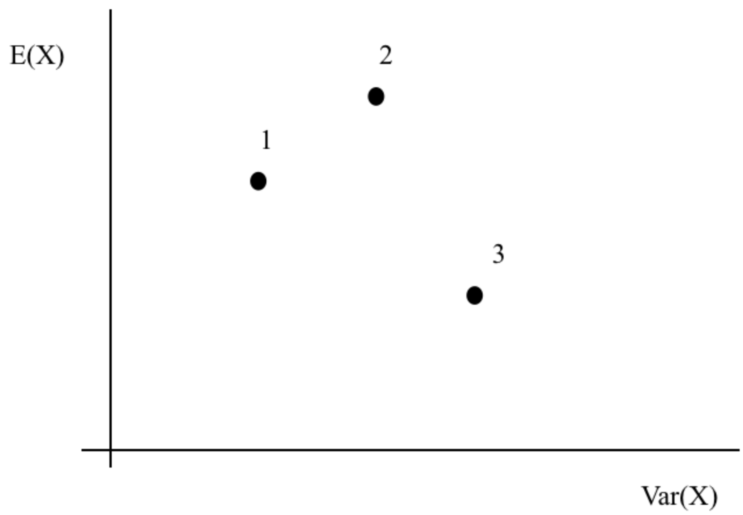

Figure 1. Three different assets mean - variance.

Secondly, in his paper, Markowitz presents the concept that in a diversified portfolio, investors should focus not only on the single asset's movements but also on each other's comovements. Every financial asset has its own price trend and returns movement that differs from the other; those fluctuations can depend on a large amount of factor, endogenous or exogenous, depending on the single security. Markowitz analyses those fluctuations in relative terms of one asset to the others. He computes covariances and the correlations of the assets' returns in the portfolio to detect how much they are related and how they move together.

In his approach, Markowitz uses the mean-variance *not only* for inducing, when possible, an ordering in the assets, *but also* for choosing the portfolio. The way he uses the mean – variance consists in detecting the set of portfolios which minimise the variance of its rate of return for a given mean of its rate of return. In other words, for every investor's possible expected returns, he wants to be able to find the portfolio with the lowest possible variance.

Knowing the mean and variance of every asset's return and the covariance between each pair of them, Markowitz computes the efficient frontier that is a *hyperbola* in the mean – standard deviation plain, in which every point represents an efficient portfolio. An efficient portfolio is a combination of the assets that give the lowest variance for the given amount of return.

The first assumption of the model is that the investor wants to maximise the utility of his final wealth. This assumption is represented by the *utility function*; that function depends on the investor's personal expected value and the investor's risk aversion (A). A simple utility function is the quadratic one; it represents a set of curves:

$$U = E(r) - A\sigma^2$$

The tangency point between the efficient frontier, the hyperbola representing the set of the efficient portfolios, and the utility function, the curve representing the investor's personal taste in terms of expected return and risk aversion, gives the *optimal portfolio* for the investor.

Figure 2 presents as example an efficient frontier (red line) and the assets[4] hold in the portfolios (black dots). As can be seen, all the assets are "under" the efficient frontier, except for the two at the extremes. That means that every asset classes combination proposed in the efficient portfolios results in better risk-return profiles than the single asset classes, lowering the risk or raising the expected value. The general idea is that with a proper combination, all the single asset classes can be dominated by better portfolios.

That represented a breakthrough; previously, there was the idea of diversification but without any theoretical formalisation. Markowitz gives an explanation of how diversification can work.

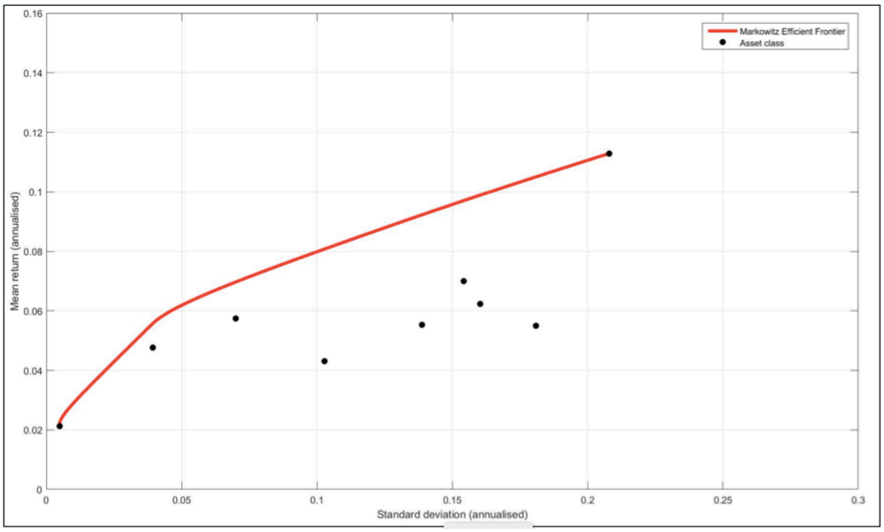

Figure 2. The Markowitz Efficient Frontier

2.1.2 The role of the correlation

The correlation is a statistical relationship between two random variables or two data and represents their linear relation. It goes from -1, the two random variables are perfectly negatively correlated, and +1, the two random variables are perfectly positively correlated. In the case of negative correlation, two assets' prices tend to move in the opposite sense, while in the case of positive correlation, assets prices tend to move together in the same path.

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[4] The asset included in the graph are some global stock and bond indices: BOFA ML euro government bill, JPM Emu government all maturities, Citi non-European government bond, BOFA ML Global Corporate Bond, MSCI EMU, MSCI Europe ex EMU, MSCI North America, MSCI Pacific, MSCI Emerging Markets.

Although challenging to realise with real-life assets, it is crucial to notice that, theoretically, investing in more securities, depending on the correlations of the return of the assets, the investors might also be able to generate portfolios with a variance that is smaller than the one of the less risky assets.

According to Markowitz, the diversification comes not simply from a large casual amount of assets but also from negative or slightly positive correlated ones.

Let us assume the simple case of two risky assets x_1, x_2, their correlation is $\rho_{1,2}$ and neither of one is dominated by the other[5].

The case for $\rho_{1,2} = +1$

This is the case of perfect positive correlation. In this case, it is possible to demonstrate that the efficient frontier is a straight line that goes from asset 1 to asset 2. The efficient frontier is simply a linear combination of the two assets.

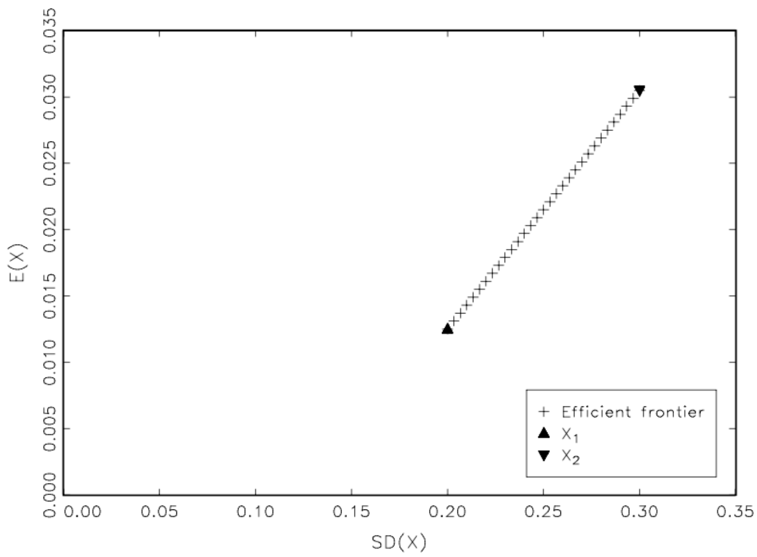

Figure 3. The case ρ=1. Source: Financial Economics, Prof. Marco Corazza, course slide

The case for $\rho_{1,2} = -1$

On the contrary, this is the case of a perfect negative correlation. This is the perfect risk reduction situation. Figure 4 shows the efficient frontier in that situation. It is possible to construct a portfolio that benefits from diversification, with zero variance and so zero risks.

[5] Otherwise, the investor would hold only the asset that dominates the other.

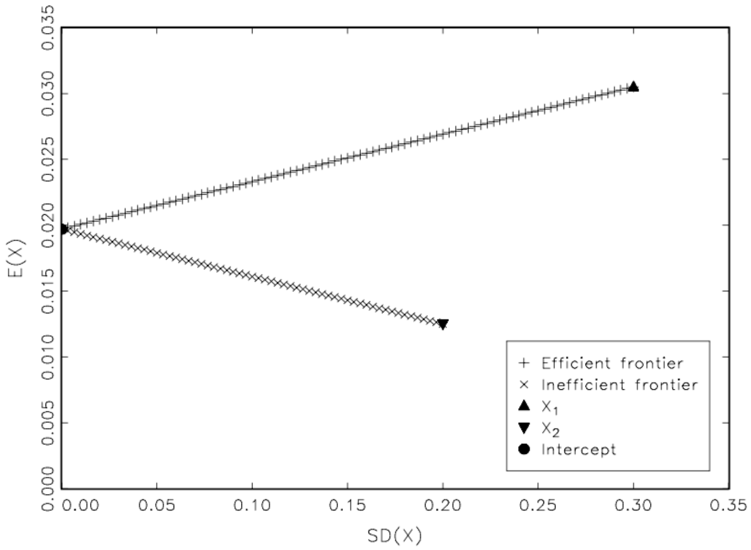

Figure 4. The case ρ= -1. Source: Financial Economics, Prof. Marco Corazza, course slide

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The case for $-1 < \rho_{1,2} < 1$

In this case, the investor might be able to reduce the risk by appropriately combining the assets. The more the assets are negatively or slightly positively correlated, the more the risk reduction is possible. In this case, there is the possibility to obtain portfolios with a variance that is less than the minimum asset variance, i.e., the possibility of risk contraction arises, and the benefit of the diversification appears.

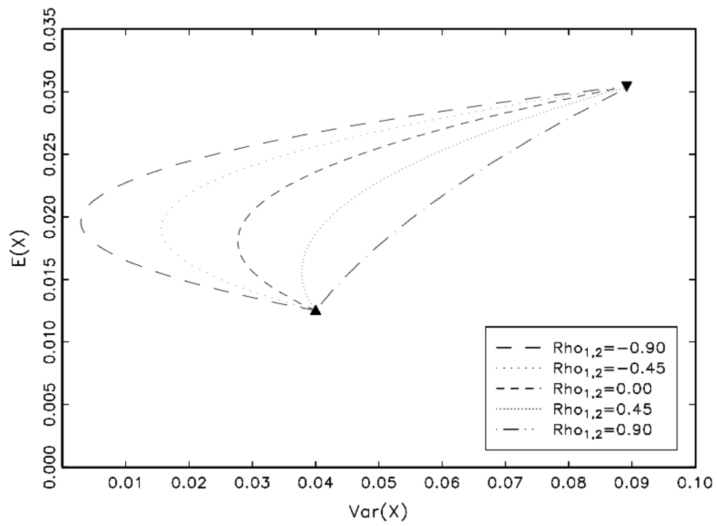

Figure 5. The case -1 < p < 1. Source: Financial Economics, Prof. Marco Corazza, course slide

2.1.3 Sharpe's contribution to the Modern Portfolio Theory

Starting with Markowitz' studies, many authors have presented new models. One of the most famous is the CAPM (Capital Asset Pricing Models), introduced by Treynor, Sharpe, Lintner and Mossin independently between the 1961 and the 1964.

The model represents an evolution of the Markowitz model. One of the main results of the CAPM is showing that not all the risk of an asset is rewarded by the market in the form of a higher return, but only that part that cannot be reduced by diversification.

The model is also important for the introduction of the reverse optimisation, a step of the procedure used by Sharpe to find the optimal portfolio. This procedure will be taken up by Black and Litterman for their model.

The Single Index Model

The presentation of Sharpe's CAPM in 1964 was anticipated in 1963 by the Single Index Model, *"A simplified model for portfolio analysis[6]",* in which he anticipates the insights later present in the CAPM. One element in particular that Sharpe demonstrates with this model is the possibility of reducing risk in the idiosyncratic component risk, but the market (or systemic) risk remains in the portfolio, and it cannot be diversified.

The intuition was the empirical evidence that when the market is bullish, the prices of most of the shares go up so that there should be a positive correlation with a unique common risk factor. This common factor is called β_i, which is the *only source of risk* in this model.

This intuition leads to a different way of thinking about the asset returns R_i:

$$R_i = a_i + \beta_i R_m$$

where:

- a_i is random variable, that represents a component independent from the market;
- R_m is the market return;

[6] Sharpe, William F. (1963). "A Simplified Model for Portfolio Analysis". Management Science. 9 (2): 277–93.

- β_i is a coefficient measuring the expected change in R_i given a change in R_m.

a_i can be decomposed as:

$$a_i = \alpha_i + e_i$$

where:

- α_i is the expected value of a_i, the stock's expected return if the market is neutral, it is the CAPM *excess return*;
- e_i is a random component (stochastic error).

Thus, the equation can be written:

$$(1.1) \quad R_i = \alpha_i + \beta_i R_m + e_i$$

It is interesting to notice the use of the model in the case of a well-diversified portfolio. The portfolio returns are:

$$R_P = \alpha_P + \beta_P R_m + e_P$$

For an equally weighted portfolio, a portfolio in which every asset has the same weight, i.e., 1/n, where n is the number of the assets, one has:

(1.2)

$$R_P = \sum_{i=1}^{N} w_i R_i = \frac{1}{n}\sum_{i=1}^{N} R_i = \frac{1}{n}\sum_{i=1}^{N}\left(\alpha_i + \beta_i R_M + e_i\right) =$$
$$= \frac{1}{n}\sum_{i=1}^{N}\alpha_i + \left(\frac{1}{n}\sum_{i=1}^{N}\beta_i\right)R_M + \frac{1}{n}\sum_{i=1}^{N} e_i$$

Comparing the equation (1.1) and (1.2), it is possible to say that the β of the portfolio is:

$$\beta_P = \left(\frac{1}{n}\sum_{i=1}^{N}\beta_i\right)$$

Which is the average of the β_i.
And that:

$$e_P = \left(\frac{1}{n} \sum_{i=1}^{N} e_i \right)$$

A zero-mean variable, which is the average of the firm – specific components. The firm – specific risk is a risk that affects the single company or asset of the portfolio, in opposition to the systemic – risk that affects the overall market.
Hence the variance of the portfolio is:

$$\sigma_P^2 = \beta_P^2 \sigma_M^2 + \sigma^2(e_P)$$

The first term on the left is the systematic risk (or market risk) component, which depends on the market movements. This term depends on the beta and on the σ_M^2, and it is unaffected by the number of securities held in the portfolio, so it is unaffected by diversification. The second term is the firm – specific components e_i, in opposition to the first, it can be called also non – systematic component. Because the e_i are each other independent and have zero mean, the law of large number can be applied to conclude that as more and more securities are added to the portfolio, the firm-specific components tend to cancel out, resulting in firm – specific.

In the equally-weighted portfolio, because the e_i are each other uncorrelated, the variance of the firm – specific components can be defined as:

$$\sigma^2(e_P) = \sum_{i=1}^{N} \left(\frac{1}{n} \right)^2 \bar{\sigma}^2(e_i) = \frac{1}{n} \bar{\sigma}^2(e \ )$$

Where $\overline{\sigma}^2(e)$ is the average of firm-specific variances.

As shown in Figure 6, since this variance is independent of n, when n gets larger, $\sigma^2(e_P)$ becomes negligible, the figure shows that as more and more stocks are combined in a portfolio, the variance of the portfolio decreases due to the diversification of firm-specific risk. However, the power of diversification is limited. Even for very large n, some of the risk remains due to the exposure of almost all assets to the common or market factor. Therefore, this systematic risk is said to be non-diversifiable.

This is very important evidence in terms of risk management and diversification.

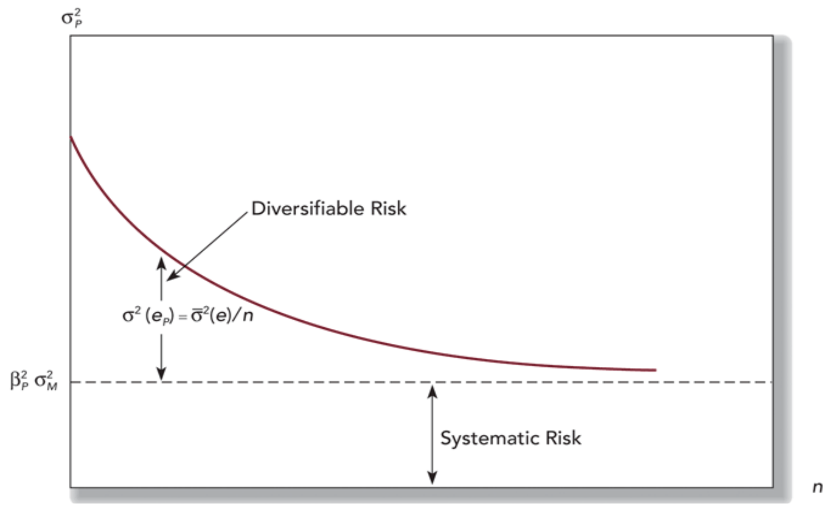

Figure 6. Single Index Model risk diversification. Source: Financial Economics, Prof. Loriana Pelizzon, course slides.

William Sharpe proved that only a specific part of risk could be reduced, the *idiosyncratic risk*, but the *market (or systemic) risk* remains in the portfolio, and it cannot be diversified.

The CAPM

The following year, in 1964, William Sharpe introduced the Capital Asset Pricing Model[7], one of his most famous contribution. The Capital Asset Pricing Model is a centrepiece of modern financial economics.

For Sharpe, since all the investors have the same information and they optimise their portfolio in the same way, every investor holds the *same portfolio* of risky assets, the so-called *market portfolio* that contains all the assets on the market, weighted by their market capitalisation.

Moreover, Sharpe introduced the idea of the use of the risk – free asset. Every investor combines the market portfolio with a certain amount of the risk – free asset, the weight of the free risk asset depends on the individual investor's risk aversion and his

[7] "Capital Asset Prices: A Theory of Markets Equilibrium Under Conditions of Risk" Journal of Finance (1964).

own investing objective. The more she accepts risks, the lower she holds the risk-free assets, and vice versa. In this way, Sharpe eliminates the utility curves of the Markowitz's model, that are unique for every single investor, in order to select the portfolio, simplifying and reducing the subjectivity of the model.

In that model, the assumptions are the same as Markowitz's model, except for the following additions:

- Investments are limited to publicly traded assets (stocks, corporate and government bonds);
- The information is free and available to all investors;
- The expectations are homogeneous, i.e., investors have the same expectations for all the input entering the optimisation process.

In addition, Sharpe posed three conditions to get the equilibrium.

- Individual investors maximise their utility;
- The demand is equal to the supply for *all* the risky assets;
- The aggregate borrowing is equal to the aggregate lending.

The basis relation for the asset's return of the model is:

$$E(R_i) = \beta_i \left[E(R_m) - R_f \right] - R_f$$

where:

- r_i is the return of the i-asset;
- $\beta_i = \frac{cov(r_i, r_m)}{var(r_m)}$ is a coefficient that measures the systematic risk;
- r_f is the return of the risk-free asset;
- r_m is the market return.

The relation can be rewrite as:

$$E(R_i) - R_f = \beta_i \left[E(R_m) - R_f \right]$$

where the firs term is the excess return, that is given by the risky asset return with respect to the risk – free asset return. Since the investors are risk averse, if the excess return would be zero, none of the investor would buy the risky asset, since it would be dominated by the risk – free asset in terms of mean - variance. If the excess return is positive then the investors buy the risky assets, if it is negative or zero, then they invest in the risk – free asset since they are risk adverse. So, it represents the risk premium to the investor for investing in the risky asset.

As seen before for the Single Index Model, the firm-specific risks can be eliminated, while the risk associated with the beta is called non-diversifiable risk.

Sharpe Ratio

Another significant contribution of Sharpe to financial theory is the so-called "Sharpe ratio", a ratio proposed by the scholar in 1966.

The numerator is the extra return of the portfolio compared to the risk-free asset, the denominator is the volatility of the portfolio:

$$Sharpe\ ratio = \frac{R_P - R_f}{\sigma_P}$$

where:

- R_P is the portfolio return;
- R_f is the risk - free asset return;
- σ_P is the portfolio volatility.

The ratio indicates the portfolio excess return obtained as a percentage per unit of risk. It is thus possible to compare different investments, making them comparable in terms of risk - return.

It is widely used in finance to analyze and compare different investments, in the last paragraph of the thesis, it will be proposed to analyze the results obtained.

2.1.4 Modern Portfolio Theory's issues

Markowitz was the first who formalised, from a quantitative point of view, the problem of the portfolio's selection in mean variance terms. He had the merit of introducing the risk in his model, using first a statistical measure. He gave an explanation of how diversification can work and created the Modern Portfolio Theory.

The estimation risk related to the model parameters

To compute the optimal portfolio, the investor needs a significant amount of data. He needs to calculate n expected value, n variance and $(n^2-n)/2$ covariances. The values are relatively few in a simple and small portfolio, but the value is greater for more extensive and better-diversified portfolios. A fund can easily have more than hundreds of securities. For $n=100$, the Markowitz model needs 100 expected values, 100 variances and 4950 covariances, a total amount of 5150 estimations. Generally, the estimation derives from historical data or subjective information, and the estimation error generally degrade the properties of the efficient frontier. In this way, the investor could introduce in the measure of performance and risk a large amount of errors estimations. This significant number of possible errors leads to a wrong selection of the optimal portfolio.

The mean-variance approach tends to overweight assets characterised by large expected values, negative correlations, and small variances. Generally, those assets are the ones likely to be bear significant estimation errors[8]. In this way, a great part of the portfolio can be represented by a small number of securities, penalising diversification.

The instability of the mean-variance solutions

Another important issue is represented by the instability of the portfolio over time. As we have seen, the portfolio choice can change a lot due to the change in the estimates over time; a small change in the estimations leads to a significant change in the portfolio composition. Neighbouring portfolios in the frontier can have a completely different composition. This is a problem also for portfolio rebalancing. Investors update the portfolio from time to time, including new expectation on the means, variances and covariances of the assets. Due to the mean-variance instability, this process could significantly shift the weights of the securities held in the portfolio, with possibly high transaction cost.

[8] (Michaud, 1989).

The operative choice of the target returns

Finally, the investor is assumed to choose the expected return desired. This can be difficult for an investor to specify a coherent return with the current financial environment. The specified return could be too high, and the investor takes too many risks, or too low, and the investor loses potential easy returns.

The absence of investor views.

Every investor has his own views and expectations on the financial markets and on the single assets. Moreover, every investor has different information and a different degree of trust in them. Different investors should be allowed to generate different portfolio following their own expectation.

However, in the models just seen, investors cannot insert their views within the model. Furthermore, there is no possibility to differentiate between the views on which the investor is strongly convinced and those on which he has only a weak conviction. The process does not distinguish between the different levels of confidence associated with the input variables.

An investor can enter absolute views regarding the return expectation of a stock, which can be positive or negative but cannot in any way enter relative views, i.e., comparison of performance between different stocks.

In the CAPM, all the investors hold the same market portfolio composition, with a different amount of the risk-free asset, depending on the individual risk aversion. This comes from the assumption that the investors have homogeneous expectations and the same information. Although it helps to explain the portfolio choice, it is unrealistic, and it leads to a perfect theoretical result, but in the real world, this assumption cannot hold.

2.2 Black and Litterman

As we have seen, starting from Markowitz's work, the asset allocation field was rapidly developing, with many contributions, improvements, and new proposals. Although these models worked pretty well in principle, their practical applications on the actual financial markets were not large due to their drawbacks, as we have seen before.

To overcome the situation, institutional investors started trying to improve the models to obtain better portfolios for their activity. In the early 90s, at Goldman Sachs, Fischer Black and Robert Litterman proposed a new model, the Black-Litterman Model

(BL). The model combines Markowitz’s mean-variance optimisation and the CAPM equilibrium portfolio, providing the possibility to add the *investor’s view* to make more stable and efficient portfolios in terms of customisation.

These characteristics will be exploited in this work to generate a portfolio that can work well with real-life assets, thus obtaining a benefit for a retail investor.

Black and Litterman made some assumptions for their model:

- The assets returns are normally distributed;
- Investors are rational and risk-averse;
- The wealth marginal utility is decreasing;
- In the model, the risk of the portfolio is a function of the variance and covariance of the assets.
- The market is frictionless (no tax or transactions costs considered).

The basic idea of the model is to start from the condition of equilibrium on the market. Like in the CAPM, starting from the weights of the assets in equilibrium conditions, it is possible to find equilibrium returns with reverse optimisation.

At this point, the investor without views can use these returns in a Markowitz optimisation system and thus find her optimal portfolio. If, on the other hand, the investor had personal views regarding the assets in the portfolio, before optimising on mean - variance with Markowitz, she has the opportunity to express them and mix them with the equilibrium returns.

In this sense, the views can be

- absolute (i.e., regarding the performances of the single asset);
- relative (i.e., regarding a comparison between the performances of different assets).

Once equilibrium returns are combined with the investor's views, thanks to the Bayes' theorem, the vector of expected returns obtained will be more shifted towards equilibrium returns or views depending on how much confidence the investor has placed in them and how extreme the views are.

The mix generates returns used in an optimisation process to get the optimal Black and Litterman portfolio.

2.2.1 Model

Firstly, we define the returns, their specification is the basis of the Black and Litterman model. The two authors assume that the expected returns are distributed as a normal with mean μ and with variance Σ, where μ, the average of expected returns, is defined as a normally distributed random variable. Since μ cannot be estimated with certainty, then it is modelled as a random variable whose dispersion represents the possible estimation error.

So, the returns are distributed as:

$$E(R) \sim N(\mu, \Sigma)$$

where μ is distributed as:

$$\mu \sim N(\pi, \Sigma_\pi)$$

π represents the estimate of the mean, which is the best approximation to μ.
Σ_π is the variance of this estimation, it can be interpreted as the variance of the estimate of the mean of the returns μ.

The equilibrium

In the book "Modern Investment Management – An Equilibrium Approach"[9], Robert Litterman explains why they choose the equilibrium conditions. The financial market is a dynamic environment, and the equilibrium conditions assumed in many models are only theoretical; however, it seems reasonable to consider it in equilibrium in the long run. That represents a neutral[10] starting point, if all the investors had the same information and views, then all of them simply hold the *market portfolio*. If all the investors have the same information and use the same computational techniques, then the market demand-supply is in equilibrium.

[9] (Litterman & Quantitative Resources Group, 2003)

[10] Black and Litterman define "neutral" as a set of expected returns that bring supply and demand to be in equilibrium. This occurs when all investors have the same information and no views.

To compute the equilibrium returns, it can be used different methods. Black and Litterman suggest the use of the CAPM model, and the *reverse optimisation* introduced by Sharpe. In Markowitz, the returns are the input and the optimal weights of the assets the model's output. However, as in the CAPM, in the BL model, using the reverse optimisation, the returns are considered unknown, and the weights are the starting point.

More in details, Black and Litterman use the inverse optimisation technique, derived from the CAPM, to calculate the expected excess returns of the equilibrium portfolio[11], which, compared to the returns model specified previously, represent the mean of μ.

From the CAPM, it is possible to express the equilibrium excess returns using the inverse optimisation method where the vector of the equilibrium excess returns is expressed as:

$$E(R_i) - R_f = \Pi = \delta\Sigma w$$

where:

- Π represents the unknown market equilibrium returns - (Nx1);
- δ represents the risk aversion coefficient;
- Σ represents the variance-covariance matrix - (NxN).

The reverse optimisation starts working backward, assuming that the weights are optimal and try to find the optimal returns. If the market is efficient, it maximises the following quadratic utility function, maximising $U(w)$ with respect to the optimal weights:

$$U(w) = w\Pi - \frac{\delta}{2}\mathrm{w}\Sigma w$$

where:

- w in the vector of the market's weights of the assets - (Nx1).

[11] The market portfolio is made up of all the assets present in the market in the quantity of their capitalization compared to the market capitalization.

$w'\Pi$ is the expected return, and $w'\Sigma w$ is the variance of the portfolio, so an increase in the returns increase the utility, vice versa, an increase in the variance decreases the utility.

The risk aversion coefficient δ is a positive constant determined by the ratio between the excess return of the market portfolio and its variance and is the rate at which the investor would give up the expected return for a lower variance. We will see it more in detail further on.

Solving an inverse maximisation problem in order to maximise $U(w)$ with respect to the weights, setting the first derivative of the utility function equal 0, it is possible to find the returns, for given weights:

$$U'(w) = 0$$

$$\Pi - \delta\, \Sigma w = 0$$

so, the solution to the problem is:

$$\Pi = \delta\, \Sigma w$$

These weights can be computed or commonly can be approximate by the use of market indices. In that case, if the investor has no view, he will be aligned with the index.

Keeping in mind the starting returns, it is now possible to calculate the variance of the estimate of the average of returns, $\Sigma\pi$. Black and Litterman assume that the structure of this variance – covariance matrix is proportional to the variance matrix covariance of returns, that is Σ.

They created a parameter, τ, modeled as a constant for proportionality. Meucci suggested to set τ as $\frac{1}{n}$ with n is the number of assets used to generate the covariance matrix[12].

[12] (Meucci, 2006).

Given this assumption:

$$\Pi = \delta \, \Sigma w$$

the distribution of the average of market returns become:

$$\mu \sim N(\Pi, \tau\Sigma)$$

It is a vector of long-term excess returns,

The views

The model's breakthrough is the possibility given to the investor to introduce his personal feelings about the market. If the investor has no views, then she will use the previous information to get his portfolio, which will be the market one. Otherwise, he could introduce his personal views to align the portfolio to his expectations. The model gives great freedom to the user. The investor can have no views, can have a view for every asset, or have a view only for some asset.

Two different types of views can be introduced in the Black Litterman model:

- Absolute views;
- Relative views.

Absolute views concern the future return of a specific asset on which that is formulated. Some examples may be: "Asset A will have a return of 5%", or "Asset B will increase the past return of 2%", and so on. The second type, the relative views, ensures a very interesting possibility. It gives the possibility to manage views on one asset with respect to one other. Typically:

- Qualitative "Asset *A* will perform better than asset *B*";
- Quantitative "Asset *A* will perform 3% better than asset *B*".

In order to include them in the model, the views are included in a matrix called $P\ (KxN)$, in which K is the number of views and N is the number of assets.

Every single value of the matrix can be 0 if there are no views on the particular asset, or it can be different from 0 if there is a view on the asset. If the view on an asset is absolute, then the sum of the row's value will be 1 (row 1 in the below figure). If the asset's view is relative, then the sum will be 0 (row 3 in the below figure).

	N Assets					*Sum*
Absolute View	1	0	0	0	0	1
No view	0	0	0	0	0	0
Relative View	-1/4	1	-1/4	-1/4	-1/4	0

Figure 7. Different types of views

The views are presented in a linear combination of these matrices, as function of the unknown expected returns μ:

$$P\,\mu = Q + \varepsilon^{v}$$

The expected returns of the view are inserted in a vertical vector named $Q\ (Kx1)$. Referring to table 7, for demonstrating Q, for example, we can hypothesize, that the asset 1 has a return of 7% (absolute view) and asset 2 has a yield that is 0.5% higher than the other assets (relative view). So, in that case, Q will be:

$$Q = \begin{bmatrix} 0.07 \\ 0 \\ 0.005 \end{bmatrix}$$

The stochastic errors that affect the estimate of the views are in the vector called $\varepsilon,\ a\ (Kx1)$ vector. ε is normally distributed, with zero mean, and its variance is the views variance-covariance matrix, named Ω, a KxK matrix, that represents the investor uncertainty on every view and their covariances. The investor is asked to define its confidence level on the views, it can be defined as the standard deviation of the expected return of each view, according to which the certainty that the return falls within the range is approximately 2/3, which is a known range of probabilities for a normal distribution. A greater confidence level causes a more considerable change in the portfolio composition from the equilibrium one, vice versa lower confidence level produces small changes in the equilibrium portfolio.

The errors of the views

Obviously, the investor's views could be wrong. The degree of uncertainty of the investor with respect to her views permits to include the error term measures.

The value of the error is proportional to the uncertainty in the view. The greater uncertainty, the higher the value of the error: ε^{υ}.

ε^{υ} is distributed as:

$$\varepsilon^{\upsilon} \sim N(0, \Omega)$$

where Ω is the variance covariance matrix (k x k) of the views, which represents the investor's uncertainty for each view.

Ω is a diagonal matrix since the views must be independent and uncorrelated with each other. Not all the values on the diagonal need to be different from 0.

Obtaining Ω is one of the most critical aspects of the whole model because calculating the matrix is the investor's task.

The Bayesian Approach

Once the investor has formulated her personal views, she has to mix them with the market returns.

Black and Litterman didn't explain how to merge them; however, different methods have been presented[13]. One of the widely used is based on the Bayes' theorem, on the conditional probability. It represents a possibility to get the necessary returns to run the model.

The theorem defines the probability of an event A, given the probability that another event B, that might be related to the first, occurs. Their jointly probability is:

$$\Pr(A|B) = \frac{\Pr(B|A)\Pr(A)}{\Pr(B)}$$

where:

- $\Pr(A|B)$ is the joint probability of A given B;

[13] See i.e., (Theil, 1971). Black and Litterman themselves, in their article in 1992, proposed another method, that uses sample from future returns.

- $\Pr(B|A)$ is the joint probability of B given A;
- Pr (A) is the probability of A;
- Pr (B) is the probability of B.

The model distinguishes between the prior and the posterior distribution.

In that case, Pr (A) represents the *prior distribution* as the probability of A does not depend on B.

$\Pr(A|B)$ is the *posterior distribution,* is the conditional probability of A given B.

The views (the investor's expected returns) are considered the prior distribution and the equilibrium return the posterior distribution (Christoduolakis, 2002), i.e.

$$\Pr(A) = \Pr(\mu) = prior\ distribution$$
$$\Pr(B) = \Pr(\Pi)$$
$$\Pr(A|B) = \Pr(\mu|\Pi) = posterior\ distribution$$
$$\Pr(B|A) = \Pr(\Pi|\mu)$$

Using the Bayes' theorem, it is possible to say that the probability of investor expected returns, given the equilibrium returns is:

$$\Pr(\mu|\Pi) = \frac{\Pr(\Pi|\mu)\Pr(\mu)}{\Pr(\Pi)}$$

As the first assumption of the model, we have mentioned is that the securities returns are normally distributed, then, the prior distribution and also the posterior distribution will be normally distributed.

$$\Pr(\mu|\Pi) \sim N$$

The mean of the distribution of the BL returns is:

$$[(\tau\Sigma)^{-1} + P\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\Pi + P\Omega^{-1}Q]$$

And its variance is:

$$[(\tau\Sigma)^{-1} + P' \, \Omega^{-1} P]^{-1}$$

These are the results that use Black and Litterman, the mean is the expected value of the returns including the views, the variance represents the variance – covariance matrix that is the input of the model.

The Bayesian approach shows that if the investor has no views, the prior and posterior distributions coincide, so he simply holds the equilibrium portfolio.

One of the most important assumptions of the BL model is that the asset returns are distributed like a normal one. For this reason, also the a priori distribution and the conditional distribution P (B | A) are distributed like a normal one, and consequently, of course, the posterior distribution will also be distributed as a normal one.

The procedure

Finally, the proceeding can be summarised in the following steps:

- The choice of the assets that will be hold in the portfolio;
- The computation of the assets expected returns and the assets covariance matrix;
- The computation of the market portfolio returns;
- The specification of the investor view (absolute or relative, and their confidence level);
- The equilibrium returns and the investor view are combined to get the BL expected returns;
- Using the BL returns and the variances in the Markowitz's optimisation process, that produces the usual efficient frontier and so the optimal portfolio.

2.3 Some Notes on the Diversification

Diversification is an important element of a portfolio, as mentioned in the introduction, in this work it will be particularly exploited to try to obtain positive performances. The focus will be on the case of the negative correlation between two assets, to try to obtain movement of opposite sign. The aim is to compensate the two opposite movements within the portfolio. To try to achieve this, this section discusses diversification and which asset can help diversify an equity component of a portfolio.

In this paragraph, firstly, it is noted that the traditional negative correlation between stocks and bonds, historically used to diversify and reduce the risk of a portfolio, seems to have disappeared in recent years. In this way, investors must find new sources for risk reduction.

From this point of view, the use of commodities can be beneficial, particularly for the possibility of using assets other than stocks and bonds, giving the possibility, as will be seen below, to manage a portfolio for tax purposes. An analysis of the previous literature is proposed here to evaluate which commodities are attractive for this work, their characteristics, and how they can be exploited.

As we have seen, Markowitz was the first to introduce a mathematical explanation of *how* diversification could work in a portfolio.

Starting from the 70s, this field grows more and more, with much new research studying the effect and effectiveness of diversification for reducing risks. (Levy & Sarnat, 1970) search international securities with negative correlation for diversifying risks, in particular, they discovered that the developing countries, in the period 1951-1967, could reduce the variance of their portfolio substantially.

Nowadays, however, three problems were highlighted.

The firs is that most of the articles notice that the globalisation is raising the value of the correlations around the world, making more difficult to ensure an efficient portfolio diversification, that could take advantages of the risks reduction due to the negative correlations (Shawky, et al., 1997), (T.C. Chiang, 2007), (Drissen & Laeven, 2007), (You & Daigler, 2010). Diversification through negative correlations is therefore becoming increasingly difficult for investing to achieve.

The second is given in (Shawky, et al., 1997), they focus their analysis on the stock markets of eleven developed countries[14], and they show that using *ex-post* data is possible to generate a portfolio that benefits of the international diversification, while *ex-ante* it is not so easy to select an optimal investment strategy that performs in a good way. So, it is even more difficult for an investor to optimize his portfolio to benefit from negative correlations

The third is pointed out by Longin and Solnik. They analyse thirty years of data and show that the correlations between seven major countries equity indices[15] *vary over*

[14] Austria, Belgium, Denmark, France, Germany, Italy, Netherlands, Norway, Spain, UK, US.

[15] Germany, United Kingdom, Canada, Japan, USA, France, Switzerland.

time, and they rejected the constant conditional correlation hypothesis (Longin & Solnik, 1995). You and Daigler confirm that the correlation between US and EU markets exhibits a positive trend over time, and they also demonstrate that the correlations are *time-varying* (You & Daigler, 2010). The variability of correlations over time is also analyzed by (Antonios Antoniou, 2007) for the US, UK, and Europe markets, the US market is the one that shows it the most. Similarly, in (T.C. Chiang, 2007), the period between 1990 and 2003 is analyzed and the change in correlations during the Asian crisis is shown, identifying two distinct periods. The first of correlations growth, the second of stable high correlations.

2.3.1 A Possible Source of Diversification: The Commodities

Since there seems to be no possibility of obtaining diversification from bonds in terms of negative correlation with the portfolio, in this section we will consider another possibility, commodities. As will be seen in the course of the discussion, commodities, in addition to their primary use, are increasingly becoming a financial asset, which can be valued like all other financial assets. Below is an analysis of the literature, to understand their characteristics and possibilities of use.

There are two main differences between commodity investing and traditional investing. Commodities do not generate cash flows and do not pay dividend or interest; instead, their returns are based only on the prices difference. For this reason, the traditional evaluation methods, such as DCF (*discounted cash flow)* or DDM (*dividend discount model)*, cannot work, and it is difficult to establish a fair price for the exchanges. The second difference is that while the traditional stocks-bonds asset classes are called *financial* assets, commodities are often defined as *real* assets because they are tangible. Commodities are strictly bounded with the real economy. It is possible to classify commodities into four major categories: *energy* (oil, gas, gasoline, …), *metals* which are divided into precious metals (gold, silver, …) and industrial metal (copper, aluminium, …), *agricultural commodities* (soybeans, corn, sugar, cocoa, coffee, …) and *livestock and meat* (cattle, hogs, pork bellies, live cattle...). As can be seen, commodities represent a vast and different asset class that is generally hard to value. Energy and industrial metals are involved in the production cycle, while agricultural and livestock are the output of the production, precious metals can be seen as an industrial commodity or a currency.

Generally, investors invest in commodity through *exchange-traded-commodity*, futures on commodity and swaps[16].

As we have seen, diversification is a key concept for the asset allocation field. Diversification can reduce risks and so permits better performances. However, in the modern financial markets, it is not easy to obtain a satisfactory diversification level due to the increasing level of financialisation of the markets. To increase diversification, financial institutions try to find new asset classes[17]. The commodities are one of these "new" asset classes proposed to the investors. The investments in commodities were rapidly grown up, in mid-2008, the total value of various commodity index-related instruments purchased by institutional investors become about $200 billion, from $15 billion estimated in 2003. (Tang & Xiong, 2012). "According to the Index Investment Data (IID) released by the U.S. Commodity Futures Trading Commission (CFTC), the total value of commodity index investment was about $160 billion by June 2015, with a record high of $256 billion in April 2011".

This large inflow of index investing is based on the perception that investors can obtain diversification benefits by including commodity futures in their portfolios"[18]. Given this interest, many studies search to find evidence in favour or against the capability of commodities to increase the diversification level. The commodity market is rapidly increasing its volume all over the world. Since March 2016, there has also been an entire journal that deals with the commodity market[19], the utility of adding commodity in a portfolio is widely debated. The commodity market appears to be interesting for investors, its returns for the period 1959-2004 is the same as the U.S. equities, with the same Sharpe ratio[20] (Gorton & Rouwenhorst, 2006).

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[16] The futures and the swaps are are financial derivatives. A financial derivative is a contract whose value derives from the value of other assets, called underlying. The futures are traded on the regulated markets, while the Swaps are traded OTC (over the counter) off the exchange. Derivatives can be used to hedge the risks or to speculate, or as for the commodities, for an asset difficult to trade otherwise in the financial markets.

[17] This is also a marketing needs, to have new and different products for the clients.

[18] Solnik, B., 1974. Why not diversify internationally rather than domestically?. *Financial Analyst Journal,* Volume 30, pp. 48-54.

[19] https://www.sciencedirect.com/journal/journal-of-commodity-markets/issues

[20] For the analysis the authors constructed an equally weights index of commodity futures.

Firstly, it appears that similarly to the general results in (Shawky, et al., 1997), also in the commodity field, the good results in terms of diversification and portfolio performances that is obtainable with the use of commodities in the in-sample tests could not be maintained in the out-of-sample ones. (Dudziński, 2010).

Many authors report that nowadays the commodities' price does not depend only on demand-supply equilibrium but also on the financial markets request for this asset class as an investment possibility (Falkowski, 2011). This is reflected in the increasing value of the commodity-stock-bond correlation (Zaremba, 2015). This effect is defined as the financialisation of commodities: the commodity that moves as the other financial assets. (Falkowski, 2011), (Zaremba, 2015), (Dudziński, 2010), (Tang & Xiong, 2012). Positive correlations arise, especially in periods of crisis, when commodity prices fall (Dudziński, 2010).

Finally, adding generical commodities in a portfolio does not induce diversification or benefits for the investors. (Thorsten, et al., 2005), (Bessler & Dominik, 2015), (Lei & Philip, 2017).

If the use of generic commodities does not seem to give certain results in terms of risk reduction, it seems instead that gold can work in this sense. As will be analyzed in the following paragraph, gold has always been viewed with interest in terms of investments. In this case the analysis is very broad and still debated, however it seems that gold can be used in this sense. (Baur & Lucey, 2010), (Junttila, et al., 2018), (Areal, et al., 2015), (Baur & McDermott, 2016).

2.3.2 The case of gold

The use of generic commodities could increase returns, as we have seen in (Gorton & Rouwenhorst, 2006), but does not reduce the risk in terms of diversification. However, it seems that gold could be a valuable asset in terms of diversification and risk reduction. There is much literature that debate gold as an asset for risk reduction and diversification. However, the results are different and often divergent. Nevertheless, the observer can notice that those studies consider different periods, different asset classes, and countries. Many authors make a difference between two possibilities: hedge and *safe asset*. An asset is a *hedge* if it is on average negative or a null correlation with other securities; thus, it can lower the portfolio's risks. The *safe asset* (or *safe heaven*) is an asset that is negative correlated with other assets, specifically during a market crisis[21]. This distinction poses an important issue: if an asset could present a positive correlation during the standard market condition and a negative correlation during turmoil, how it should be valued? In that case, its correlation could be positive, and the models will consider it useless in terms of diversification, even if it may be more useful than another asset, thanks to its safe asset behaviour. Authors should take into account this consideration in their studies. Some studies distinguish between periods of low volatility and periods with high volatility. However, a safe asset shows its quality during a crisis, and a high volatility period does not imply a financial crisis, so the results may underestimate the asset's role. Many authors also analysed a small or too specific market that generally is not included in an (occidental) investor's portfolio[22]. Safe asset quality is a specific characteristic that needs to be studied in its specific context.

Concerning the US, UK and German stock-bond market, the conclusion was that: "We find that gold is a hedge against stocks on average and a safe haven in extreme stock market conditions." (Baur & Lucey, 2010). Similar conclusions are also presented in (Junttila, et al., 2018), for the period 1989-2016, when during crisis periods, while the oil price became positively correlated with the US equity, the gold futures had a negative correlation with the US equity, supporting the safe asset hypothesis. (Areal, et al., 2015) take into consideration a long period of 37 years (1976-2013) and a sub-period (1998-2013) and claim that gold can be considered a safe asset for the U.S. market, as it always exhibits a negative correlation with the stock market in adverse market condition. Besides, gold has been a safe asset during the turmoil of September 11th 2001 and in 2007, after the Lehman bankruptcy (Baur & McDermott, 2016).

[21] To be classified as a safe haven, an asset is required to exhibit a hedge-like behavior under adverse market conditions (Areal, et al., 2015).

[22] Iqbal considers India and Pakistan, hardly anyone individual investors hold financial assets of those markets (Iqbal, 2017). Also, Billah Dar and Maitra consider Indian markets (Billah Dar & Maitra, 2017). In his work Kumar consider the Indian financial market (Kumar, 2014).

3 Portfolio construction and rebalancing for a retail Italian investor

As we have seen, a problem of the Modern Portfolio Theory is that of the practical functionality of the model when dealing with real-life assets.

In this thesis, the focus is primarily on a portfolio for a retail investor; therefore, it has been used financial instruments typical of small investors, such as mutual funds and ETFs. These assets present advantages and disadvantages, that are analysed in this chapter.

The investor is based in Italy, so it must relate to the Italian legislation regarding the taxation of instruments; each country, in fact, has different tax legislation.

3.1 Mutual funds and Asset Management

This paragraph explains why choosing to use assets such as mutual funds and ETFs, listing the advantages that these can bring to a retail investor in managing his portfolio. As will be seen later, mutual funds in Italy have drawbacks concerning the fiscal aspect, but it will be the objective of this thesis to try to overcome these limits.

Mutual funds can be beneficial for a retail investor. The American Security and Exchange Commission defines a mutual fund as a company that brings together money from many people and invests it in stocks, bonds or other assets. The combined holdings of stocks, bonds or other assets the fund owns are known as its portfolio. Each investor in the fund owns shares, which represent a part of these holdings[23].

King[24] identifies some advantages of mutual funds for a retail investor. "A mutual fund portfolio represents a pooling of the assets of many investors" so the individual can hold a well-diversified portfolio, mutual funds can hold 1000-2000 securities, and it benefits from economies of scale. Even if the investor has a small amount of money, the same diversification level is hard or impossible to reach with single socks and bonds, an investor should have thousands of individual stocks in her portfolio, which are impossible to assess initially and constantly monitor. Moreover, the transaction costs for purchasing the securities would rise enormously, risking lowering the performances. Mutual funds allow a person to invest in a specific industry, sector, or geographic area. The individual has the opportunity to concentrate his investments in a specific sector while at the same

[23] https://www.sec.gov/investor/tools/mfcc/mutual-fund-help.htm
[24] (King, 2002)

time diversifying the portfolio, reducing the risk. Another important advantage of mutual funds is their liquidity. Mutual funds shares may be redeemed on any day at the fund NAV[25]. Last, mutual funds provide investors with the benefit of professional asset management since numerous professionals manage them that an individual would not have outside mutual funds.

Assogestioni[26] lists five advantages of mutual funds[27] compared to single stocks and bonds:

- *Asset's autonomy*, the mutual fund's assets are separate from the company's assets. That protects the investors in case of company default. In fact, in case of a default of the asset manager, the company's creditors cannot ask for the mutual fund assets that belong to investors.
- *Control*, in every country, there is at least one commission that controls the fund's balance sheets and supervises, in general, the funds. The asset manager must have a bank that certifies the daily operations.
- *Diversification*, every fund contains many stocks and bonds, and security cannot have a too high weight in the portfolio. This reduces the firm-specific risks in the fund portfolio.
- *Transparency*, every day, the companies update the NAV, so investors know the value of their investments and the composition of the fund's portfolio.
- *Solidity,* the significant diversification reduces the risk and the volatility of the mutual funds concerning the single stocks. The investor emotionally suffers a smaller variation in the value of his investments.

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[25] Net Asset Value is the value of a mutual fund single share. Is computed as the total fund asset valued divided by the number of fund shares.

[26] Assogestioni is an Italian organisation born in 1984 representing the Italian asset managers or those who operate in Italy. Nowadays, the association counts over 290 members, between banks, insurance company and asset managers that operate in Italy.

[27] https://www.assogestioni.it/articolo/tutela-del-risparmio-5-buone-ragioni-per-avere-e-per-scegliere-i-fondi-comuni-di

3.1.1 *Drawback: the fiscal impact of a portfolio rebalancing*

As we have seen, mutual funds can be beneficial for an investor. However, they can have some disadvantages, specifically in the fiscal aspects that will be explaind in this paragraph.

This paragraph is essential for the thesis, since the goal of the thesis, in fact, is precisely to overcome this problem within a portfolio.

The portfolio will be optimized with Black and Litterman, however, this model considers only one investment period, thus not considering a possibility of portfolio rebalancing. When an investor holds a portfolio, he sometimes needs to rebalance it. The markets trends and the prices movements change the original optimal asset's weights. The initial historical data may have varied, as well as the investor's expectations. The investor needs to verify if the portfolio is optimal yet. If it is no longer optimal, the investor should proceed to rebalance his portfolio. For this procedure, the investor must then reformulate her assessments and recalculate the historical data, calculating the new portfolio composition. In that situation, some assets may be sold, and some others should be bought.

Depending on the prices during the portfolio rebalance, that operation could generate capital gains or capital losses. When an investment has increased its value over the investment period and it is sold, it generates a capital gain, vice versa, when an investment has reduced its value, and it is sold, it generates a capital loss.

In many financial systems worldwide, in case of capital gains, individuals and firms must pay taxes[28]. However, sometimes it is possible to *compensate* capital losses and capital gains. In that case, the investor can reduce the amount of capital gain on which he pays tax by the number of capital losses, and he pays the tax on a smaller amount of capital gain.

In the Italian financial system, capital losses and capital gains are divided into two categories: "*redditi di capitale"* and *"redditi diversi"*, as reported in Table 1; that difference is merely a formal definition related to the type of capital gain/loss. Redditi di capitale and redditi diversi cannot compensate each other. As shown in the table below,

[28] For more precise data: https://taxsummaries.pwc.com/quick-charts/capital-gains-tax-cgt-rates

for stocks, bonds, ETC[29], capital gains and capital losses are both redditi diversi so that they can compensate each other. However, for mutual funds and ETF, it is not possible, and an Italian investor rebalancing his mutual funds portfolio cannot reduce its taxes.

It is crucial to notice that a capital loss in mutual funds and ETFs is redditi diversi, and a capital gain in stocks, bonds, ETC is also redditi diversi so that they can be fiscally compensated each other's.

Redditi di Capitale	*Redditi Diversi*
Capital Gain on mutual funds	Capital Loss on mutual funds
Capital Gain on ETF	Capital Loss on ETF
	Capital Gain on ETC/ETN
	Capital Loss on ETC/ETN

Table 1. Radditi di Capitale and Redditi Diversi.

3.2 ETF, ETC

Although mutual funds were mentioned at the beginning of the chapter, in this thesis, to simplify the assumptions regarding transaction costs and management costs, we will work with ETFs. This type of product is illustrated below, and the reasons that led to this choice are explained.

ETFs and ETCs, are both ETP, *Exchange – Traded – Products,* a particular type of financial securities traded by the investors during the day in the stock market, like common stocks and bonds, while mutual funds are traded based on their price at the end of the day.

The ETF, exchange-traded fund, are very close to the mutual funds. They are also a portfolio of stocks and bonds. However, ETFs are generally a so-called passive investor; they simply replicate a financial index without the goal of overperforming it.

ETC, Exchange – Traded – Commodities, is an asset – backed[30] security that replicates the value of a commodity.

[29] Exchange Traded Commodities, a particular type of product in which the issuer invests directly in a physical commodity or in a derivative on a commodity. https://www.borsaitaliana.it/etc-etn/formazione/cosaunetc/cosaunetc.htm

[30] An asset – backed security is a financial asset that is guaranteed by the underlying assets that compose it

Costs

When dealing with funds, it is necessary to take into account their costs, the management fees. These are fees due to the professional team that manage the fund. The fees are defined as a percentage. In addition to management costs, funds often have entry costs, which are also defined as percentages. These management costs can significantly influence the investment's performance (Olson, 1998) (Ajay Khorana, 2009).

To minimise these costs, in this work, it has used ETFs instead of mutual funds. ETFs have the advantage of having significantly lower management costs than funds, and they have no entry fees. In this way, the overall impact of management fees and transactions costs is significantly reduced, so it has not been considered in work.

For ETFs, the TER, total expense ratio, the total amount of cost paid by the investor is much lower than that of mutual funds. In 2018, the TER of mutual funds was 1.67%, while that of ETFs was 0.16%, with a difference of 1.51% per year favouring ETFs[31].

3.3 Portfolio Construction

In this paragraph, the practical details of the work are briefly anticipated. The aim is not to go into the details but to give an overview of the work. In the next chapter, each phase will be specified punctually.

The logic behind the portfolio's construction, the composition of the portfolio, the views and the rebalancing procedure is now anticipated.

In this work, as stated before, I want to generate a portfolio for an Italian investor that benefits of the diversification and take advantages of funds. During the holding time of the portfolio, I try to implement a rebalancing strategy that uses the prices movements to gain a fiscal advantage for the portfolio.

The portfolio is composed of ETFs that replicate equity indices and that are diversified around the world. It has been selected some indices around the world that are the benchmark of the ETF:

- USA: SP500 index;
- Emerging markets: MSCI Emerging Markets index;

[31] https://www.adviseonly.com/capire-la-finanza/finanza-personale/investimenti-efficienti-un-confronto-tra-i-costi-dei-fondi-comuni-e-degli-etf/

- Japan: MSCI Japan index;
- Europe: S&P Europe 350 index.

Moreover, it has been selected an ETC on gold for two reasons: fiscal compensation and diversification. As seen before, it is possible to compensate a capital gain for an ETC with a capital loss for an ETF. In the first chapter, we have seen that gold is considerable as an asset negative correlated with stocks. Since they are expected to move oppositely in terms of prices, this allows them to compensate ETF and ETC fiscally.

The portfolio is Black and Litterman optimised using the software Matlab. For the ETF, my personal views favour the SP500 index over the others, followed by Europe, Emerging Markets and Japan. For the gold, a 2% annual expected return, coherent with the gold capacity to cover the inflation in the long run.

Data are from January 3rd 2005 to January 27th 2020. Data till December 31st 2009 are used Matlab to find the optimal portfolio, data from January 3rd 2010 are used in Excel for rebalancing the portfolio. Quarterly there is a portfolio check: *if at the portfolio evaluation, there is a capital gain on gold and a capital loss on the ETF,* then the portfolio is automatically rebalanced. If there is a rebalancing, the amount sold is the minimum quantity between the capital gain on ETC and the sum of capital loss on ETFs, SP500 excluded. That amount is sold from ETC and is sold from the ETFs. ETFs selling follows this order: Japan, Emerging Markets and finally Europe, the inverse of trust in the ETF. Finally, the amount obtained, on which an investor does not pay taxes, is wholly invested in the SP500.

4 Procedure

Finally, this chapter presents the work done on the model used and the results obtained.

The first three paragraphs of the chapter are dedicated to the Black and Litterman model's specification for this work.

The first paragraph illustrates the selection of the assets to be included in the portfolio, also the criterion for choosing the selected ETFs is presented.

The personal views typical of the Black and Litterman model are shown in the second paragraph, highlighting the reasons that led to the formulation of the views of this thesis.

Finally, in the third paragraph, the steps of the model on which we worked and the results obtained in terms of the optimal portfolio composition are explained. The portfolio that would have been obtained using the Markowitz model is also presented.

The next phase is that of rebalancing. Once the initial optimal portfolio has been obtained, a rule has been set for rebalancing and is periodically implemented. In the fourth paragraph, the rule for rebalancing is explained in detail, and the rebalances performed during the selected time interval are presented.

Finally, the results obtained in the thesis are presented and commented on

4.1 The data

The choice of assets on which this work is based is geographical, i.e. each ETF refers to a specific stock index of a country or group of countries. With this choice, we intend to try to obtain a portfolio as diversified as possible.

The MSCI indexes were used as a provider. MSCI is a company that provides financial services, among the best known are the financial indices, both equity and bonds and which are used by many ETFs and Mutual Funds as a benchmark. MSCI derives from Morgan Stanley Capital International, as the company was founded in 1969 by Morgan Stanley, who subsequently sold the property between 2007 and 2009, making it solely MSCI.

The largest geographic equity index proposed by MSCI is the All-Country World Index[32], ACWI, an equity index representing large and mid-cap across 23 developed

[32] https://www.msci.com/documents/10199/8d97d244-4685-4200-a24c-3e2942e3adeb

markets and 27 emerging markets, and it covers 88%[33] of the global investable equity opportunity set.

MSCI divides ACWI into two subsets: the World Index, which represents developed countries and the Emerging Markets, which represents emerging countries.

Excluded from the ACWI is the Frontier Markets, it is a group of countries that are particularly underdeveloped from a financial point of view and therefore still very marginal for the financial markets. When a country of this group starts to develop financially, it is moved to emerging markets.

The World Index and the emerging markets are further divided into American countries, European countries, and for the Word Index, Pacific countries, for the Emerging markets, Asian countries.

In the following table, it is shown a graphical representation of the subsets.

[33] https://www.msci.com/World

Table 2. MSCI ACWI composition. Source: MSCI website.

MSCI ACWI INDEX					
MSCI WORLD INDEX			MSCI EMERGING MARKETS INDEX		
DEVELOPED MARKETS			EMERGING MARKETS		
Americas	Europe & Middle East	Pacific	Americas	Europe, Middle East & Africa	Asia
Canada	Austria	Australia	Argentina	Czech Republic	China
United States	Belgium	Hong Kong	Brazil	Egypt	India
	Denmark	Japan	Chile	Greece	Indonesia
	Finland	New Zealand	Colombia	Hungary	Korea
	France	Singapore	Mexico	Kuwait	Malaysia
	Germany		Peru	Poland	Pakistan
	Ireland			Qatar	Philippines
	Israel			Russia	Taiwan
	Italy			Saudi Arabia	Thailand
	Netherlands			South Africa	
	Norway			Turkey	
	Portugal			United Arab Emirates	
	Spain				
	Sweden				
	Switzerland				
	United Kingdom				

The ACWI covers 88% of the global investable equity opportunity set and is therefore used as a starting point for this work.

For the portfolio, were selected the USA, the Europe developed countries, Japan, and the Emerging Markets.

The choice of using the USA instead of America and Japan instead of Pacific countries is due to the possibility of more easily finding historical data on long-term ETFs, which the USA and Japan, taken individually, guarantee more. Similarly, for emerging markets, it was decided to select the entire index and not the sub-indices, because as mentioned above, sometimes frontier markets countries have been added to the index, making the composition of the sub-indices more unstable. For this reason, long-term data is more available for the entire emerging countries, rather than data for the sub-indices.

In summary, here are the geographical areas and the relative equity index selected, that will be part of this work:

- USA: SP500 index;
- Japan: MSCI Japan index;
- Europe: S&P Europe 350 index;
- Emerging markets: MSCI Emerging Markets index.

The equity indices selected for Japan and Emerging countries are MSCI indices. On the other hand, for the USA and Europe, different equity indices were used, namely the SP500 and S&P 350, respectively.

SP500 is the largest stock index in the world[34] and represents the US market, managed by the S&P Dow Jones Indices. S&P 350 is an index created by the American company Standard & Poor's. These indices were chosen because they are widely used by ETFs as benchmarks.

The goal is to have the largest possible proxy of the investable equity universe. The choice of these benchmarks seems reasonable even if we observe, from the graph below, the composition by single countries of the ACWI:

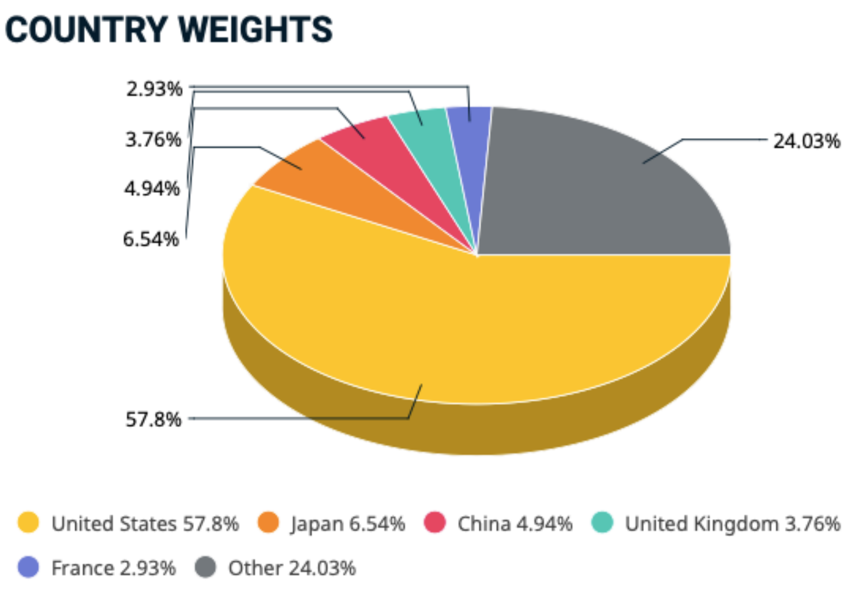

Figure 8. MSCI ACWI coutry weights. Source: MSCI website.

[34] https://www.spglobal.com/spdji/en/indices/equity/sp-500/#overview

It can be seen how the selected countries comprehend most of the index. The selection made includes 92.94% of the MSCI ACWI.

The weights are shown in the following table.

Table 3. The weights of the selected assets.

Benchmark	MSCI ACWI Weight
USA	53.64%
Europe	20.09%
Japan	7.61%
Emerging Markets	11.60%
Total	92.94%

These are, therefore, the benchmarks selected for the job to which the relevant ETFs were matched. The selected ETFs are shown in the table below. In Table 2, there are the main characteristics of the ETFs I used. The ISIN is the reference number that uniquely identifies every ETP or mutual funds. iShares is the company that administrates the ETFs, it is the biggest ETFs company in the markets[35].

Table 4. ETFs ISIN and benchmarks.

ISIN	Name	Benchmark
464287200	iShares Core S&P 500 ETF	USA
464287861	iShares Europe ETF	Europe
46434G822	iShares MSCI Japan ETF	Japan
IE00B0M63177	iShares MSCI Emerging Markets ETF	Emerging markets
IE00BVFZGK87	Boost Gold ETC	Gold

.

[35] https://www.ishares.com/us

To achieve the purpose of the work, namely tax compensation, alongside the equity ETFs, I have included an ETC that replicates the price of the gold; more precisely, it replicates the NASDAQ Commodity Gold ER Index. That is the gold quotation on the Nasdaq exchange.

I have downloaded the daily closing prices of the 4 ETFs and the ETC.

The data start from January 3rd 2005, to January 27th 2020. Yahoo Finance is the data provider of the ETF's, while SPD Gold Shares is the data provider for the ETC. I used data until December 31st 2009, to select the optimal portfolio, while I used data from January 3rd 2010, till January 27th 2020 to perform the portfolio rebalancing process.

Table 5. The first asset returns

	Prices				
Dates	SP500	Emerging markets	Japan	Europe	Gold
2010-01-04	92.682	34.662	39.480	29.666	109.800
2010-01-05	92.951	34.913	40.120	29.696	109.700
2010-01-06	93.025	34.986	40.080	29.733	111.510
2010-01-07	93.433	34.784	39.960	29.637	110.820
2010-01-08	93.751	35.060	40.440	29.874	111.370
2010-01-11	93.881	34.986	40.720	30.119	112.850
2010-01-12	93.017	34.426	41.120	29.703	110.490
2010-01-13	93.751	34.532	41.000	30.045	111.540
2010-01-14	94.028	34.443	41.640	30.059	112.030
2010-01-15	92.968	34.045	42.000	29.518	110.860
2010-01-19	94.118	34.735	41.560	29.830	111.520
2010-01-20	93.188	33.899	40.960	29.066	108.940
2010-01-21	91.361	32.876	41.360	28.168	107.370
2010-01-22	89.388	32.146	41.080	27.508	107.170
2010-01-25	89.836	32.381	41.120	27.990	107.480
2010-01-26	89.412	31.708	40.560	27.886	107.560
2010-01-27	89.869	31.627	40.240	28.027	106.530
2010-01-28	88.792	31.407	40.280	27.315	106.480
2010-01-29	87.797	31.067	39.680	26.989	105.960

2010-02-01	89.184	31.903	39.760	27.530	108.350
2010-02-02	90.301	32.162	40.160	27.975	109.130
2010-02-03	89.844	31.976	40.160	27.708	108.700
2010-02-04	87.096	30.531	39.400	26.448	104.370
2010-02-05	87.284	30.190	39.200	25.943	104.680
2010-02-08	86.607	29.890	39.160	25.654	104.040
2010-02-09	87.732	30.864	39.360	26.581	105.410
2010-02-10	87.528	30.783	39.320	26.440	105.120
2010-02-11	88.425	31.594	39.160	26.633	107.130
2010-02-12	88.393	31.196	39.080	26.403	107.040
2010-02-16	89.787	32.000	39.160	27.011	109.620
2010-02-17	90.187	32.114	39.680	26.982	109.250
2010-02-18	90.758	32.235	39.560	27.115	109.980
2010-02-19	90.945	31.959	39.040	27.100	109.470
2010-02-22	90.953	32.000	39.440	27.137	109.070
2010-02-23	89.812	31.261	39.400	26.566	107.890
2010-02-24	90.635	31.562	39.320	26.818	107.360
2010-02-25	90.562	31.302	39.200	26.581	108.310
2010-02-26	90.562	31.618	39.640	26.715	109.430
2010-03-01	91.516	32.154	40.120	26.774	109.430

And the final data:

Table 6. The last assets return.

2019-12-05	311.834	41.924	60.110	44.892	139.000
2019-12-06	314.617	42.169	60.510	45.290	137.620
2019-12-09	313.683	42.003	60.520	45.091	137.580
2019-12-10	313.355	42.150	60.310	45.111	137.970
2019-12-11	314.150	42.766	60.190	45.329	138.920
2019-12-12	316.932	43.511	60.140	45.707	138.430
2019-12-13	317.041	43.491	60.410	46.124	139.050
2019-12-16	319.320	43.860	60.070	46.740	139.040
2019-12-17	319.350	44.260	60.120	46.410	139.010
2019-12-18	319.370	44.500	59.840	46.360	139.020

2019-12-19	320.740	44.520	59.780	46.370	139.380
2019-12-20	322.380	44.610	59.840	46.530	139.520
2019-12-23	322.610	44.670	59.680	46.640	139.950
2019-12-24	322.650	44.570	59.570	46.640	141.270
2019-12-26	324.320	44.890	59.610	46.820	142.380
2019-12-27	324.260	45.070	59.740	47.000	142.330
2019-12-30	322.510	44.770	59.540	46.680	142.630
2019-12-31	323.240	44.870	59.020	46.960	142.900
2020-01-02	326.320	45.780	59.800	47.330	143.950
2020-01-03	323.810	44.930	59.130	46.820	145.860
2020-01-06	325.090	44.820	59.130	46.990	147.390
2020-01-07	324.200	44.790	59.700	46.720	147.970
2020-01-08	325.850	45.050	59.400	46.810	146.860
2020-01-09	328.050	45.350	59.860	46.950	146.030
2020-01-10	327.140	45.590	59.780	46.720	146.910
2020-01-13	329.430	46.300	59.530	46.930	145.820
2020-01-14	328.920	46.040	59.710	46.960	145.690
2020-01-15	329.660	45.680	59.570	46.930	146.540
2020-01-16	332.400	45.970	59.610	47.190	146.310
2020-01-17	333.500	46.230	59.700	47.420	146.580
2020-01-21	332.780	45.060	59.670	47.040	146.740
2020-01-22	332.850	45.430	59.810	47.070	146.790
2020-01-27	324.870	43.060	58.600	45.810	148.990

4.2 The Views

The Black and Litterman model in this thesis is functional to the possibility for the investor to express her own views on the assets.

In this way, the assets with the worst views will be compensated with gold, while the asset with the best views will be subject to additional payments.

The preferences I expressed in this work for ETFs allowed me to indicate an order among the assets for operations in rebalancing.

The order obtained is therefore as follows:

- USA;

- Europe;
- Emerging Markets;
- Japan.

In this paragraph, the reasons that led to the order for this work are illustrated.

The Table 7 shows the data of the reference benchmarks. In the first columns there is the returns, then the volatility calculated with the annualised standard deviation over different time horizons and finally the Sharpe Ratio.

Table 7. An analysis of the assets returns, standard deviation and Sharpe ration for differents time intervals.

	3Y Returns	5Y Returns	10Y Returns	3Y Standard Deviation	5Y Standard Deviation	10Y Standard Deviation	Sharpe Ratio
Europe	6.28	8.84	5.71	19.14	16.15	16.65	0.37
SP500	17.44	16.75	14.06	18.75	15.14	13.76	0.97
Japan	6.7	10.86	7.52	15.23	12.87	13.42	0.55
Emerging markets	6.48	12.07	3.65	19.16	16.41	17.7	0.25

The standard deviations are similar across assets, which is not surprising given that they are equity indices. However, the most significant differences are represented by past returns and, therefore, the Sharpe ratio. Following the Sharpe ratios of the last five years, we can put the countries in the following order:

- USA;
- Japan;
- Europe;
- Emerging Markets.

However, taking a longer time horizon, starting from March 2006, it is possible to see how the index on the Japan is the one that has achieved the worst performance. The figures below show the performance of the various MSCI indices.

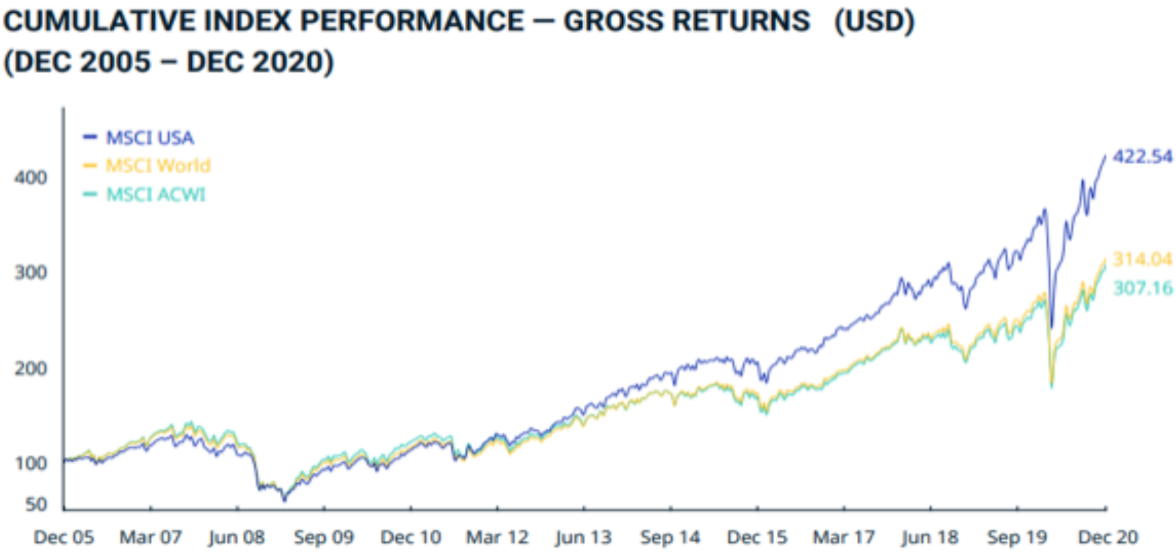

Figure 9. USA equity line since 2006. Source: MSCI website.

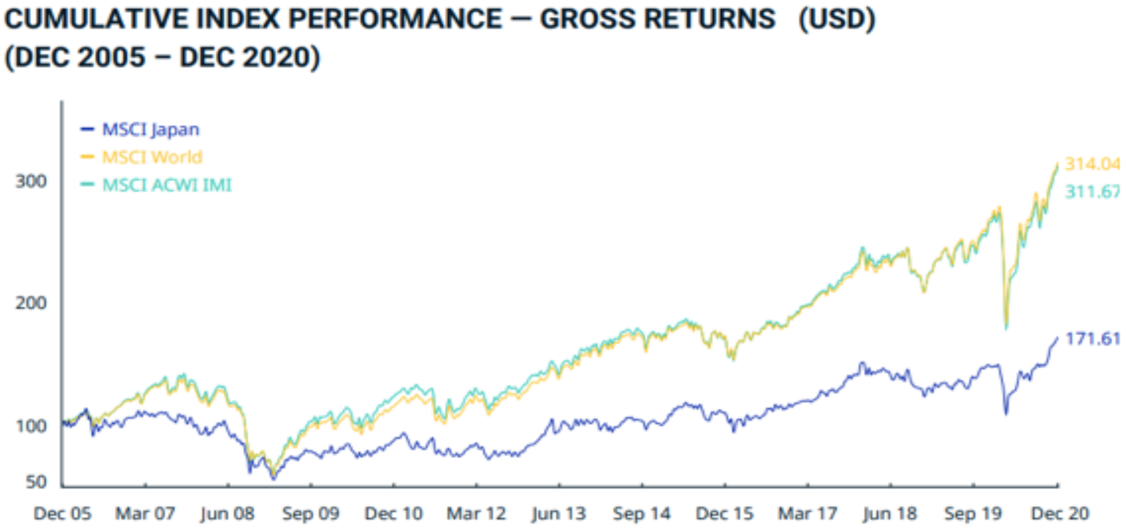

Figure 10. Japan equity line since 2006. Source: MSCI website.

Figure 11. Europe equity line since 2006. Source: MSCI website.

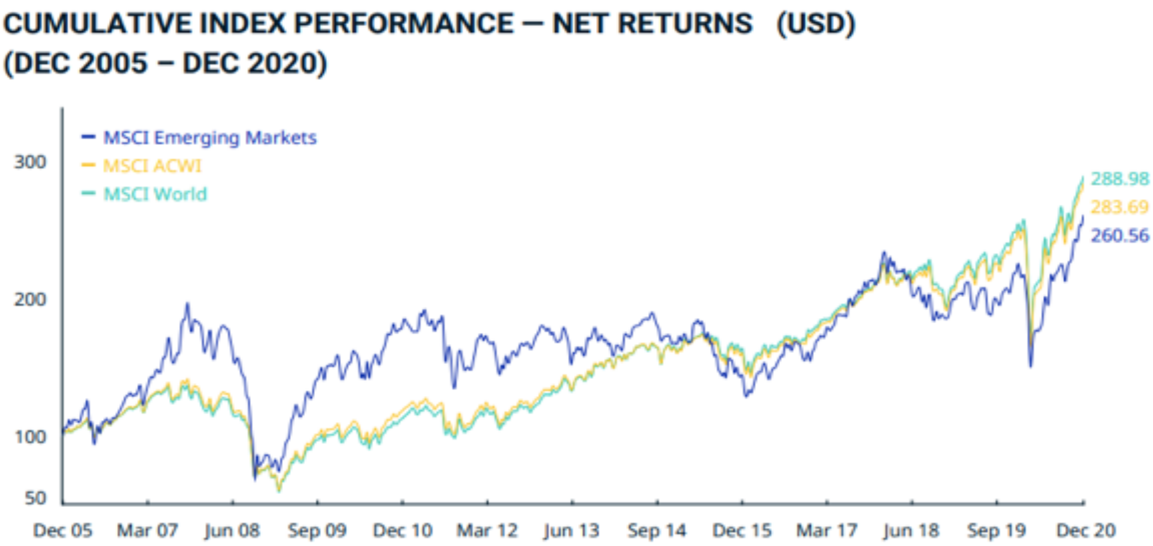

Figure 12. Emerging markets equity line since 2006. Source: MSCI website.

In this longer time horizon, volatilities are once again similar, with Japan's Sharpe index falling significantly compared to the one of the other nations.

From an economic point of view, Japan has experienced a challenging situation in recent decades, which has been identified as a "lost decade", precisely the period of crisis was around the 80s and 90s. However, it does not seem that the country has still definitively solved its problems, having ended up in a long period of stagflation.

In expressing my personal views, I make Japan demote, reducing my degree of confidence in it. The final order of preferences is, therefore, this:

- USA;

- Europe;
- Emerging Markets;
- Japan.

Through the views, I also intend to resolve the question of expected returns. The data used to find the optimal portfolio are included between 2005 and 2009, ending precisely during the 2007 crisis period. For this reason, as can be seen in the Table 8, the expected value of the returns of that period is far from the long-term historical returns observed at the beginning of this paragraph.

Table 8. The selected assets return in the period 2005 - 2009.

Index	Returns
SP500	0.32%
Emerging markets	9.30%
Japan	-1.31%
Europe	2.22%

The goal is to give the US, the country with the best view, the highest yield, and gradually reduce the view on yield.

As seen above, the US has outperformed MSCI ACWI in recent years, which has grown by an average of 7-8% in the past.

(Roger G. Ibbotson, 2003) determine a "long-term equity risk premium" of around 6%. So, 7% as a long-term global market return appears to be a reasonable return[36]. However, it must be taken into account that it is a net return, while as mentioned in the previous chapter, even if low, ETFs have a cost, and although minimal, there are also transaction costs. To take these costs into account, it was decided to set the long-term return of the equity investment of 6% to the USA and the other ETFs a lower return to scale.

- SP500 6% expected return;

[36] https://www.evidenceinvestor.com/whats-the-long-run-rate-of-return-on-equities/ here is a brief analysis of the historical returns of the SP500 from 1825 to 2019 https://advisor.visualcapitalist.com/historical-stock-market-returns/

- Europe 5.5% expected return;
- Emerging Markets 5% expected return;
- Japan 4.5% expected return.

I have also added an absolute view for the gold, referring to the literature cited in the first chapter about it, it has the ability in the long run to cover inflation. In line with the ECB's objectives of keeping inflation just below 2%, I have set an expected return of 2% per annum for gold.

Table 9. The investor's views

4×7 table

USA	EM	EWJ	EU	GOLD	View_Return	View_Uncertainty
1	0	0	0	0	0.06	0.0001
0	1	0	0	0	0.05	0.001
0	0	0	1	0	0.055	0.001
0	0	1	0	1	0.02	0.0001

In the figure above, there is a resume of the views. The view on gold has greater confidence due to the literature supporting the hypothesis. Also, the view for the SP500 has higher confidence since it has the long – run returns more stable than the others.

4.3 Model

Starting from the prices, the daily logarithmic returns of the assets were calculated. Expressed as:

$$R_t = \ln\left(\frac{P_t}{P_{t-1}}\right)$$

where:

- R_t is the daily return at time t;
- P_t is the daily price at time t.

The historical variance - covariance matrix, Σ (NxN), was calculated from the returns obtained. The variances of the ETPs are indicated in the diagonal and the covariances between the assets in the other areas.

Table 10. Historic variance - covariance matrix.

	SP500	*Emerging markets*	*Japan*	*Europe*	*Gold*
SP500	1.62704E-05				
Emerging markets	1.91243E-05	3.34489E-05			
Japan	6.33317E-06	9.8562E-06	2.00373E-05		
Europe	1.83969E-05	2.56653E-05	9.6927E-06	2.88671E-05	
Gold	-3.50099E-07	2.98017E-06	-4.31022E-07	1.57322E-06	1.79893E-05

In this work, τ was set equal to $\frac{1}{n}$, with n equal to the number of data points used to generate the covariance matrix.

At this point, the market portfolio was searched, representing the starting point for the investor's portfolio. In the absence of views, this would be the portfolio held by the investor. A market portfolio is not uniquely defined, but it is up to the investor to define his market. As we are dealing with a geographically distributed portfolio worldwide, the market portfolio should be a global portfolio. Furthermore, since gold is inserted, an asset inserted to diversify and reduce the portfolio's risk, the market portfolio should also be a diversified portfolio and not just equity.

Therefore, it was decided to use a mutual fund, the Transamerica Asset Allocation fund. It is a balanced mutual fund consisting of 50% shares and 50% bonds and invests globally. The time series consistent with the time series of the ETFs were also used for this mutual fund. It could be argued that the fund invests in bonds and not gold, while in our portfolio, it is the other way around. However, from this fund, we get a unique figure, namely the NAV. The values of shares and bonds are, as mentioned, therefore incorporated into a single data, which can therefore be seen as a value that the investor could obtain from a global equity investment, balanced by a prudential component, in this case, represented by bonds.

This fund was, therefore, the starting point and benchmark for obtaining the market portfolio. To make this step, the investor was required to be fully invested and long-only, there is no possibility of short selling.

Through reverse optimisation, it is possible to find the equilibrium optimal returns.

Once the prior belief was obtained, the data was used to calculate the Black and Litterman returns. Of these, through the following formulas, the expected value and the covariance can then be calculated.

$$\mu = [P^T \Omega^{-1} P + C^{-1}]^{-1} [P^T \Omega^{-1} q + C^{-1} \pi]$$

$$cov(\mu) = [P^T \Omega^{-1} P + C^{-1}]^{-1}$$

Table 11. The prior belied of expected return vs Black Litterman expected returns.

5×3 table

Asset_Name	Prior_Belief_of_Expected_Return	Black_Litterman_Blended_Expected_Return
"USA"	-0.007897	0.015958
"EM"	-0.013498	0.027055
"EWJ"	-0.004661	0.0073786
"EU"	-0.0095164	0.018864
"GOLD"	-0.0031625	0.0031718

This result is interesting for two reasons. Firstly, it confirms that the data range can lead to a biased portfolio without a correction due to the period considered. Second, it confirms one of the problems mentioned for the Modern Portfolio Theory: the Markowitz model can create very concentrated portfolios even in a single asset.

In fact, since the data around the 2007 crisis, the portfolio optimised in average variance with Markowitz is all concentrated on 80% gold, which is a period of crisis, therefore, shows good performance and on emerging countries 19%. In fact, in those years, stock indices had suffered heavy drawdowns, and gold had proved a safe asset, and the emerging markets had the highest returns, increasing their weight in the Markowitz model.

For this reason, if the Markowitz model had been used without correcting the views, the portfolio would have been highly concentrated in these two assets, making the rebalancing phase very difficult and reducing the monetary amounts subject to rebalancing, with a reduction in the utility of the rebalancing process.

The figure 13 shows the compositions of the portfolios obtained with Markowitz and Black and Litterman.

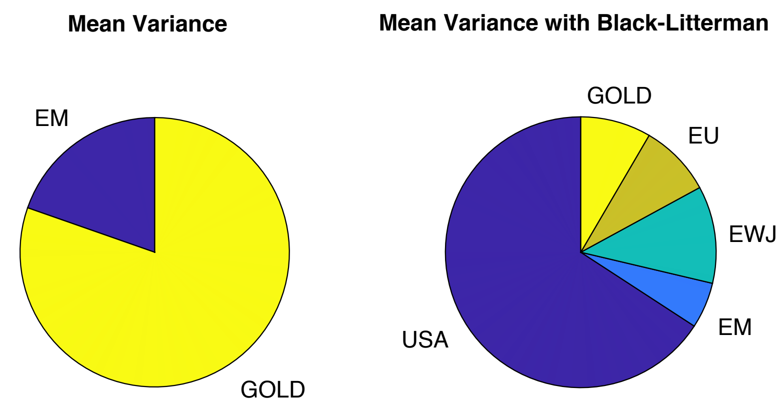

Figure 13. Mean - variance and Black Litterman optimal portfolio.

This portfolio, extremely concentrated on an asset (80%), brings out the drawbacks of the classic portfolio theory, which concerns the preference for assets that perform better in mean-variance, generating a great concentration on them.

While the addition of views allows for greater diversification in the portfolio with Black Litterman, so the resulting portfolio is:

- iShares Core S&P 500 ETF: 65,82%;
- iShares MSCI Emerging Markets ETF: 5,49%;
- iShares MSCI Japan ETF: 11,62%;
- iShares Europe ETF: 8,64%;
- Boost Gold ETC: 8,42% .

These weights can be seen, in comparison with those of Markowitz, in the Table 12.

Table 12. Mean - variance and Black Litterman optimal portfoliol.

5×3 table

AssetName	Mean_Variance	Mean_Variance_with_Black_Litterman
"USA"	2.2271e-13	0.6582
"EM"	0.19644	0.0549
"EWJ"	1.9736e-13	0.1162
"EU"	1.1509e-12	0.086435
"GOLD"	0.80356	0.08426

4.4 The rebalancing

As mentioned above, an investor must cyclically rebalance his portfolio to keep it updated to market trends.

This second paragraph, therefore, illustrates the procedure that was used to perform the rebalancing.

As mentioned earlier, the data for this part of the work is from January 3rd, 2010, to 2020.

4.4.1 The rule

Quarterly control windows have been set, four times a year it is the investor's responsibility to check the performance of the portfolio. Considered 20 trading days per month, so the check is performed every 60 trading days. Furthermore, rebalancing is not always performed.

The event that triggers the rebalancing event is illustrated in the following paragraph.

How and when the portfolio is rebalanced

The goal is to follow quarterly the trend of the portfolio and perform a rebalancing when the following condition occurs:

There is a
capital gain on the ETC on gold
AND there is a
capital losses on Japan, Emerging Market and Europe.

Conversely, if the condition is not verified, the portfolio remains unchanged, and there is no rebalancing, and the next check is carried out.

If, on the other hand, the rebalancing occurs, all the losses between the ETFs subject to rebalancing are added together, and the quantity subject to rebalancing will be:

$$\text{amount to compensate} = \min[\textit{capital gain on gold}, \textit{capital losses on ETFs}]$$

to sell only the gain or the loss and no more, since a larger sale would no longer constitute a capital gain or loss, therefore it would not be possible to fiscally compensate it.

The loss is calculated for each ETF, starting from the first one compares the loss with the amount to be compensated, and the compensation amount will be:

$$\min\,[\textit{amount to compensate}, \textit{capital loss on the ETF}]$$

If the amount to be compensated is greater than the loss, proceed with the following loss-making ETF and repeat the procedure. In this case, the amount to be compensated will be equal to the initial amount reduced by the loss already compensated.

$$\min\,[\textit{residual amount to compensate}, \textit{capital loss on the ETF}]$$

Losses will be sold, subject to availability, in this order:

- first Japan;
- then Emerging markets;
- finally, Europe.

From the ETF with a lower expected value to the one with a higher expected value.

Thus, the monetary amount obtained from ETFs and ETC, on which no taxes will have to be paid[37], will then be fully reinvested in the ETF on the SP500.

The optimal portfolio by Black and Litterman was expressed in monetary terms rather than percentages to simplify the calculation of capital gains and losses.

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[37] It is important to notice that operationally an investiture must sell the capital loss first, this is recorded in its fiscal backpack, a fiscal archive where the capital losses are recorded, and at that point it is possible to sell the capital gain and offset the positions for tax purposes.

A portfolio of € 100,000 was therefore assumed distributed as follows:

Table 13. The composition of the starting BL optimal portfolio expressed both in weights and in monetary amount.

	Starting Portfolio				
	SP500	Emerging markets	Japan	Europe	Gold
BL Weights:	0.6582	0.0549	0.1162	0.086435	0.08426
In money:	65,820.00 €	5,490.00 €	11,620.00 €	8,643.50 €	8,426.00 €

Quarterly, every 60 days, the new value of the ETFs was calculated and updated, compounding the quarter returns, which serves as a starting point for the new quarter.

4.4.2 The actual rebalances

In the period, only five times a rebalance occur. The rebalancing dates, the composition of the portfolio at the time of rebalancing and the new composition are shown below:

June 25th, 2010

	SP500	Emerging markets	Japan	Europe	Gold
2010-06-25	64,649.21 €	5,316.07 €	11,429.95 €	8,003.37 €	8,832.80 €
	65,462.82 €	5,142.14 €	11,239.90 €	7,960.55 €	8,426.00 €

Gold capital gain		406.80 €
ETFs total losses	-	1,004.11 €
Amount rebalanced		406.80 €
Japan loss	-	190.05 €
Emerging markets loss	-	173.93 €
Europe loss	-	640.13 €
Japan compensation		190.05 €
Residual amount to compensate		216.75 €
Emerging markets compensation		173.93 €
Residual amount to compensate		42.83 €
Europe compensation		42.83 €

- € 406.80 have been on gold;
- € 190.05 have been sold on Japan;
- € 173.93 have been sold on Emerging markets;
- € 42.82 have been sold on Europe;
- € 813.60 have been reinvested in the USA.

September 21rst, 2010

	SP500	Emerging markets	Japan	Europe	Gold
2010-09-21	67,172.32 €	5,365.11 €	11,396.16 €	8,400.48 €	8,521.62 €
	67,363.56 €	5,365.11 €	11,300.54 €	8,400.48 €	8,426.00 €

Gold capital gain		95.62 €
ETFs total losses	-	711.73 €
Amount rebalanced		95.62 €
Japan loss	-	536.10 €
Emerging markets loss	-	91.72 €
Europe loss	-	83.91 €
Japan compensation		95.62 €
Residual amount to compensate		- €
Emerging markets compensation		- €
Residual amount to compensate		- €
Europe compensation		- €

- € 95.62 have been sold on gold;
- € 95.62 have been sold Japan;
- 191.24 have been reinvested in the USA.

March 14th, 2011

	SP500	Emerging markets	Japan	Europe	Gold
2011-03-14	71,437.17 €	5,526.77 €	11,295.90 €	8,738.76 €	8,784.57 €
	72,154.30 €	5,365.11 €	11,195.65 €	8,642.11 €	8,426.00 €

Gold capital gain	358.57 €
ETFs total losses	399.69 €
Amount rebalanced	358.57 €
Japan loss	- 100.26 €
Emerging markets loss	161.66 €
Europe loss	338.28 €
Japan compensation	100.26 €
Residual amount to compensate	258.31 €
Emerging markets compensation	161.66 €
Residual amount to compensate	96.65 €
Europe compensation	96.65 €

- € 358.57 have been sold on gold;
- € 100.26 have been sold Japan;
- € 161.66 have been sold Emerging markets;
- € 96.65 have been sold Europe;
- € 717.14 have been reinvested in the USA.

June 8th, 2011

	SP500	Emerging markets	Japan	Europe	Gold
2011-06-08	71,888.56 €	5,406.51 €	11,298.12 €	8,781.37 €	8,703.75 €
	72,444.06 €	5,406.51 €	11,020.36 €	8,781.37 €	8,426.00 €

Gold capital gain	277.75 €
ETFs total losses	- 283.50 €
Amount rebalanced	277.75 €
Japan loss	- 413.63 €
Emerging markets loss	- 95.17 €
Europe loss	225.30 €
Japan compensation	277.75 €
Residual amount to compensate	- €
Emerging markets compensation	- €
Residual amount to compensate	- €
Europe compensation	- €

- € 277.75 have been sold on gold;
- € 277.75 have been sold Japan;
- € 555.5 have been reinvested in the USA.

September 1rst, 2011

	SP500	Emerging markets	Japan	Europe	Gold
2011-09-01	70,714.74 €	5,188.18 €	10,890.29 €	8,230.92 €	9,054.26 €
	71,971.26 €	4,965.54 €	10,484.67 €	8,230.92 €	8,426.00 €

Gold capital gain		628.26 €
ETFs total losses	-	1,252.05 €
Amount rebalanced		628.26 €
Japan loss	-	405.62 €
Emerging markets loss	-	338.59 €
Europe loss	-	507.84 €
Japan compensation		405.62 €
Residual amount to compensate		222.64 €
Emerging markets compensation		222.64 €
Residual amount to compensate		- €
Europe compensation		- €

- € 628.26 have been sold on gold;
- € 405.62 have been sold Japan;
- € 222.64 have been sold Emerging markets;
- €1256.52 have been reinvested in the USA.

4.5 Comparison

Rebalancing and compounding of returns were simulated for ten years, from January 4th to January 8th, 2020.

To verify the portfolio's performance, in the same time interval, two other portfolios were assessed. The first is identical to the initial portfolio but without rebalancing, only capitalised to test the effectiveness of rebalancing.

The second, on the other hand, is without gold and its share distributed among the funds. This portfolio also has not undergone any rebalancing to verify without rebalancing and, in the absence of gold, what the performances could be.

These are its shares:

- iShares Core S&P 500 ETF: € 67,925.00;
- iShares MSCI Emerging Markets ETF: € 7,595.00;
- iShares MSCI Japan ETF: € 13,725.00;
- iShares Europe ETF: € 10,745.00.

The results of the three portfolios were:

- € 153,634.14 for the rebalanced portfolio;
- € 151,568.00 for the non-rebalanced wallet, with gold;
- € 142,201.61 for the non-rebalanced and gold-free portfolio.

The overall returns of the three portfolios were, therefore:

- 53,634% for the rebalanced portfolio;
- 51.568% for the non-rebalanced portfolio, with gold;
- 42-201% for the non-rebalanced and gold-free portfolio.

The first is, therefore, the portfolio that had the highest overall final return.

To verify the performance with respect to risk, the standard deviation and the Sharpe Ratio were calculated, shown in the following table.

	Rebalance	No rebalance	No Rebalance and No Gold
Standard deviation	2.358%	2.327%	2.339%
Sharpe Ratio	0.54135	0.532765	0.51574

Table 6. Portfolio's standard deviations and Sharpe ratios.

The portfolio with rebalances, therefore, had higher volatility than the others, this perhaps due to the significant changes in its composition on the occasion of the rebalances. On the other hand, the portfolio without gold achieved greater volatility than the one with gold, a sign that gold has helped reduce volatility.

Finally, comparing the Sharpe ratios, the portfolio with the rebalancing is the portfolio that has obtained the best performances, superior to other portfolios.

4.6 Conclusion

Markowitz in 1952 revolutionised the financial sector with his *Portfolio Selection* work, creating a scientific method for researching and managing an investment portfolio. We have seen how starting from that date, the classic portfolio theory, the Modern Portfolio Theory, began to develop and improve, to try to find a way to generate efficient portfolios, taking into account various elements, both subjective: the aversion to risk of the individual investor, both objective: such as the introduction of the risk-free rate. These models have been fundamental in establishing the initial pillars of research and establishing concepts that have remained unchanged to this day, such as demonstrating the role of diversification in risk reduction and the difference between systematic and idiosyncratic risk.

While excellent in modelling, Modern Portfolio Theory has shown some problems in its practical implementation. It was discussed in the paragraph dedicated to its issues, which we quickly report here:

- The estimation risk related to the model parameters;
- The instability of the mean-variance solutions;
- The operative choice of the target returns;
- The absence of investor views.

Therefore, the turning point was in the 90s, when Black and Litterman published their model for portfolio optimisation. Black and Litterman use the previous theory to take advantage of Sharpe's reverse optimisation and the Markowitz model to create a more efficient model. It is undoubtedly a complex model due to the mathematical/statistical passages to put its factors together, but which nevertheless seems to provide solutions to the problems presented above.

The model's great turning point is the inclusion of investor views, combined with historical estimated returns, generating a new set of expected returns, capable of improving and stabilising the portfolio's performance as seen above.

This work uses this new model's basis, taking advantage of the investor's views, to express preferences on the portfolio's assets.

A fundamental element in finance is diversification because it allows reducing the overall risk of a portfolio. A paragraph has been dedicated to diversification, particularly on commodities as an element of diversification within an investment portfolio. As we have seen, commodities are a vast set of assets that have different characteristics and properties. Quoting what was written previously:

"It is possible to classify commodities into four major categories: energy (oil, gas, gasoline, ...), metals which are divided into precious metals (gold, silver, ...) and industrial metal (copper, aluminium, ...), agricultural commodities (soybeans, corn, sugar, cocoa, coffee, ...) and livestock and meat (cattle, hogs, pork bellies, live cattle...). As can be seen, commodities represent a very large and different asset class that is hard to value."

Each category has different characteristics and must be analysed individually. Sometimes even elements of the same category have different or opposite characteristics.

One element of interest is gold. It has been seen that there is much literature regarding the characteristics of gold as an asset, even with different interpretations, and the solution does not seem to have arrived yet. Those on which we relied for this work are those that seemed most convincing, in particular the fact that gold, in the long run, can equalise inflation and that gold can be a safe haven in times of financial crisis.

Starting from the Black and Litterman model and gold as a source of financial diversification, a portfolio was created consisting of 5 total assets, including 4 ETFs, on the USA, Japan, Emerging Countries and Europe and an ETC on gold. As explained, we chose to use ETFs and ETFs for fiscal reasons, trying to exploit the characteristics of gold to minimise taxes and maximise returns and for cost containment, both transition and management.

Thanks to the model's views, preferences were given to the portfolio's assets, specifically a greater preference for the USA and a lower preference for Europe, Emerging Markets, Japan, respectively, and an annual +2% for gold, as covering the inflation expectations. This made it possible to set up the quarterly rebalancing cases. In the event of a simultaneous loss on ETFs with low views and capital gains on gold, the model proceeded to sell and offset the portfolios' quantities, shifting the amount of money obtained towards ETFs with more positive views.

The model invests 100,000.00 € starting from January 3rd, 2010 in conclusion at the end of the period: January 27th, 2020, the final amount achieved with this method was 153,634.14 €, against 151,568.00 € that would have been obtained with the same initial portfolio without carrying out rebalancing, and 142,201.61 € for the last portfolio without rebalance and without gold. So, the method seems to have generated good results.

The Table 14 shows the results in terms of the total final amount obtained, the standard deviation and the Sharpe ratio of the portfolios.

Table 14. Final results.

	Rebalance	No rebalance	No Rebalance and No Gold
Final amount	153,634.14 €	151,568.00 €	142,201.61 €
Standard deviation	2.358%	2.327%	2.339%
Sharpe Ratio	0.54135	0.532765	0.51574

The motivation for this result could be that the USA, which has the most positive views, and to which the money is therefore shifted, has the highest overall return in the historical series that have been used. So, briefly, the procedure shifts money from a low-yielding asset to a higher-yielding asset, increasing the portfolio's final performance.

However, by comparing the volatility of portfolios, it can be seen how the presence of gold contributes to increasing diversification and also contributes to improving the Sharpe ratio's overall performance.

A mix of different assets was used to achieve this result, including gold, which, with its characteristics, allows generating a more efficient portfolio. The same result in terms of risk-return would not have been obtained by inserting only the USA in the portfolio because with the highest historical return, obtaining, in this case, a significant increase in risk.

The tax issue is, as seen, an essential element to take into account for an investment, and its minimisation or optimisation must undoubtedly be taken into account.

I think that there are two different in which develop the work.

The first is to insert an asset in the portfolio two assets that are *negatively correlated*, instead of that one safe asset as gold, to balance them, to see if the negative correlation works better than a safe asset. The mix of the two assets reduces the portfolio volatility, and their potential compensation could increase the portfolio's overall return.

The second is the possibility to use a different type of assets for whom capital gain is classified as Redditi diversi. This work is based on an ETC on gold, but also stocks and bonds could work. Obviously, it is more complicated to value a stock or a bond then gold, both in terms of correlation with other assets and in terms of expected value in the long run. Moreover, the bonds seem to have lost their negative correlation with stocks, that for a long period, it has played a crucial role in investors' portfolio risk reduction.

5 Appendix – Matlab code

```
close all hidden
clear all
clc

T = readtable('Prezzi.xlsx');

%% Define the Asset name, and the benchmark (IMOAX)
assetNames = ["USA", "EM", "EWJ", "EU", "GOLD"];
% 4 ETF, the benchamrks are:
% USA = SP500
% EM = MSCI Emerging Markets
% EWJ = MSCI Japan
% EU = S&P Europe 350, UK and Switzerland included
% GOLD = Gold price

benchmarkName = "IMOAX";

% IMOAX is a well diversified mutual found that invest both in stocks and
% bond, that will serve as benchmark
head(T(:,["Dates" benchmarkName assetNames]))

%% Compute the log returns

retnsT = tick2ret(T(:, 2:end));
assetRetns = retnsT(:, assetNames);
benchRetn = retnsT(:, "IMOAX");
numAssets = size(assetRetns, 2);

%% Introduce the views
v = 5;  % total 5 views
```

```
P = zeros(v, numAssets);
q = zeros(v, 1);
Omega = zeros(v);

% View 1
% Absolute view
P(1, assetNames=="USA") = 1; % Absolute view
q(1) = 0.06; %USA will performe 6% annual
Omega(1, 1) = 1e-4;

% View 2
% Absolute view
P(2, assetNames=="EM") = 1; % Absolute view
q(2) = 0.05; %EM will performe 5% annual
Omega(2, 2) = 1e-3;

% View 3
% Absolute view
P(3, assetNames=="EU") = 1; % Absolute view
q(3) = 0.055; %EU will performe 5.5% annual
Omega(3, 3) = 1e-3;

% View 4
% Absolute view
P(4, assetNames=="EWJ") = 1; % Absolute view
q(4) = 0.045; %Jap will performe 4.5% annual
Omega(4, 4) = 1e-3;

% View 5
% Absolute view
P(4, assetNames=="GOLD") = 1; % Absolute view
q(4) = 0.02; %GOLD will performe 2% annual
Omega(4, 4) = 1e-4;
```

```
%% Table of the views
array2table([P q diag(Omega)], 'VariableNames', [assetNames "View_Return"
"View_Uncertainty"])


%% Convert the annual views into daily views
bizyear2bizday = 1/252;
q = q*bizyear2bizday;
Omega = Omega*bizyear2bizday;

%% The covariance of the historial asset return
Sigma = cov(assetRetns.Variables);

%% Define C (the uncertainty, that depend on tau)
tau = 1/size(assetRetns.Variables, 1);
C = tau*Sigma;

%% The function regress the benchmark on the assets, to find a market portfolio
[wtsMarket, PI] = findMarketPortfolioAndImpliedReturn(assetRetns.Variables,
benchRetn.Variables);

%% Estimated mean returns and the covariance
mu_bl = (P'*(Omega\P) + inv(C)) \ ( C\PI + P'*(Omega\q));
cov_mu = inv(P'*(Omega\P) + inv(C));

%% The prior belief of the expected return and the Black Littermna returns
table(assetNames', PI*252, mu_bl*252, 'VariableNames', ["Asset_Name", ...
    "Prior_Belief_of_Expected_Return",
"Black_Litterman_Blended_Expected_Return"])

%% The final portfolio, compared to the Markowitz one
port = Portfolio('NumAssets', numAssets, 'lb', 0, 'budget', 1, 'Name', 'Mean Variance');
port = setAssetMoments(port, mean(assetRetns.Variables), Sigma);
wts = estimateMaxSharpeRatio(port);
```

```
portBL = Portfolio('NumAssets', numAssets, 'lb', 0, 'budget', 1, 'Name', 'Mean Variance with Black-Litterman');
portBL = setAssetMoments(portBL, mu_bl, Sigma + cov_mu);
wtsBL = estimateMaxSharpeRatio(portBL);

ax1 = subplot(1,2,1);
idx = wts>0.001;
pie(ax1, wts(idx), assetNames(idx));
title(ax1, port.Name ,'Position', [-0.05, 1.6, 0]);

ax2 = subplot(1,2,2);
idx_BL = wtsBL>0.001;
pie(ax2, wtsBL(idx_BL), assetNames(idx_BL));
title(ax2, portBL.Name ,'Position', [-0.05, 1.6, 0]);

%Mean-Variance portfolio represents the Markowitz portfolio's weights
%Mean-Variance portfolio represents the solution with the expected returns with the investor's views

%% The 2 portfolios graphically
table(assetNames', wts, wtsBL, 'VariableNames', ["AssetName", "Mean_Variance", ...
    "Mean_Variance_with_Black_Litterman"]) %we see that BL portfolio avoid the concentration in only 2 assets
```

```
function [wtsMarket, PI] = findMarketPortfolioAndImpliedReturn(assetRetn,
benchRetn)
% Local function
    Sigma = cov(assetRetn);
    numAssets = size(assetRetn,2);
    LB = zeros(1,numAssets);
    Aeq = ones(1,numAssets);
    Beq = 1;
    opts = optimoptions('lsqlin','Algorithm','interior-point', 'Display',"off");
    wtsMarket = lsqlin(assetRetn, benchRetn, [], [], Aeq, Beq, LB, [], [], opts);
    shpr = mean(benchRetn)/std(benchRetn);
    delta = shpr/sqrt(wtsMarket'*Sigma*wtsMarket);
    PI = delta*Sigma*wtsMarket;
end
```

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8 Figures and Tables