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# Final Thesis

**A Univariate Volatility Modelling combining the GARCH model with the Volatility Index VIX**


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"We all know that art is not truth. Art is a lie that makes us realize truth, at least the truth that is given us to understand. The artist must know the manner whereby to convince others of the truthfulness of his lies". Pablo Picasso. 1923

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**Abstract**

In this thesis I shall examine the characteristics of the volatility in the financial markets. In particular, the volatility is extrapolated both from the historical volatility by time series of past market prices and from derivative instruments providing an implied volatility. The first part explores the causes of volatility, especially volatility clustering, and explain the behavioural reactions of the stockholders. It is a well-known fact that there are GARCH models and many others that are accurate and useful to estimate the conditional variance. Anyway, looking the historical returns could be not be enough to fit the model on the data. Our purpose is to create a non-linear Univariate model to evaluate the financial markets using the Generalised Autoregressive Conditional Heteroskedasticity (GARCH) model with the CBOE Volatility Index (VIX) as a exogenous variable. The exogenous variable VIX is independent of the GARCH model but is included in the new model we want to realize. Using the daily rates of return of 10 major indices, we want to determine if the new model created, adding an exogenous variable, is better than the single GARCH model. Therefore, the empirical analysis analyses the volatility implementing the GARCH with the exogenous implied volatility, determined look forward in time, being derived from the market price of a market-traded derivative. It is using the Variance Swaps, based on the S&P 500 Index, the core index for U.S. equities, and estimates expected volatility by aggregating the weighted prices of S&P 500 puts and call plain vanilla options over a wide range of strike prices. By empirically examining the time series of different world indices we hope to produce a more complete understanding of the utility of the VIX into the GARCH models.

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# 1 Introduction and brief history

The volatility is defined by the standard deviation of the trading price series over time and is essentially a measure of deviation from the expected value. It is a symmetric measure, which consider both positive and negative deviation. We can calculate the volatility using the historical volatility, which measures a time series of past market prices, or with the implied volatility, using a derivative instrument.

The objective of this thesis is to study a method for modelling the volatility of some indices returns. A peculiar characteristic of stock volatility is that is not directly observable, but can be estimated from historic data, or using an option pricing model. From the returns is not possible to observe the daily volatility because there is only one observation in a trading day but is possible to estimate the daily volatility using the intraday data of the stock. Obviously, the high-frequency intraday returns contain only very limited information about the overnight volatility. The returns can be analysed using different time orders like years, months, days, hours or minutes but each one have some problems and choose the accurate one depends of the model. There are various characteristics that are commonly seen in asset returns. The most important is the volatility clustering where the volatility may be high for certain time periods and low for other periods; but the volatility follows also other properties such as the volatility evolves over time with a continuous behavior, but does not diverge to infinity. We notice that there is also a leverage effect, meaning the volatility react differently to a big price increase or a big price drop.

In option markets, if we consider that the prices are governed by an econometric model such as an option pricing model by using Black-Scholes formula, then we can use the price to obtain the implied volatility. The implied volatility is derived under the assumption that the price of the underlying asset follows a Geometric Brownian Motion and that might be different from the actual volatility. The volatility index (VIX) of a market has become a financial instrument. The VIX volatility index compiled by the Chicago Board of Option Exchange (CBOE) is an implied volatility, started to trade in futures on March 26, 2004.[1] One characteristic of financial time series is that they suffer from heteroskedasticity, but we can solve this problem using the ARCH (Autoregressive Conditional Heteroskedasticity) model which capture this characteristic of heteroskedasticity by conditioning variance. The implementation of the ARCH model is the GARCH (generalized ARCH) providing a better model, more complex but also more parsimonious. The GARCH model is a non linear statistical model for time series that describes the variance of the current term or innovation as a function of the actual sizes of the previous time periods error terms. It is widely used in modelling and forecasting volatility. The GARCH model depends only on information available on the time series itself and not on any exogenous infor-

[1] See, Ruey S. Tsay. (2010).[1]

mation.[2]. The idea of the new model is to use the GARCH model with the implied volatility as a exogenous variable. The use of the VIX with the GARCH is because the volatility index, seen as a general fear index for a stock market, has relevant information due to globalization and interdependence. Intuitively it is reasonable to consider VIX as an additional variable in our forecast exercise. This is because VIX is defined as a benchmark of expected short-term market volatility and provides a forward-looking measure of volatility. The research and the analysis of the relationship between the volatility of prices and their returns cause many debates between Econometrics, Finance and Statistics Scholars. The uncertainty of the future price is hard to forecast and the better forecast of the future is analysing the past time series. Anyway, only an analysis of the past time series is not enough to forecast the future price because there are many others variables that influence the market like the irrationality of the financial market participants, macro and micro economics data and the financial market as a whole.

[2]See, Vega Ezpeleta. (2015).[2]

# 2 Chapter 1: The structure of the model

## 2.1 Time Series

Financial data are usually presented as a set of observations $y_1, y_2, ..., y_i$, ..., $y_n$ where $i$ is a time index. This kind of data are usually referred as a time series.[3] The analysis of time series is essential when we fit a model that explain how the observed data works. The time series is defined as the realization of a finite stochastic process, therefore a sequence of observations of some quantity or quantities taken over time. When we observe a time series, the fluctuations appear random, but often with the same type of stochastic behaviour from one time period to the next. One of the most useful methods for obtaining parsimony in a time series model is to assume stationarity. Weak stationarity means that mean, variance, and covariance are unchanged by time shifts. Thus, the mean and variance do not change with time and the correlation between two observations depends only on the lag, the time distance between them. A time series $\{y_t\}$ is said to be strictly stationary if the joint distribution of $(y_{t1}, ....., y_{tk})$ is identical to that of $(y_{t1+t}, ....., y_{tk+t})$ for all t, where k is an arbitrary positive integer and $(t_1, ....\ t_k)$ is a collection of k positive integers. In other words, strict stationarity requires that the joint distribution of $(y_{t1}, ...., y_{tk})$ is invariant under time shift. This is a very strong condition that is hard to verify empirically. So, usually is enough to have a weakly stationary time series when the both mean of $y_t$ and the covari-

[3]See, Pastore (2018).[3]

ance between $y_t$ and $y_{t-\iota}$ are time invariant, where $\iota$ is an arbitrary integer. Therefore $\mathrm{E}(y_t) = \mu$, which is a constant, and $\mathrm{Cov}(y_t, y_t t - \iota) = \lambda_\iota$.

When we analyse a time series, we can never be absolutely certain that it is stationary but there are many different tests to evaluate if the time series is stationary or not. Otherwise, there are model fitted to time series data either to better understand the data or to predict future points in the series. Usually, a good approach is to use the ARMA (autoregressive integrated moving average) model applying the differencing step and modifying the parameters to explain a stationary and ergodic time series. The model can be implemented with seasonality and/or with other variants. The stationarity aids to evaluate the time series because the statistical properties of the process do not change over time and consequently is much easier to model and investigate. Anyway, different models has constraints or tests to evaluate and confirm that the time series is stationary.

Many statistical models, analysing the time series, assume that a random sample comes from a normal distribution. It well known that from empirical evidences the time series follows a leptokurtic form (knowing that the time series randomly display outliers) therefore to use the normal distribution could be an erroneous fit. Therefore, is necessary to investigate how the distribution of the data differs from a normal distribution and evaluate it.[4]

[4]See, Ruppert D., Matteson D.(2015).[4]

## 2.2 Volatility

Financial markets can move quite dramatically both up and down and the stock prices may appear too volatile to be justified by changes in fundamentals. Volatility as a concept as well as a phenomenon remain a hot topic to modern financial markets and academic research. Justified volatility can form the basis for efficient price discovery, as the relationship between risk and return, while volatility dependence implies predictability, which is welcomed by traders and medium-term investors. Equilibrium prices, obtained from asset pricing models (CAPM), are affected by changes in volatility, investment management lies upon the mean-variance theory, while derivatives valuation joints upon reliable volatility forecasts. The stockholders as the portfolio managers, risk arbitrageurs and corporate treasurers closely watch volatility trends, as changes in prices could have a major impact on their investment and risk management decisions. the information are asymmetric on conditional volatility and early evidences uveils that bad news in the Futures market increase volatility in the cash markets more than good news. The asymmetry is documented as "Leverage effect". The leverage hypothesis is not the only force behind asymmetries but many others may well contribute to the rise of asymmetries as noise trading, high-frequency trading, flash crash and irrational behaviour.

Tipically, as a measure of volatility is used the standard deviation of

[5] See, J. M. Bland, D. G. Altman (1996).[23]

the returns between daily close prices of instruments. [5] In the mathematical approach the standard deviation is simply the square root of the variance. The square root of the variance is easier to interpret because is restored to the original data, while the variance is no longer in the same unit of measurement as the original data. Therefore, the standard deviation is a measure of the amount of variation or dispersion of a set of values. A high standard deviation indicates than the values are spread out over a wide range increasing the risk, while a low standard deviation means that the values tend to be close to the mean of the set. The standard deviation is often used as a measure of risk associated with the fluctuations of the price of a given asset or, more globally, the risk of a portfolio assets. Portfolio managers evaluate accurately the risk because is an important factor in determining how the efficiently manage a portfolio of invetsments. Analyze the risk determines the variation in returns on an asset or portfolio and gives stockholders a mathematical basis for investment decisions. As the capital asset pricing model, if the risk increases, the expected return on an investment should increase as well, an increase known as the risk premium. So, if the investment carries a higher level of risk or uncertainty stockholders expected an higher return. In substance, the uncertainty is compensated with the return. When evaluating investments, stockholders should estimate both the expected return and the uncertainty of future returns. Standard deviation provides a quantified estimate of the uncertainty of future returns.

Let X be a random variable with mean value $\mu$ :

$$E[X]=\mu \tag{1}$$

Here the operator $E$ denotes the average of expected value of $X$. Then the standard deviation of $X$ is the quantity:

$$\sigma = \sqrt{E[(X-\mu)^2]} \tag{2}$$

$$= \sqrt{E[X^2]+E[-2\mu X]+E[\mu^2]} \tag{3}$$

$$= \sqrt{E[X^2]-2\mu E[X]+\mu^2} \tag{4}$$

$$= \sqrt{E[X^2]-2\mu^2+\mu^2} \tag{5}$$

$$= \sqrt{E[X^2]-\mu^2} \tag{6}$$

$$= \sqrt{E[X^2]-(E[X])^2} \tag{7}$$

In other words, the standard deviation $\sigma$ (sigma) is the square root of the variance of X. The standard deviation of a univariate probability distribution is the same as that of a random variable having that distribution. Anyway, not all random variables have a standard deviation, since these expected values need not exist.

### 2.3 Volatility Clusters

As documented in Bollerslev (1987)[6] the general conclusion to emerge from most of many studies is that price changes and rates of return are approximately uncorrelated over time and well described by a unimodal symmetric distribution with fatter tails than the normal, However, even though the time series are serially uncorrelated, they are not independent. As noted by Manderlbrot[7]Hinich and Patterson[8],

*"..., large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes,..."*

This behaviour might very well explain the rejection of an independent increments process for daily stock returns in Hinich and Patterson.

The model of this thesis works when there is volatility clustering therefore when there is period of higher, and of lower, variation within each series. Volatility clustering does not indicate a lack of stationarity but rather can be viewed as a type of dependence in the conditional variance of each series. As such, the variance of daily returns can be high one month and show low variance the next. This occurs to such a degree that it makes and i.i.d. (independent and identically distributed) model of log-prices or asset returns unconvincing. The market respond

[6]See, Bollerslev(1987).[5]
[7]See, Manderlbrot(1963).[6]
[8]See, Hinich and Patterson(1985).[7]

to new information with large price movements, these high-volatility environments tend to ensure for a while after that first shock. In other words, when a market suffers a volatile shock, more volatility should be expected. The volatility clustering is a non-parametric property meaning that there is not a request that the data being analyzed meet certain assumptions, or parameters.[9] A standard way to remove volatility clusters is by modeling returns with a GARCH(1,1) specification, model delevoped in Bollersvel (1986).[10] The conditional error distribution is normal, but with conditional variance equal to a linear function of past squared errors. Thus, there is a tendency for extreme values to be followed by other extreme values, but of unpredictable sign.

### 2.4 Why does volatility change?

The models focus on providing a statistical description of the time-variation of volatility, but does not go into depth on why volatility varies over time. A number of explanations have been proffered to explain this phenomenon, although treated individually, none are completely satisfactory.

- News Announcements: The arrival of unexpected news forces stockholders to update beliefs, changing the strategies of the investment or modifying the weight of the asset portfolio. Many institutional investors rebalance his portfolios and high periods of volatility cor-

[9] See, Rama Cont(2005).[8]
[10] See, Bollerslev(1986).[9]

respond to stockholders dynamically solving for new asset prices. Additionally, news-induced periods of high volatility are generally short and the apparent resolution of uncertainty is far too quick to explain the time-variation of volatility seen in asset prices.

- Leverage: When firms are financed using both debt and equity, only the equity will reflect the volatility of the firms cash flows. However, as the price of equity falls, a smaller quantity must reflect the same volatility of the firm's cash flows and so negative returns should lead to increases in equity volatility.

- Volatility Feedback: Volatility feedback is motivated by a model where the volatility of an asset is priced. When the price of an asset falls, the volatility must increase to reflect the increased expected return (in the future) of this asset, and an increase in volatility requires an even lower price to generate a sufficient return to compensate an investor for holding a volatile asset.

- Illiquidity: Intuitively, if the market is oversold, a small negative shock will cause a small decrease8 in demand. However, since there are few participants willing to buy (sell), this shock has a large effect on prices. If the market is overbought, a small positive shock will cause a small decrease in demand but also here there will be a large effect on prices if there are few participants.

- State Uncertainty: When the state is uncertain, like the actual situation due to Covid-19, slight changes in beliefs may cause large shifts in portfolio holdings which in turn feedback into beliefs about the state.

The actual cause of the time-variation in volatility is likely a combination of these and some not present.[11]

## 2.5 Univariate Models

In Univariate models the volatility is a scalar ($h_t$) because it works only with an unique variable, while in the multivariate models is represented with a symmetric square matrix, positively semi-defined with a dimension equal to the variables analyzed. Although a univariate time series data set is usually given as a single column of numbers, time is in fact an implicit variable in the time series. The market volatility is a latent variable meaning that it is not directly observable, unlike market price. If prices fluctuate a lot, we know volatility is high, but we cannot ascertain precisely how high. Therefore the volatility must be forecast by a statistical model, a process that inevitably entail making strong assumptions. Indeed, volatility modelling is quite demanding, and often seems to be as much an art as a science because of challenges posed by the presence of issues such a nonnormalities, volatility clusters and structural breaks.

The presence of volatility clusters suggests that it may be more effi-

[11] See, Sheppard(2020)[10]

cient to use only the most recent observations to forecast volatility, or assign a higher weight to the most recent observations.

The most commonly used models are:

1. Moving average (MA).
2. Exponentially weighted moving average (EWMA).
3. GARCH and its extension models.
4. Stochastic volatility.
5. Implied volatility.
6. Realized volatility.

Our model is an hybrid model that combine the GARCH including the implied volatility using the VIX Index as a exogenous variable. We usually assume the mean return is zero. While this is obviously not correct, the daily mean is orders of magnitude smaller than volatility and therefore can usually be safely ignored for the purpose of volatility forecasting. Conditional volatility, $\sigma_t$, is typically, but not always, obtained from application of a statistical procedure to a sample of previous return observations, making up the estimation windows. Such methodologies provide conditional volatility forecasts, represented by:

$\sigma_t \mid \text{past returns and a model} = \sigma(y_{t-1}, \ldots, y_{t-W_E})$

where various methods are used to specify the function $\sigma(\cdot)$. [12]

[12] See, Danielsson(2011).[11]

### 2.6 Moving Average models

The most obvious and easy way to forecast volatility is simply to calculate the sample standard error from a sample of returns. Over time, we would keep the sample size constant, and every day add the newest return to the sample and drop the oldest. This method is called the Moving Average (MA) model. The observations are equally weighted, which is problematic when financial returns exhibit volatility clusters, since the most recent data are more indicative of whether we are in a high-volatility or low-volatility cluster.

### 2.7 EWMA model

The MA model can be improved by exponentially weighting returns, so that the most recent returns have the biggest weight in forecasting volatility. The best known such model is the exponentially weighted moving average (EWMA) model.

$$\hat{\sigma}_t^2 = (1-\lambda) y_{t-1}^2 + \lambda \hat{\sigma}_{t-1}^2 \tag{8}$$

where $0 < \lambda < 1$ is the decay factor, $\hat{\sigma}_t^2$ the conditional volatility forecast on day t. This model si not optimal because use only a $\lambda$ that is constant and identical for all assets. Obviously is not realistic that $\lambda$ is constant but can be implemented easily and using multivariate forms.[13]

[13] See, Ruppert D., Matteson D.(2015).[4]

## 2.8 The ARCH/GARCH Model

Our model use the GARCH (Generalised Autoregressive Conditional Heteroskedasticity) model with the implementation of VIX. First of all, I explain what is the GARCH model and why it is used. The model is useful for the volatility clustering, therefore when the market goes through periods of high volatility and other periods when volatility is low. The variance of this model is not constant and use the conditional volatility, defined as volatility in a given time period, conditional on what happened before. These models are based on using optimal exponential weighting on historical returns to obtain a volatility forecast. The first such model was the Autoregressive Conditional Heteroskedasticity (ARCH) proposed by Engle (1982) [14] but the generalized ARCH model (GARCH) by Bollerslev(1986)[9] is the common denominator for most volatility models. The GARCH framework builds on the notion of volatility dependence to measure the impact of last period's forecast error and volatility in determining current volatility. Returns on day t are a function of returns on previous days, where older returns have a lower weight than more recent returns. The parameters of the model are typically estimated with the maximum likelihood. We want to study the statistical properties of returns given information available at time t-1 and create a model of how statistical properties or returns evolve over time.

The principle of the model is the conditional volatility of the ran-

[14] see, Engle(1982).[13]

dom variables $Y_t$ and is useful separate estimation of the mean from the volatility estimation to use a more efficient model. So, the $\mathrm{E}(Y_t)$ is equal to zero, called de-meaned. The return on day t can be indicated by

$$Y_t = \sigma_t \epsilon_t \qquad \epsilon_t \sim N(\mu, \sigma_t^2) \tag{9}$$

In this case, the distribution of $\epsilon_t$ is normal but there is no need to make any further assumptions about the distribution because is possible change the distribution that fit the model, like the Student-t.

#### 2.8.1 ARCH Model

The ARCH model is a Autoregressive Conditional Heteroskedasticity model therefore the variance change over time in a time series.

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i Y_{t-i}^2 \tag{10}$$

where p is the number of lags. Setting the lag to one in the formula will result in the ARCH(1) model which states that the conditional variance of today's return is equal to a constant, plus yesterday's return squared; that is:

$$\sigma_t^2 = \omega + \alpha Y_{t-1}^2 \tag{11}$$

The unconditional volatility of the ARCH(1) model is given by:

$$\sigma_t^2 = \frac{\omega}{1-\alpha} \tag{12}$$

The most common distributional assumption for residuals $\epsilon_t \sim N(0,1)$ In this case, conditional returns are conditionally normal. However, the unconditional distribution of the return will be fat, easily demonstrated by showing that unconditional excess kurtosis exceeds zero:

$$Kurtosis = \frac{E(Y^4)}{(E(Y^2))^2} \tag{13}$$

where, at the end, the unconditional kurtosis is:

$$Kurtosis = \frac{3(1-\alpha^2)}{1-3\alpha^2} > 3 \qquad if \quad 3\alpha^2 < 1. \tag{14}$$

There are two main restrictions that are often imposed on the parameters of the ARCH model:

To ensure positive volatility forecasts for the ARCH:

$$\forall i = 1, \ldots\ldots, p, \quad \alpha_i, \omega > 0 \tag{15}$$

To ensure covariance stationarity so that unconditional volatility is defines, impose:

$$\sum_{j=1}^{p} \alpha_i < 1. \tag{16}$$

It is only the nonnegativity constraint that always has to be imposed and, depending on the final application, we may or may not want to impose covariance stationarity. In case of the ARCH(1) model, if $\alpha \geq 1$ the unconditional volatility is no longer defined, as is clear from the unconditional volatility of ARCH(1).

One of the biggest problems with the ARCH model concerns the long lag lengths required to capture the impact of historical returns on current volatility. By including lagged volatility during ARCH model creation, it has the potential to incorporate the impact of historical returns. The result is a GARCH model.

#### 2.8.2 The GARCH model

Therefore, the GARCH (p,q) model is:

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i Y_{t-i}^2 + \sum_{j=1}^{q} \beta \sigma_{t-j}^2 \tag{17}$$

where p,q are the numbers of lags. Setting the lag to one means that the model is the GARCH(1,1). The most common version of the GARCH is the GARCH (1,1) and Akgiray (1989)[15] demonstrated that a GARCH(1,1) model is sufficient to capture all volatility clustering.

The unconditional volatility of the GARCH(1,1) is given by:

$$\sigma_t^2 = E(\omega + \alpha Y_{t-1}^2 + \beta \sigma_{t-1}^2) = \omega + \alpha \sigma^2 + \beta \sigma^2 \tag{18}$$

[15] See, Akgiray (1989).[14]

where

$$\sigma_t^2 = \omega + \alpha\sigma^2 + \beta\sigma^2 \tag{19}$$

So,

$$\sigma_t^2 = \frac{\omega}{1-\alpha-\beta} \tag{20}$$

There are two main restrictions that are often imposed on the parameters of the GARCH model and are:

- to ensure positive volatility forecasts $\omega, \alpha, \beta > 0$

- to ensure covariance stationarity $\alpha + \omega < 1$

Therefore, unconditional variance is infinite when $\alpha + \beta = 1$ and undefined when $\alpha + \beta > 1$. We should not impose the constraint when all we need is a forecast of conditional volatility, but it is necessary to predict unconditional volatility.

## 2.9 Stochastic Volatility

The stochastic volatility models are those in which the variance of a stochastic process is itself randomly distributed. The name derives from the model's treatment of the underlying security's volatility as a random process, governed by state variables such as the price level of the underlying security, the tendency of volatility to revert to some long-run mean value, and the variance of the volatility process itself, among others. The volatility process is a function of an exogenous shock as well as past volatilities, so the process $\sigma_t$ is itself random, with an innovation term that is not known at time t. However, these models cannot explain long-observed features of the implied volatility surface such as volatility smile and skew, which indicate that implied volatility does tend to vary with respect to strike price and expiry. [16]

## 2.10 The VIX, the Index of the implied volatility

In 1993, CBOE Global Markets, Incorporated introduced the CBOE Volatility Index (VIX), which was originally designed to measure the market's expectation of 30-day volatility implied by at-the money S&P 100 index option prices. in 2003, CBOE together with Goldman Sachs, updated the VIX index to reflect a new way to measure expected volatility, one that continues to be widely used by financial theorists, risk managers and volatility traders alike. The new VIX index is based on the S&P 500 Index, the core index for U.S. equities, and estimates expected volatility by aggregating the weighted prices of S&P 500 puts and calls over a wide range of strike prices. In 2014, Cboe enhanced the VIX index to include series of S&P 500 Weekly's. It allows the VIX index to be calculated with S&P 500 index option series that most precisely match the 30-day target timeframe for expected volatility that the VIX index is intended to represent. This extensive data set provides investors with a useful perspective of how option prices have behaved in response to a variety of market conditions.

[16] See, Gatheral (2006).[12]

The VIX is a volatility index comprised of options rather than stocks, with the price of each option reflecting the market's expectation of future volatility. Like conventional indices, the VIX Index calculation employs rules for selecting component options and a formula to calculate index values. Some different rules and procedures apply when calculating the VIX index value to be used for the final settlement value of VIX futures and options.

More specifically, the VIX index is intended to provide an instantaneous measure of how much the market expects the S&P 500 Index will fluctuate in the 30 days from the time of each tick of the VIX Index.

Intraday VIX Index values are based on snapshots of SPX option bid/ask quotes every 15 seconds and are intended to provide an indication of the fair market price of expected volatility at particular points in time. VIX is the most popular indicator of volatility and has been regarded as the world's premier barometer of investors' sentiment and market volatility. [17] Implied volatility, as measured by VIX, reflects expectations, hence its market name as "the fear gauge". In general, VIX, starts to rise during time of financial stress and lessens as investors become complacent. Thus, the greater the fear, the higher the volatility index would be.

[17] see CBOE VIX (2019).[15]

#### 2.10.1 VXO

The first VIX was renamed VXO, the old VIX index based on the Black-Scholes-Merton implied volatility of S&P 100 options. To construct the old VIX, two puts and two calls for strikes immediately above and below the current index are chosen. Near maturities (greater than eight days) and second nearby maturities are chosen to achieve a complete set of eight options. By inverting the BSM pricing formula using current market prices, an implied volatility is found for each of the eight options. An iterative search procedure can be used to find the implied $\sigma$. These volatilities are then averaged, first the puts and the calls, then the high and low strikes. Finally, an interpolation between maturities is done to compute a 30 calendar day (22 trading day) implied volatility. Because the BSM model assumes the index follow a geometric Brownian motion with constant volatility, when in fact it does not, the old VIX will only approximate the true risk-neutral implied volatility over the coming month. In reality the price process is likely more complicated than geometric Brownian motion.[18]

The Black-Scholes-Merton model and its extensions assume that the probability distribution of the underlying asset at any given future time is lognormal. This assumption is not the one made by traders. They assume the probability distribution of an equity price has an heavier left tail and a less heavy right tail than the lognormal distribution. Traders

[18] See, McAleer (2007).[24]

use volatility smiles to allow for non-lognormality. The volatility smile defines the relationship between the implied volatility of an option and its strike price. [19]

The implied volatility used to create the VXO are as follow:

| Exercise price | Nearby contract(1) | | second Nearby contract(2) | |
|---|---|---|---|---|
| | Call | Put | Call | Put |
| $X_l(< S)$ | $\sigma_{c,1}^{X_l}$ | $\sigma_{p,1}^{X_l}$ | $\sigma_{c,2}^{X_l}$ | $\sigma_{p,1}^{X_l}$ |
| $X_u(> S)$ | $\sigma_{c,1}^{X_u}$ | $\sigma_{p,1}^{X_u}$ | $\sigma_{c,2}^{X_u}$ | $\sigma_{p,2}^{X_u}$ |

where the $X_l$ is the strike below the price of the current index, S is the share price of the index at maturity (higher than 8 days) and the $X_u$ is the strike above the price of the current index. In the table there are their relative options.

The first step is to average the put and call implied volatilitIes for each strike and maturity to reduce the number of volatilitIes to 4. Compute:

$$\sigma_1^{X_l} = \frac{(\sigma_{c,1}^{X_l} + \sigma_{p,1}^{X_l})}{2}; \quad \sigma_1^{X_u} = \frac{(\sigma_{c,1}^{X_u} + \sigma_{p,1}^{X_u})}{2}; \quad \sigma_2^{X_l} = \frac{(\sigma_{c,2}^{X_l} + \sigma_{p,2}^{X_l})}{2}; \quad \sigma_2^{X_u} = \frac{(\sigma_{c,2}^{X_u} + \sigma_{p,2}^{X_u})}{2} \tag{21}$$

Now average the implied volatility above and below the index level as

[19]Hull (2018).[16]

follows:

$$\sigma_1 = \sigma_1^{X_l}\left(\frac{X_u - S}{X_u - X_l}\right) + \sigma_1^{X_u}\left(\frac{S - X_l}{X_u - X_l}\right); \qquad \sigma_2 = \sigma_2^{X_l}\left(\frac{X_u - S}{X_u - X_l}\right) + \sigma_2^{X_u}\left(\frac{S - X_l}{X_u - X_l}\right) \tag{22}$$

The final step is calculating the VXO is to interpolate between the two maturities to create a 30 calendar day (22 trading day) implied volatility index.

$$VXO = VIX_{old} = \sigma_1\left(\frac{N_{t2} - 22}{N_{t2} - N_{t1}}\right) + \sigma_2\left(\frac{22 - N_{t2}}{N_{t2} - N_{t1}}\right) \tag{23}$$

where $N_t1$ and $N_t2$ are the number of trading days to maturity of the two contracts. [20]

#### 2.10.2 VIX as a Variance Swaps

The new VIX is coNstructed as a Variance Swaps. Three important changes are being made to update and improve VIX. The new VIX is calculated using a wide range of strike prices in order to incorporate information from the volatility skew. The VXO used only at-the money options. The new VIX uses a newly developed formula to derive expected volatility directly from the prices of a weighted strip of options. The VXO extracted implied volatility from an option-pricing model (BSM model). The new VIX uses options on the S&P 500 Index, which is the primary U.S. stock market benchmark. So, the new VIX provide a more precise

[20] see, HaoNewVix, Hao Zhou and Matthew Chesnes. (2003).[19]

and robust measure of expected market volatility and to create a viable underlying index for tradable volatility products. The VIX is more practical and simpler because it uses a formula that derives the market expectation of volatility directly from index option prices rather than an algorithm that involves backing implied volatility out of an option-pricing model.

We compute the VIX starting from a Markov process. A Markov process is a particular type of stochastic process where only the actual value of a variable is relevant to forecasts the future. The past history of the variable and the way that the present has emerged from the past is irrelevant. Predictions for the future are uncertain and must be expressed in terms of probability distributions. The Markov property implies that the probability distribution of the price at any particular future time is not dependent on the particular path followed by the price in the past. We consider a particular type of Markov stochastic process, the Wiener process, with a mean change of zero and a variance rate of 1 per year expressed formally, a variable z follows a Wiener process if it has the following two properties:

Property 1. The change $\Delta z = \epsilon \sqrt{\Delta t}$ where $\epsilon$ has a standard normal distribution $\phi(0,1)$.

Property 2. The values of $\Delta z$ for any two different short intervals of time, $\Delta t$, are indipendent.

It follows from the first property that $\Delta z$ itself has a normal distribution with

mean = 0

standard deviation = $\sqrt{\Delta t}$

The second property implies that z follows a Markov process.

Consider the change in the value of z during a relatively long period of time, T. This can be denoted by z(T) - z(0). It can be regarded as the sum of the changes in z in N small time intervals of length Δt, where:

$$N = \frac{T}{\Delta t}$$

Thus,

$$z(T) - z(0) = \sum_{i=1}^{N} \epsilon_i \sqrt{\Delta t} \tag{24}$$

where the $\epsilon$ (i= 1,2,....N) are distributed $\phi(0,1)$. We know from the second property of Wiener processes that the $\epsilon_i$ are independent of each other. It follows that z(T) -z(0) is normally distributed, with mean equal to zero and standard deviation $\sqrt{T}$.

The d means that the value y in Δ is in the limit as Δ y → 0.

The mean change per unit time for a stochastic process is known as the drift rate and the variance per unit time is known as the variance rate. So, a generalized Wiener process for a variable x can be defined in terms on $dz$ as

$$dx = a\,dt + b\,dz \tag{25}$$

where a and b are constants.

The $a\,dt$ term implies that x has an expected drift rate of $a$ per unit of time. The $b\,dz$ term can be regarded as adding noise or variability to the path followed by $x$. The amount of this noise or variability is $b$ times a Wiener process. A Wiener process has a variance rate per unit time of 1.0. It follows that $b$ times a Wiener process has a variance rate per unit time of $b^2$.[21]

In a small time interval Δt, the change Δx in the value of $x$ is given as

$$\Delta x = a\Delta t + b\epsilon\sqrt{\Delta t} \tag{26}$$

where, as before, $\epsilon$ has a standard normal distribution $\phi(0,1)$. Thus Δx has a normal distribution with mean = $a$ Δt and standard deviation = $b\ \sqrt{\Delta t}$.

Similar arguments to those given for a Wiener process show that the change in the value of x in any time interval T is normally distributed with mean = $aT$ standard deviation =$b\ \sqrt{T}$.

To summarize, the generalized Wiener process has an expected drift rate of $a$ and a variance rate of $b^2$.

Now, we discuss the stochastic process usually assumed for the price of a non-dividend-paying stock. The stock price follows a generalized

[21] See, Reto R. Gallati, (2003).[22]

Wiener process but the assumption of constant expected drift rate is inappropriate and needs to be replaced by the assumption that the expected return is constant. If S is the stock price at time t, then the expected drift rate in S should be assumed to be $\mu$S for some constant parameter $\mu$. This means that in a short interval of time, $\Delta$t, the expected increase in S is $\mu$S$\Delta$t. The parameter $\mu$ is th expected rate of return on the stock. The uncertainty of the process, the standard deviation, should be proportional to the stock price and therefore [16]:

$$\frac{dS}{S} = \mu dt + \sigma dz \tag{27}$$

The variable $\mu$ is the stock' expected rate of return. The variable $\sigma$ is the volatility of the stock price. The model represents the stock price process in the real world.

Applying Ito's formula, it get:

$$d(lnS_t) = (\mu - \frac{\sigma^2}{2})dt + \sigma dZ_t \tag{28}$$

By subtracting these two equations it obtain

$$\frac{dS_t}{S_t} - d(lnS_t) = \frac{\sigma^2}{2}dt \tag{29}$$

Integrating between time 0 and time T, the realized average variance rate, $\bar{V}$, between time 0 and time T is given by

$$\bar{V} = \frac{1}{T}\int_0^T \sigma^2 dt = \frac{2}{T}\left(\int_0^T \frac{dS_t}{S_t} - ln\left(\frac{S_t}{S_0}\right)\right) \tag{30}$$

Taking expectations in a risk-neutral world

$$\hat{E}(\bar{V}) = \frac{2}{T} ln\frac{F_0}{S_0} - \frac{2}{T}\hat{E}\left(ln\frac{S_t}{S_0}\right) \tag{31}$$

Where $F_0$ is the forward price of the asset for a contract maturing at time T.

Consider

$$\int_{K=0}^{S^*} \frac{1}{K^2} max(K - S_T, 0) dK \tag{32}$$

for some value $S^*$ of S. When $S^* < S_T$ this integral is zero. When $S^* > S_T$ it is

$$\int_{K=S_T}^{S^*} \frac{1}{K^2}(K - S_T, 0) dK = ln\frac{S^*}{S_T} + \frac{S_T}{S^*} - 1 \tag{33}$$

Consider next

$$\int_{K=S^*}^{\infty} \frac{1}{K^2} max(S_T - K, 0) dK \tag{34}$$

When $S^* > S_T$ this is zero. When $S^* < S_T$ it is

$$\int_{K=S^*}^{S^T} \frac{1}{K^2}(S_T - K, 0) dK = ln\frac{S^*}{S_T} + \frac{S_T}{S^*} - 1 \tag{35}$$

From these results it follows that

$$\int_{K=0}^{S^*} \frac{1}{K^2} max(K - S_T, 0) dK + \int_{K=S^*}^{\infty} \frac{1}{K^2} max(S_T - K, 0) dK = ln\frac{S^*}{S_T} + \frac{S_T}{S^*} - 1 \tag{36}$$

for all value of $S^*$ so that

$$ln\frac{S_T}{S^*} = \frac{S_T}{S^*} - 1 - \int_{K=0}^{S^*} \frac{1}{K^2} max(K - S_T, 0) dK - \int_{K=S^*}^{\infty} \frac{1}{K^2} max(S_T - K, 0) dK \tag{37}$$

This shows that a variable that pays off in $S_T$ can be replicated using options. This result can be used in conjunction with equation $\bar{V}$ (1) to provide a replicating portfolio for $\bar{V}$.

Taking expectation in a risk-neutral world

$$\hat{E}\left(ln\frac{S_T}{S^*}\right) = \frac{F_0}{S^*} - 1 - \int_{K=0}^{S^*} \frac{1}{K^2} e^{RT} P(K) dK - \int_{K=S^*}^{\infty} \frac{1}{K^2} e^{RT} C(K) dK \tag{38}$$

where C(K) and P(K) are the prices of European call and put options with strike price K and maturity T and R is the risk-free interest rate for a maturity of T. Combining equations 31 and for 38 and noting that

$$\hat{E}\left(ln\frac{S_T}{S_0}\right) = \frac{S^*}{S_0} + \hat{E}\left(ln\frac{S_T}{S^*}\right) \tag{39}$$

$$\hat{E}(\bar{V}) = \frac{2}{T} ln \frac{F_0}{S_0} - \frac{2}{T} ln \frac{S^*}{S_0} - \frac{2}{T}\left[\frac{F_0}{S*} - 1\right] + \frac{2}{T}\left[\int_{K=0}^{S^*} \frac{1}{K^2} e^{RT} P(K) dK + \right.$$

$$\left. + \int_{K=S*}^{\infty} \frac{1}{K^2} e^{RT} C(K) dK \right] \tag{40}$$

which reduces to

$$\hat{E}(\bar{V}) = \frac{2}{T} ln \frac{F_0}{S*} - \frac{2}{T}\left[\frac{F_0}{S*} - 1\right] + \frac{2}{T}\left[\int_{K=0}^{S^*} \frac{1}{K^2} e^{RT} P(K) dK + \int_{K=S*}^{\infty} \frac{1}{K^2} e^{RT} C(K) dK \right] \tag{41}$$

This result is the Variance Swaps where we find the VIX formula. The Variance swaps offer straightforward and direct exposure to the volatility of an underlying such as a stock or index. They are swap contracts where the parties agree to exchange a pre-agreed variance level for the actual amount of variance realised over a period.

The strike of a variance swap, not to be confused with the strike of an option, represents the level of volatility bought of sold and is set at trade inception. The strike is set according to prevailing market levels so that the swap initially has zero value. If the subsequent realised volatility is above the level set by the strike, the buyer of a variance swap will be in profit; and if realised volatility is below, the buyer will be in loss. A buyer of a variance swap is therefore long volatility. Similarly, a seller of a variance swap is short volatility and profits if the level of variance sold (the variance swap strike) exceeds that realised.

The $ln$ function can be approximated by the first two terms in a series expansion:

$$\left(\frac{F_0}{S*}\right)=\left(\frac{F_0}{S*}-1\right)-\frac{1}{2}\left(\frac{F_0}{S*}-1\right)^2 \tag{42}$$

This means that the risk-neutral expected cumulative variance is calculated as

$$\hat{E}(\bar{V})T=\sigma^2=VIX^2=\frac{2}{T}\sum_i\frac{\Delta K_i}{K_i^2}e^{RT}Q_i(K_i)-\frac{1}{T}\left[\frac{F}{K_0}-1\right]^2 \tag{43}$$

Where:

$\sigma$ *100 $= VIX$

T =Time to expiration

F = Forward index level derived from index option prices $F = K_0 + e^{RT}(C_0 - P_0)$

$K_0$ = First strike below the forward index level, F

$K_i$ = Strike price of $i^{th}$ out-of-the-money option, a call if $K_i > K_0$ and a put if $K_i < K_0$, both put and call if $K_i = K_0$.

$\Delta K_i$ = interval between strikes prices -half the difference between the strike on either side of $K_i$: $\frac{K_{i+1}-K_{i-1}}{2}$

R = Risk free interest rate to expiration

Q ($K_i$) The midpoint of the bid-ask spread for each option with strike $K_i$.

So, the constant 30-day volatility index VIX:

$$VIX = 100 * \sqrt{\left\{ T_1\sigma_1^2\left(\frac{N_{T_2} - N_{30}}{N_{T_2} - N_{T_1}}\right) + T_2\sigma_2^2\left(\frac{N_{30} - N_{T_1}}{N_{T_2} - N_{T_1}}\right)\right\} * \frac{N_{365}}{N_{30}}}; \qquad (44)$$

Where $N_T$ is the number of minutes.

#### 2.10.3 Interpretation of the VIX

The VIX is listed in percentage points and represents the expected range of movement in the S&P 500 index over the next month, at a 68% confidence level (one standard deviation of the normal probability distribution). For example, if the VIX is 30, this represents an expected annualized change, with a 68% probability, of less than 30% up or down. The expected volatility range for a single month can be calculated by dividing the VIX by $\sqrt{(12)}$ which would imply a range of +/- 8.67% over the next 30-day period. Similarly, expected volatility for a week would be 30 divided by $\sqrt{(52)}$, or +/- 4.16%. The VIX uses calendar day annualization so the conversion of 30% is 30 divided by $\sqrt{(365)}$, or +/- 1.57% per day. The calendar day approach does not account for the number of when the financial markets are open in a calendar year. In the financial markets trading days typically amount to 252 days out of a given calendar year.

### 2.10.4 Alternatives of the VIX

The VIX is not the only volatility index but there are many different instrument calculated according to a variance swap stile calculation (VIX for S&P 500, VXN for the Nasdaq, VSTOXX for the Euro Stoxx 50, VDAX for DAX and VSMI for the SMI). They represent the theoretical level of a rolling 1-month (30 calendar day) maturity variance swap, based on traded option prices. In fact, theoretical variance swap levels are first calculated for listed option maturities, and then the 30-day index level is interpolated. So, each index represents the risk-neutral expected variance of the underlying over the next month.

## 2.11 Realized Volatility

Realized volatility measures what actually happened in the past and is based on taking intraday data, sampled at regular intervals (e.g., every 10 minutes), and using the data to obtain the covariance matrix. The main advantage is that it is purely data driven and there is no reliance on parametric models. The downside is that intraday data need to be available; such data are often difficult to obtain, hard to use, not very clean and frequently very expensive. In addition, it is necessary to deal with diurnal patterns in volume and volatility when using realized volatility (i.e., address systematic changes in observed trading volume and volatility throughout the day). Moreover, the particular trading platform in use is likely to impose its own patterns on the data. All these issues complicate the implementation of realized volatility models.

### 2.12 Intro of the model

The objective is computing a model that combines the Generalised Autoregressive Conditional Heteroskedasticity model (GARCH model) with the Volatility Index (VIX) as a exogenous variable. We build a model that use the historical volatility and the implied volatility using their respective models. The GARCH model is useful when there is volatility clustering and use the time series. The VIX is useful during periods of high uncertainty and the value of the Index increase rapidly. The formula of the VIX use the S&P 500 option prices with, on average, a 30 days maturity.

Now we compute the GARCH model adding as an exogenous variable the VIX and we evaluate if this model increase the performance in the empirical analysis. This model is a plain vanilla GARCH(1,1) model with the VIX(1) as an exogenous variable in the variance equation. The model is given by

$$\sigma_t^2 = \omega + \alpha Y_{t-1}^2 + \beta \sigma_{t-1}^2 + \delta VIX_{t-1} \tag{45}$$

Conditions for non-negative conditional variance are the same of the GARCH with the implementation of the parameter $\delta > 0$ that capture the effect of the exogenous variable VIX.

Therefore,

$$\alpha \geq 0; \tag{46}$$

$$\beta \geq 0; \tag{47}$$

$$\alpha + \beta < 1; \tag{48}$$

$$\omega + \delta > 0 \tag{49}$$

So, like in the GARCH model, all variables must be positive and the variance cannot be negative. The new model created is an evolution of the analysis of the conditional heteroskedasticity using an independent variable implementing the reactions of the volatility and could be useful to forecast volatility clustering. The GARCH model could be not enough to fit a model in a time series because it analyses the historical variance but not consider a derivative instruments. A GARCHVIX approach could anticipate the volatility using the options as an exogenous indicator. We evaluate the model with the EWMA and the GARCH model to determine if the new model created is more accurate than the others already available. An empirical analysis can help to determine the accuracy of the model.

# 3 Chapter 2: Empirical Research

## 3.1 Data

As a demonstration of the faithfulness of the model, we consider the daily log returns series of each closing price for the 10 major global indices, using as data source Bloomberg. I downloaded the Future contracts of the following 10 equity indices:

DAX 30 (Germany)

CAC 40 (France)

FTSE 100 (England)

FTSE MIB (Italy)

S&P 500 (USA)

NASDAQ (USA - Tech)

HANG SENG (Hong Kong)

FTSE CHINA A50 (China)

TOPIX (Japan)

SPI 100 (Australia)

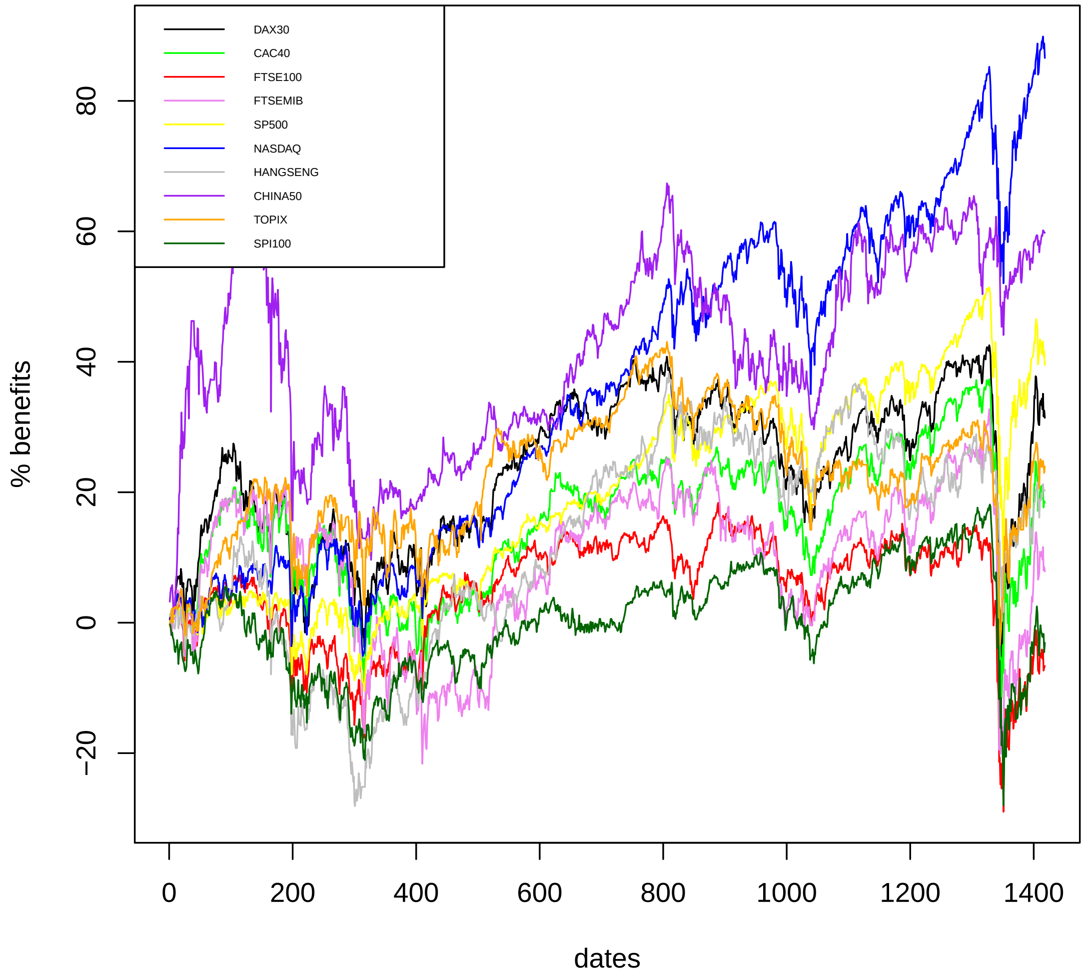
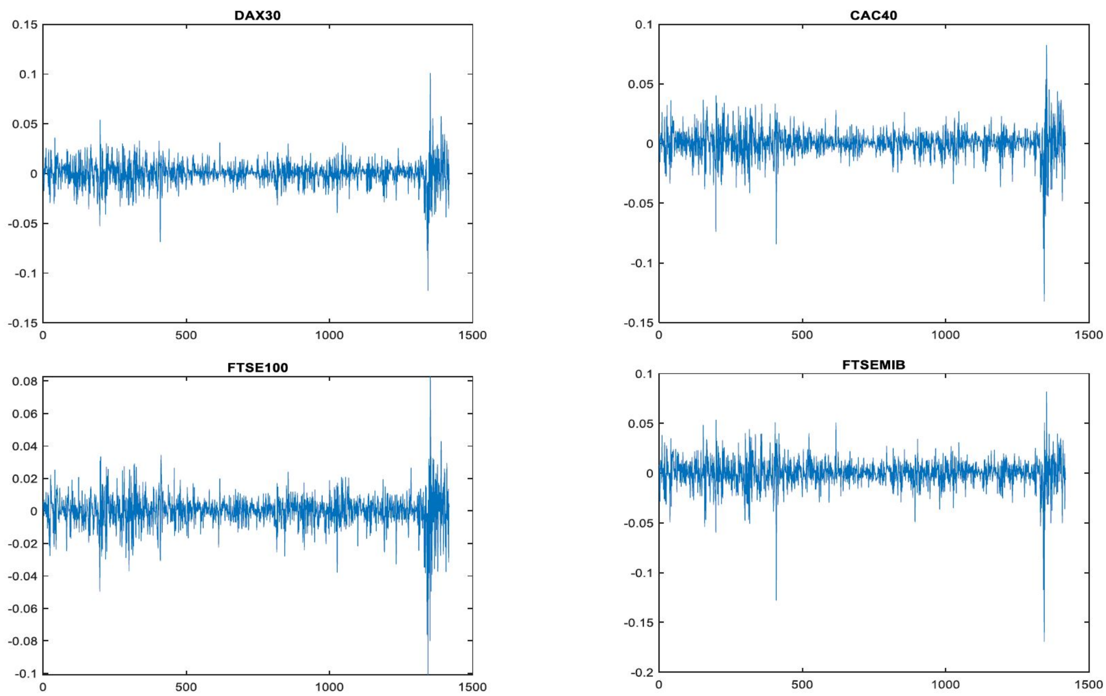



Figure 1: Time series Indices

In selecting the data, I diversified from different markets around the world in order to test the model with different time series, looking to diversify and reduce the correlation among them. Every index is composed with different sector and Iights and the comparison among them could bend the truth entailing wrong assumptions. We downloaded Futures of the indices because they are highly liquid ensuring smaller transaction costs due to bid/ask spreads and an efficient asset pricing. The period analyzed starts the 10, November 2014 and ends the 26, June 2020 with 1418 observations. I managed the data computing the logarithmic daily returns of every time series and fitted data among them with the VLOOKUP function of Excel. I have also used MATLAB and R studio to manage and analyze the empirical research. So, for a better interpretation of data graphs, the x axis show values starting from 0, concerning the 10 October 2014, the 500 concerning the 10 October 2016, 1000 concerning the 29 October 2018 and to the ends the 1418 observation concerning to the 26 June 2020.

**Daily log returns of the indices**



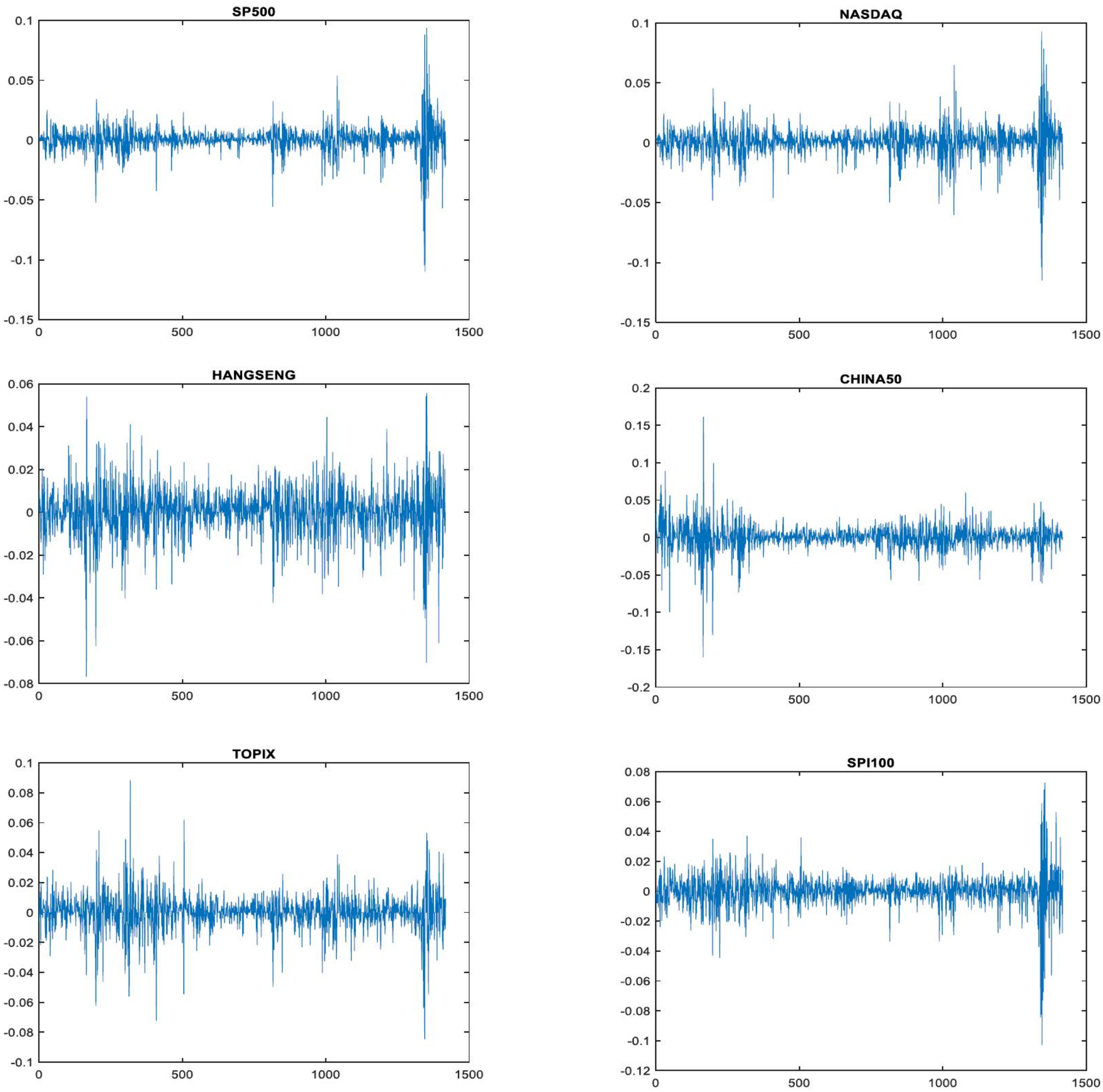



Figure 2: Logarithmic daily returns

As we can see from figure 2, the returns of the indices are plotted over time and the volatility tends to vary. The differenced daily log returns series show higher degree of volatility in the last 6 Months than relative to the entire time series analyzed. This is happened due to Covid-19, a pandemic virus that, as a black swan, destabilized the global markets increasing the uncertainty and, therefore, the volatility. Transforming data permits to remove the trend of the time series and make a time series stationary. The graphs show a different range of daily returns exhibit different volatility among them, but the volatility is different also inside every single time series showing a possible volatility clustering. Usually the returns of time series are assumed to follow a normal distribution, but the empirical analysis show how assume the normality is wrong and is almost impossible find data that roughly fits a bell curve shape. A common characteristic of asset returns is that they have fat-tailed distribution, presence of positive or negative skewness and suffer from leptokurtosis.

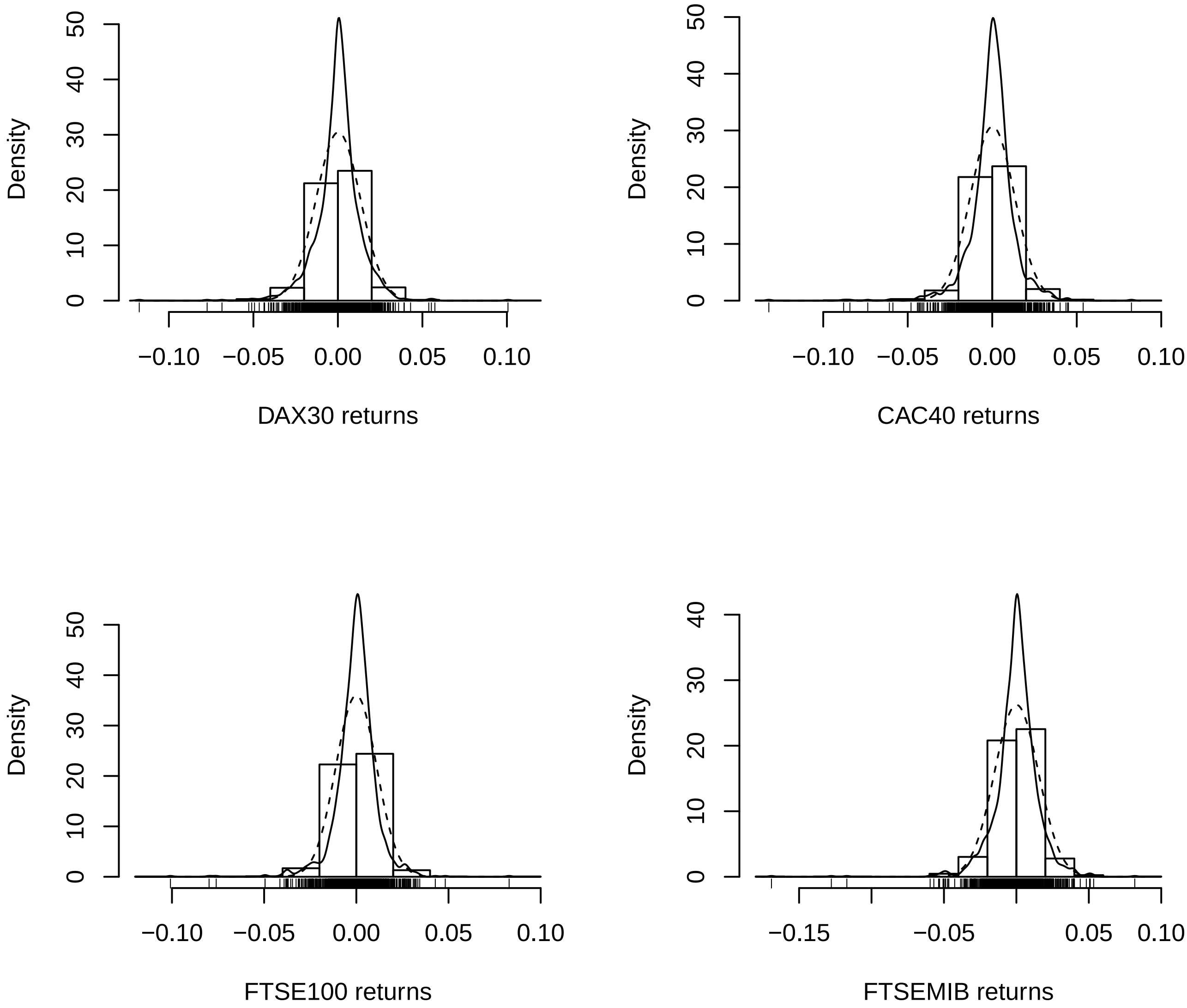
### 3.2 Distribution of Returns

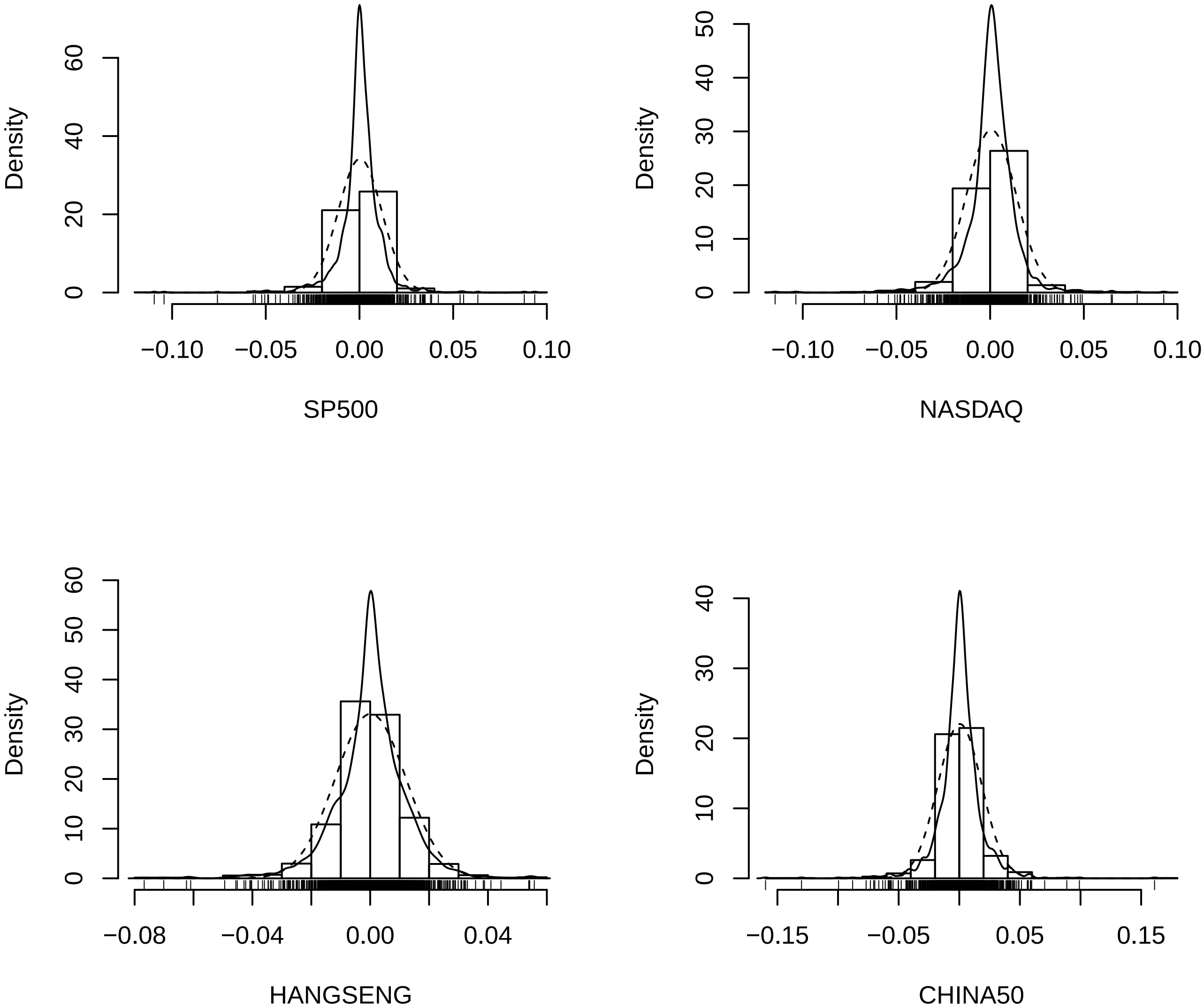
To determine the distribution there are many statistical tests useful to determine if the returns follows a normal distribution or not. Anyway, there are many parametric and non-parametric distribution useful to fit the model in the returns. We computed the Jarque-Bera test that analyze the normality distribution. It is a goodness-of-fit test of whether sample data have the skewness and kurtosis matching a normal distribution. A normal distribution has a skewness of 0 and a kurtosis of 3. If the test statistics is far from zero, it signals the data do not have a normal distribution. the Jarque-Bera statistic asymptotically has a chi-squared distribution with 2 degrees of freedom, so the statistic can be used to test the hypothesis that the data are from a normal distribution. the null hypothesis is a joint hypothesis of the skewness being zero and the excess kurtosis being zero. To avoid the risk to take to falsely reject $H_0$ or do not reject $H_0$ is important to consider the p-value or significance level alpha of the statistics. For example, if the p-value of a test statistic result is 0.059, then there is a probability of 5.9% that I falsely reject $H_0$ because the p-value is higher than 0.05.

Table 1: Jarque-Bera test

| | X-squared | df | p-value |
|---|---|---|---|
| DAX 30 | 5313.1 | 2 | 0 |
| CAC 40 | 10652 | 2 | 0 |
| FTSE 100 | 8339.6 | 2 | 0 |
| FTSE MIB | 19397 | 2 | 0 |
| S&P 500 | 21717 | 2 | 0 |
| NASDAQ | 8910.4 | 2 | 0 |
| HANG SENG | 1291.3 | 2 | 0 |
| CHINA 50 | 10884 | 2 | 0 |
| TOPIX | 3381.6 | 2 | 0 |
| SPI 100 | 12911 | 2 | 0 |

As we can see, the test exhibit high signals that the distribution is not normal. The p-value is statistically significant to reject the null hypothesis. We can investigate which is the empirical distribution and how differs from a normal distribution. First of all, I compute the histogram of the time series, then I calculate the normal density function with the sample mean and the sample standard deviation of the given series (dashed line) and, last but not least, I compute the estimate of the density from a sample (solid line).


Density
DAX30 returns
Density
CAC40 returns
Density
FTSE100 returns
Density
FTSEMIB returns


Density
0
20
40
60
−0.10
−0.05
0.00
0.05
0.10
SP500
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0
10
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30
40
50
−0.10
−0.05
0.00
0.05
0.10
NASDAQ
Density
0
10
20
30
40
50
60
−0.08
−0.04
0.00
0.04
HANGSENG
Density
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10
20
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40
−0.15
−0.05
0.05
0.15
CHINA50

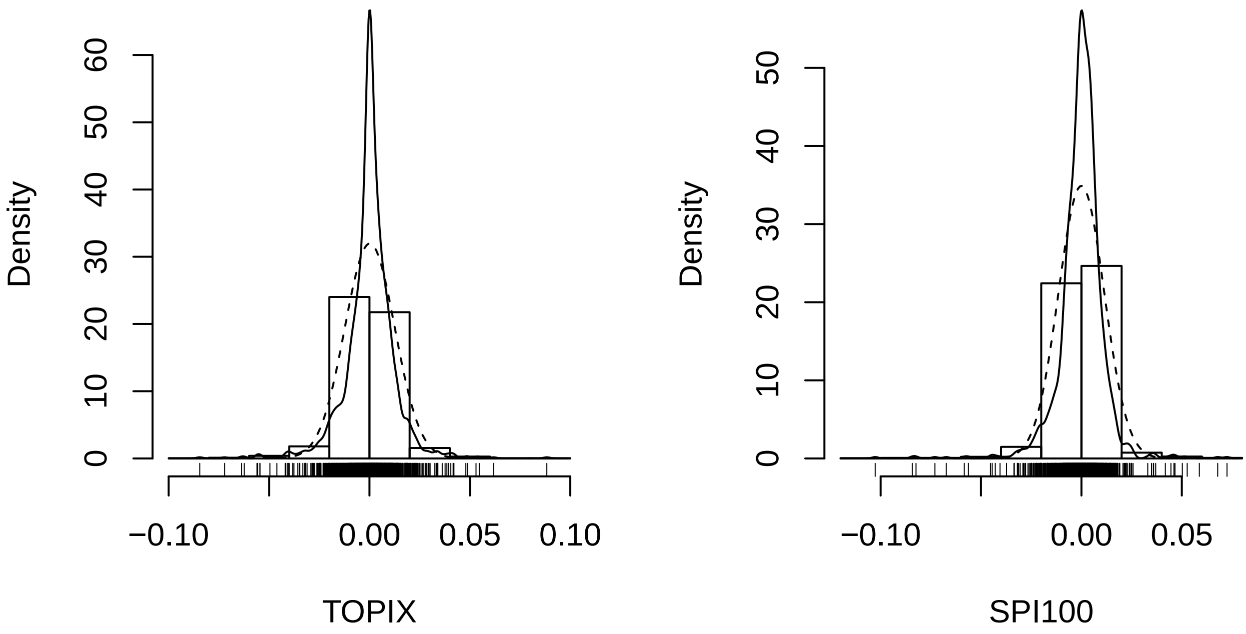



Figure 3: Distribution analysis

So, we have the histograms that present the returns of the indices showing outliers. Then, we have a dashed line that represent the normal distribution. Finally we have a solid line that represent the empirical distribution, completely different from a normal distribution. The figures shows that a normal distribution does not fits well the sample estimate of the density of returns for the Major Indices. With the Shapiro test every p-value was less than a given confidence level (0.05) then the null hypothesis was rejected for every time series. The t-distribution have played an extremely important role in classical statistics because of their use in testing and confidence intervals when the data are modelled as having normal distributions. More recently, t-distributions have gained added importance as models for the distribution of heavy-tailed phenomena such as financial markets data. Anyway, using the Student-t distribution might need at least 3000 observations and in our model I have observed only 1418 observations.

### 3.3 Descriptive Statistics

Another approach to evaluate the time series is simply realize a table that describe the statistics indicators. The descriptive statistics determine the mean, variance, maximum and minimum variables, skewness and kurtosis. Moreover, I provided the Ljung-Box test with the p-value and in autoror the five lags of the AutoCorrelation Function (ACF). The Ljung-Box test is a type of statistical test of whether any of a group of autocorrelations of a time series are different from zero. Instead of testing randomness at each distinct lag, it tests the overall randomness based on a number of lags, and is therefore a portmanteau test. The null hypothesis of the Ljung-Box test is not rejected when the data are independently distributed, and the critical region for rejection of the hypothesis of randomness is determined with a chi-squared distribution with h degrees of freedom. (lags=degree of freedom).

The following table 1 shows the output for each time series.

Table 2: Descriptive statistics and Ljung-box test

| | DAX 30 | CAC 40 | FTSE 100 | FTSE MIB | S&P 500 | NASDAQ | HANG SENG | CHINA 50 | TOPIX | SPI 100 |
|---|---|---|---|---|---|---|---|---|---|---|
| mean | 2.21E-04 | 1.30E-04 | -4.69E-05 | 5.54E-05 | 2.78E-04 | 6.11E-04 | 1.43E-04 | 4.22E-04 | 1.69E-04 | -2.18E-05 |
| std | 0.0131 | 0.013 | 0.0111 | 0.0152 | 0.0117 | 0.0132 | 0.012 | 0.0181 | 0.0125 | 0.0114 |
| min | -0.1175 | -0.1322 | -0.1008 | -0.169 | -0.1095 | -0.1148 | -0.0768 | -0.1598 | -0.0845 | -0.1027 |
| max | 0.1006 | 0.0824 | 0.0829 | 0.0817 | 0.0935 | 0.0927 | 0.0557 | 0.1611 | 0.0883 | 0.0725 |
| skewness | -0.5952 | -1.2946 | -0.9968 | -1.7525 | -0.9001 | -0.8143 | -0.5607 | -0.4358 | -0.4123 | -1.1113 |
| kurtosis | 12.4079 | 16.1753 | 14.7122 | 20.777 | 22.0872 | 15.1721 | 7.5384 | 16.5448 | 10.5202 | 17.6142 |
| | | | | | | | | | | |
| Ljung Box | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.0000 | 1.0000 |
| P-value | 0.2273 | 0.4217 | 0.8822 | 0.3527 | 1.41E-11 | 2.01E-14 | 0.0028 | 8.32E-05 | 0.122 | 1.44E-08 |
| Stats | 6.9114 | 4.9526 | 1.7524 | 5.5482 | 59.688 | 73.4015 | 18.0867 | 26.1569 | 8.6912 | 45.0106 |
| autoror | : | : | : | : | : | : | : | : | : | : |
| lag1 | 0.0060 | -0.0006 | -0.0248 | -0.0299 | -0.1584 | -0.1927 | -0.0821 | -0.0932 | -0.0031 | -0.1593 |
| lag2 | 0.0679 | 0.0529 | -0.0024 | 0.0462 | 0.0782 | 0.0653 | 0.0446 | -0.0502 | 0.0187 | 0.0457 |
| lag3 | 0.0137 | 0.0189 | 0.0205 | 0.0233 | 0.0232 | 0.0367 | 0.0265 | -0.0005 | -0.0583 | 0.0208 |
| lag4 | 0.0056 | -0.0003 | -0.0137 | -0.0067 | -0.085 | -0.0737 | -0.0565 | 0.0724 | -0.0341 | -0.0268 |
| lag5 | -0.0001 | -0.0183 | 0.0023 | 0.017 | 0.0561 | 0.0599 | 0.0105 | -0.0446 | -0.0347 | 0.0558 |

As we can see in the table, every time series shows negative skewness, meaning longer left tails than a normal and asymmetrical distribution. The kurtosis is high thus the data seems to have a leptokurtosis distribution (fat-tailed), due to extreme outliers. The time series are similar about the skewness and the kurtosis following the fact that they are similar in terms of structure of data. The Ljung-Box test (lbq test) has as output the Hypothesis, the pvalue, the stats in terms of chi-squared distribution and five autolags considered as h degrees of freedom. The Ljung-Box test show that there is no serial dependence in the DAX 30, CAC 40, FTSE 100, FTSE MIB indices and in the TOPIX. It is determined because the test do not reject the null hypothesis, the p-value is higher than alpha (the significance level) 0.05 and the critical region for rejection is lower than 11.07, determined in the chi-squared distribution table. Otherwise, we reject the null hypothesis in the S&P 500, NASDAQ, HANG SENG, CHINA 50 AND SPI 100 indices, showing how the observed results are statistically significant, because the observed p-value is less than the pre-specified significance level alpha. The t-statistics show that the values are higher than the 11.07 and the ACF of lags 5 are autocorrelated. In these cases, we need to assume further assumptions when we create a model.

### 3.4 Correlation

It has been frequently observed that USA markets leads other developed markets in Europe or Asia, and that at times the leader becomes the follower. With different markets, some assets' returns are observed to behave like other assets' returns, or completely opposite. The study of the possible interdependence between two financial time series is calculated with the correlation of their returns. The correlation is just a particular symmetric relation of dependency among stochastic variables, and so to know the degrees of asymmetry in the dependence of one financial time series with respect to another, whereby we may determine who leads and who follows, we must study measures of causality.

Anyway, I calculated the correlation matrix among the indices and there is evidence of high correlation between the European indices but they present also high correlation with the USA markets.

Table 3: Correlation Matrix

| | DAX30 | CAC40 | FTSE100 | FTSEMIB | SP500 |
|---|---|---|---|---|---|
| DAX 30 | 1 | 0.924282 | 0.820164 | 0.837192 | 0.582559 |
| CAC 40 | 0.924282 | 1 | 0.849198 | 0.861904 | 0.617962 |
| FTSE 100 | 0.820164 | 0.849198 | 1 | 0.743503 | 0.588729 |
| FTSE MIB | 0.837192 | 0.861904 | 0.743503 | 1 | 0.566213 |
| S&P 500 | 0.582559 | 0.617962 | 0.588729 | 0.566213 | 1 |
| NASDAQ | 0.537395 | 0.565518 | 0.534707 | 0.501671 | 0.918291 |
| HANG SENG | 0.448886 | 0.484345 | 0.481356 | 0.394928 | 0.296158 |
| CHINA 50 | 0.25078 | 0.267338 | 0.27243 | 0.185092 | 0.208006 |
| TOPIX | 0.355731 | 0.38675 | 0.347864 | 0.31926 | 0.212698 |
| SPI 100 | 0.387 | 0.423385 | 0.459289 | 0.358352 | 0.417533 |

| | NASDAQ | HANGSENG | CHINA50 | TOPIX | SPI100 |
|---|---|---|---|---|---|
| DAX 30 | 0.537395 | 0.448886 | 0.25078 | 0.355731 | 0.387 |
| CAC 40 | 0.565518 | 0.484345 | 0.267338 | 0.38675 | 0.423385 |
| FTSE 100 | 0.534707 | 0.481356 | 0.27243 | 0.347864 | 0.459289 |
| FTSE MIB | 0.501671 | 0.394928 | 0.185092 | 0.31926 | 0.358352 |
| S&P 500 | 0.918291 | 0.296158 | 0.208006 | 0.212698 | 0.417533 |
| NASDAQ | 1 | 0.305527 | 0.210748 | 0.171838 | 0.355453 |
| HANG SENG | 0.305527 | 1 | 0.596897 | 0.491856 | 0.470971 |
| CHINA 50 | 0.210748 | 0.596897 | 1 | 0.290561 | 0.262706 |
| TOPIX | 0.171838 | 0.491856 | 0.290561 | 1 | 0.440931 |
| SPI 100 | 0.355453 | 0.470971 | 0.262706 | 0.440931 | 1 |

As we can see, the correlation increase with the status of the development economics and also with the geographic areas.

Then, we can show the correlation between the indices and the VIX to analyse how the Volatility index used show negative correlation with the indices and the differences among them.

Table 4: Correlation with the VIX

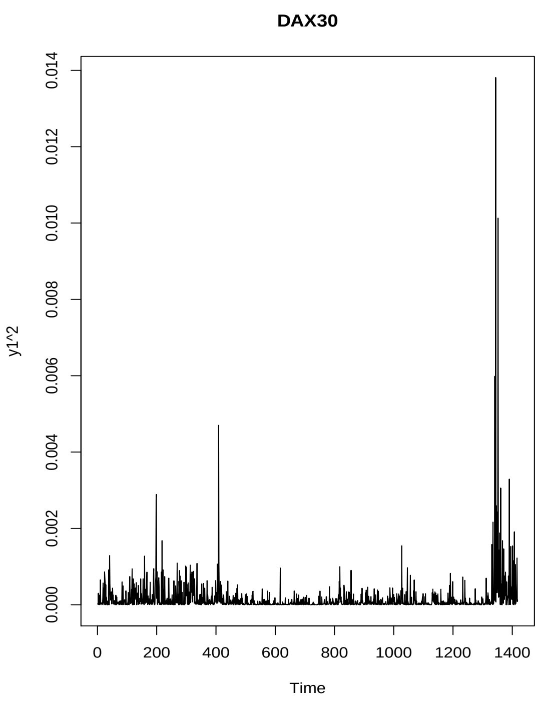
| | VIX |
|---|---|
| DAX 30 | -0.4246994 |
| CAC 40 | -0.4430257 |
| FTSE 100 | -0.3792228 |
| FTSE MIB | -0.4219787 |
| S&P 500 | -0.6659271 |
| NASDAQ | -0.6443102 |
| HANG SENG | -0.2205619 |
| CHINA 50 | -0.1266067 |
| TOPIX | -0.1659127 |
| SPI 100 | -0.1865044 |

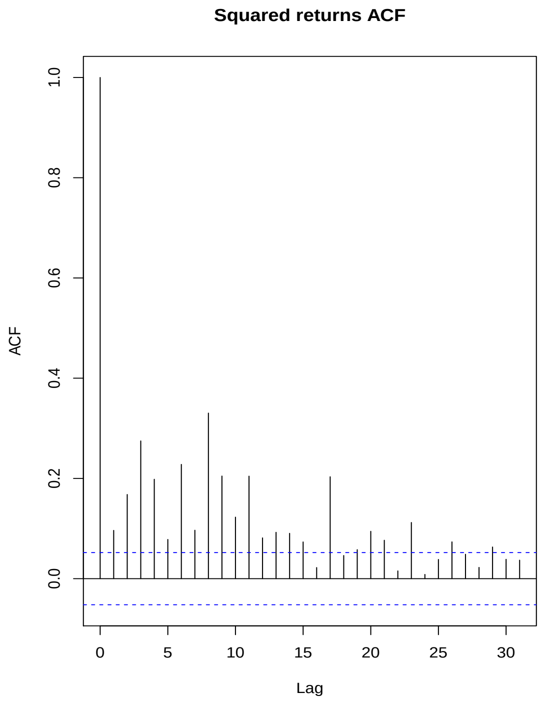
In the table 4 every time series analyzed shows negative correlation but is highly relevant in the Standard & Poor index and in the Nasdaq. In the European Countries is relevant and is low in the others.

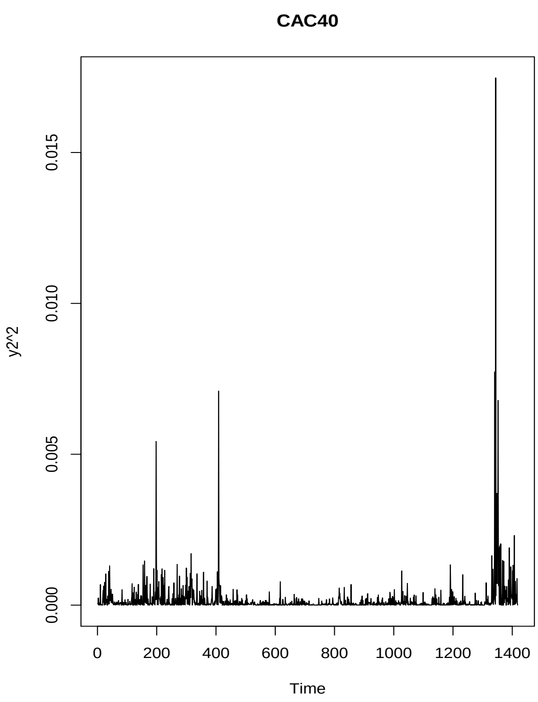
### 3.5 Squared Residuals

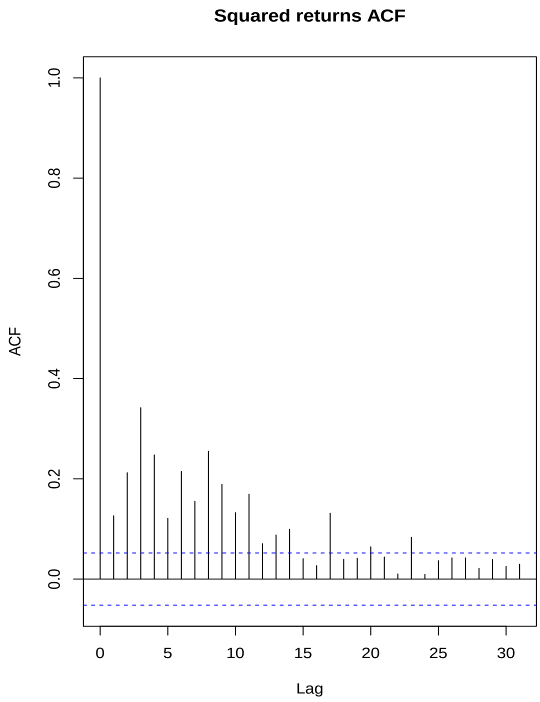
Then, after a general descriptive statistics, I focus on the heteroskedasticity and an useful graphical analysis is check the squared residuals.

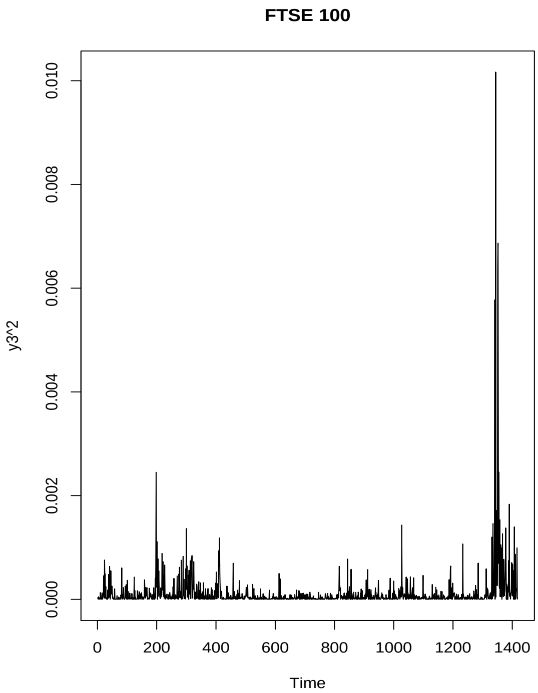
First of all, I plot a graph of the squared time series to analyze if exhibit volatility clustering, then I evaluate the squared residuals using the Autocorrelation function (ACF). We shown below the 10 analysed indices in exam.

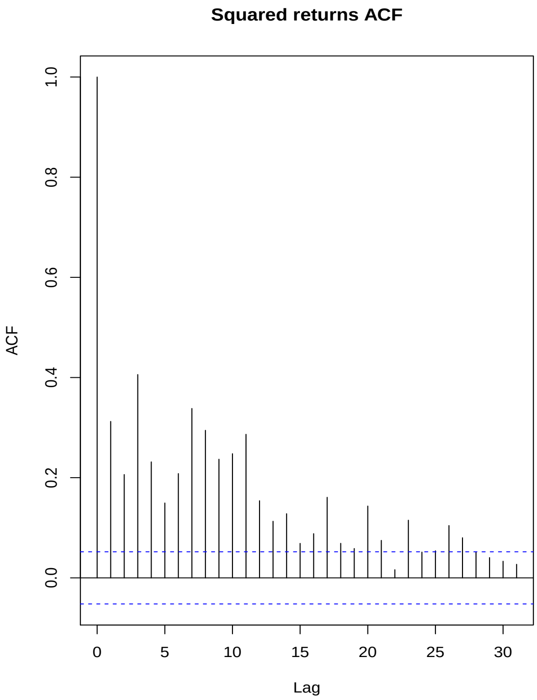


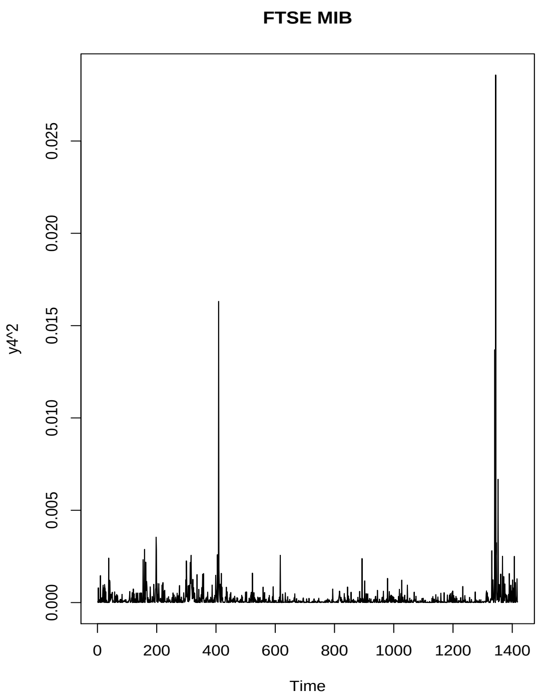



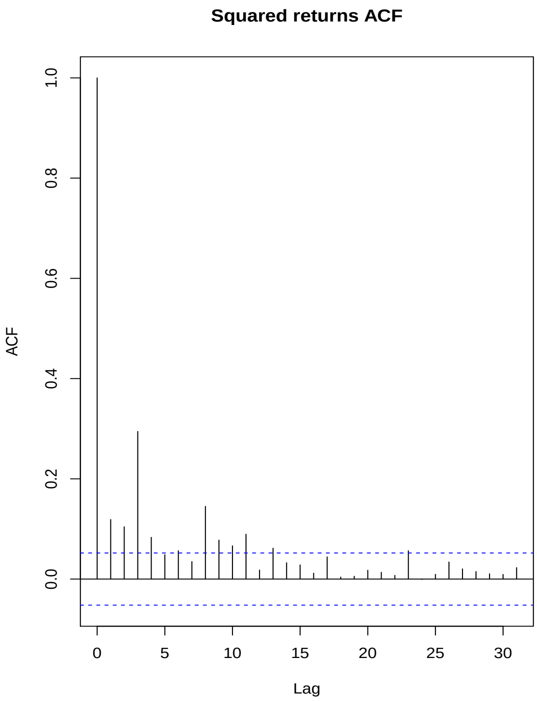



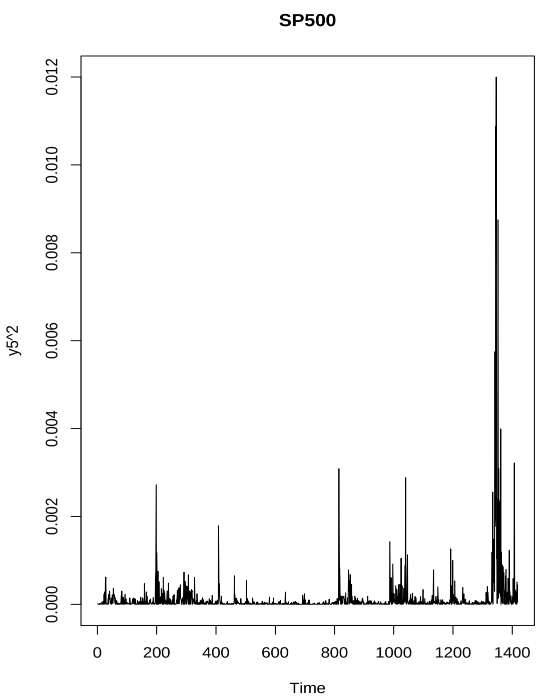



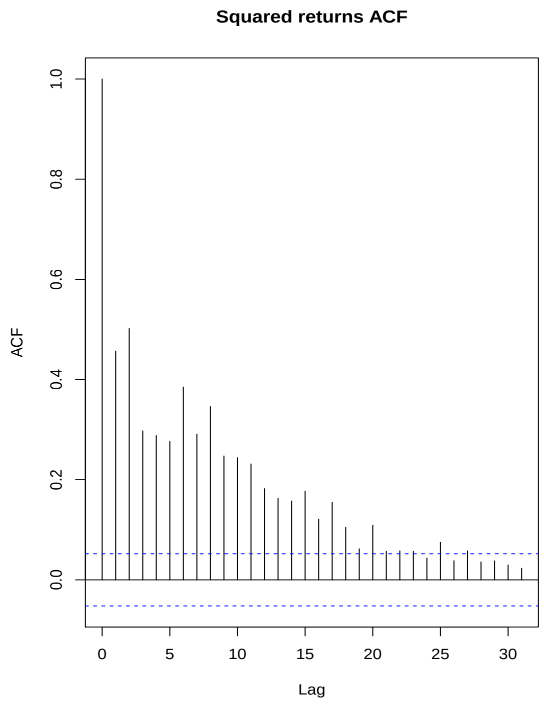

FTSE 100
y3^2
0.000
0.002
0.004
0.006
0.008
0.010
0 200 400 600 800 1000 1200 1400
Time

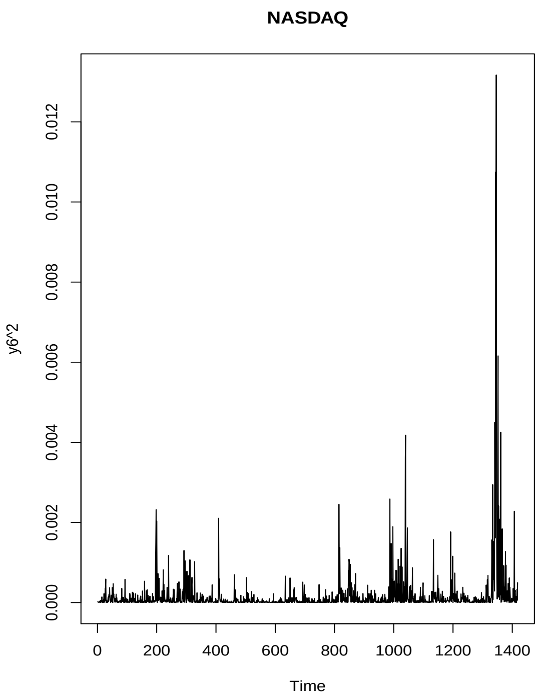

Squared returns ACF
ACF
0.0
0.2
0.4
0.6
0.8
1.0
0 5 10 15 20 25 30
Lag

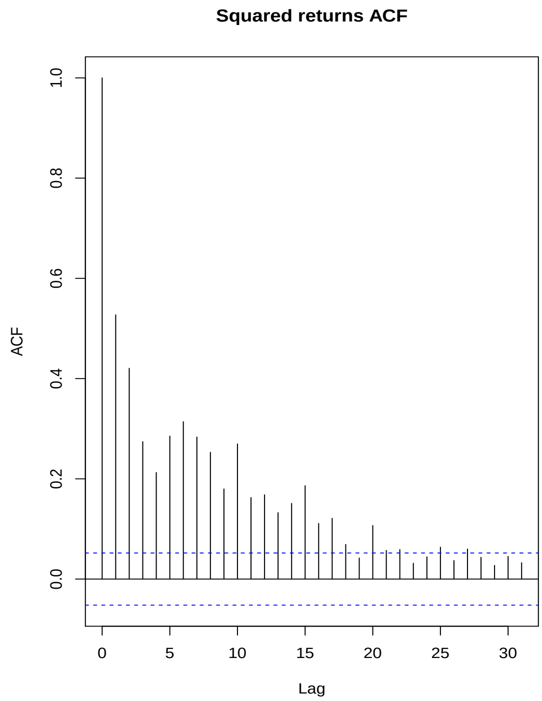

FTSE MIB
y4^2
0.000
0.005
0.010
0.015
0.020
0.025
0 200 400 600 800 1000 1200 1400
Time

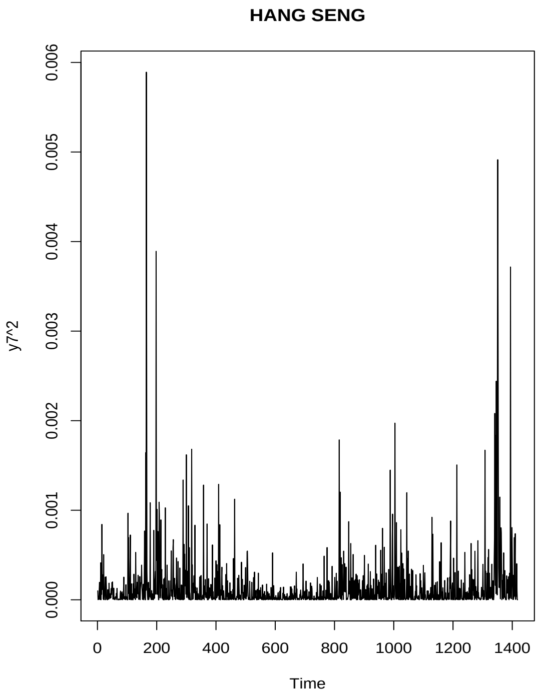
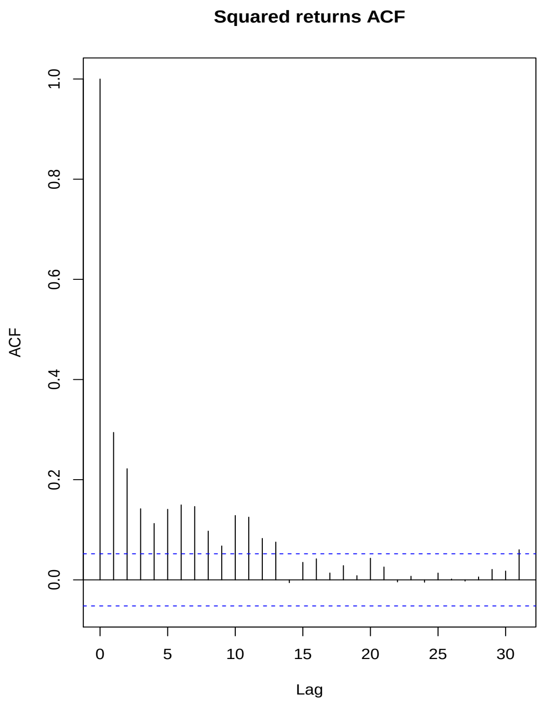
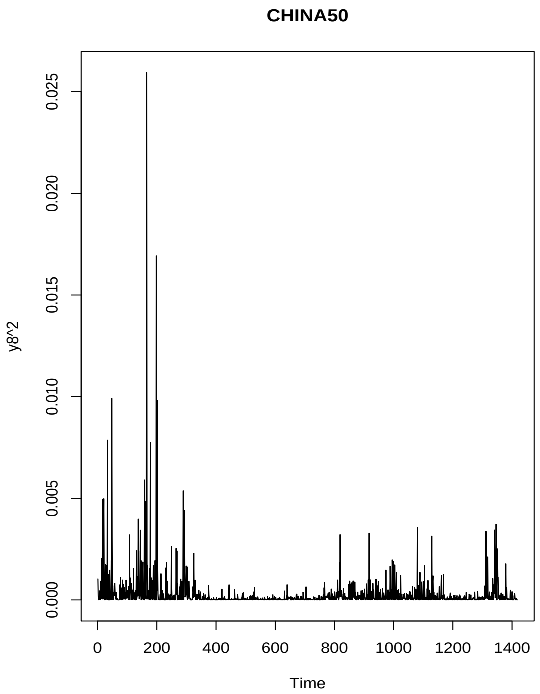
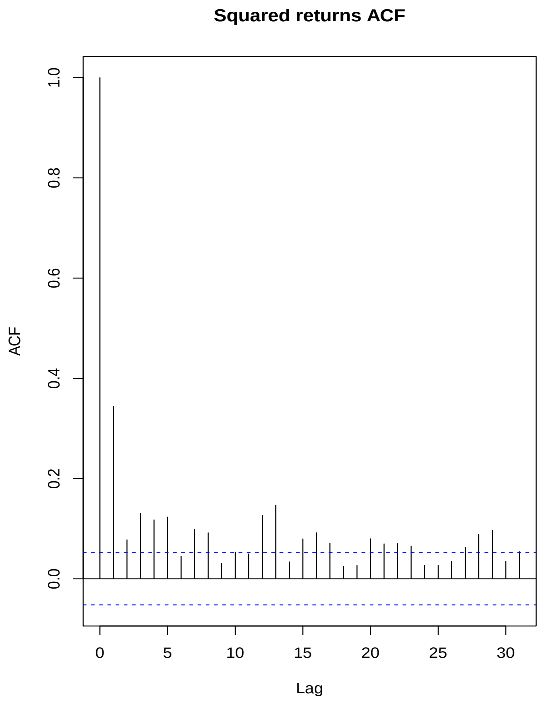

Squared returns ACF
ACF
0.0
0.2
0.4
0.6
0.8
1.0
0 5 10 15 20 25 30
Lag


SP500
y5^2
0.000
0.002
0.004
0.006
0.008
0.010
0.012
0
200
400
600
800
1000
1200
1400
Time


Squared returns ACF
ACF
0.0
0.2
0.4
0.6
0.8
1.0
0
5
10
15
20
25
30
Lag


NASDAQ
y6^2
0.000
0.002
0.004
0.006
0.008
0.010
0.012
0
200
400
600
800
1000
1200
1400
Time


Squared returns ACF
ACF
0.0
0.2
0.4
0.6
0.8
1.0
0
5
10
15
20
25
30
Lag


HANG SENG
y7^2
Time
0.000
0.001
0.002
0.003
0.004
0.005
0.006
0
200
400
600
800
1000
1200
1400

Squared returns ACF
ACF
Lag
0.0
0.2
0.4
0.6
0.8
1.0
0
5
10
15
20
25
30

CHINA50
y8^2
Time
0.000
0.005
0.010
0.015
0.020
0.025
0
200
400
600
800
1000
1200
1400

Squared returns ACF
ACF
Lag
0.0
0.2
0.4
0.6
0.8
1.0
0
5
10
15
20
25
30

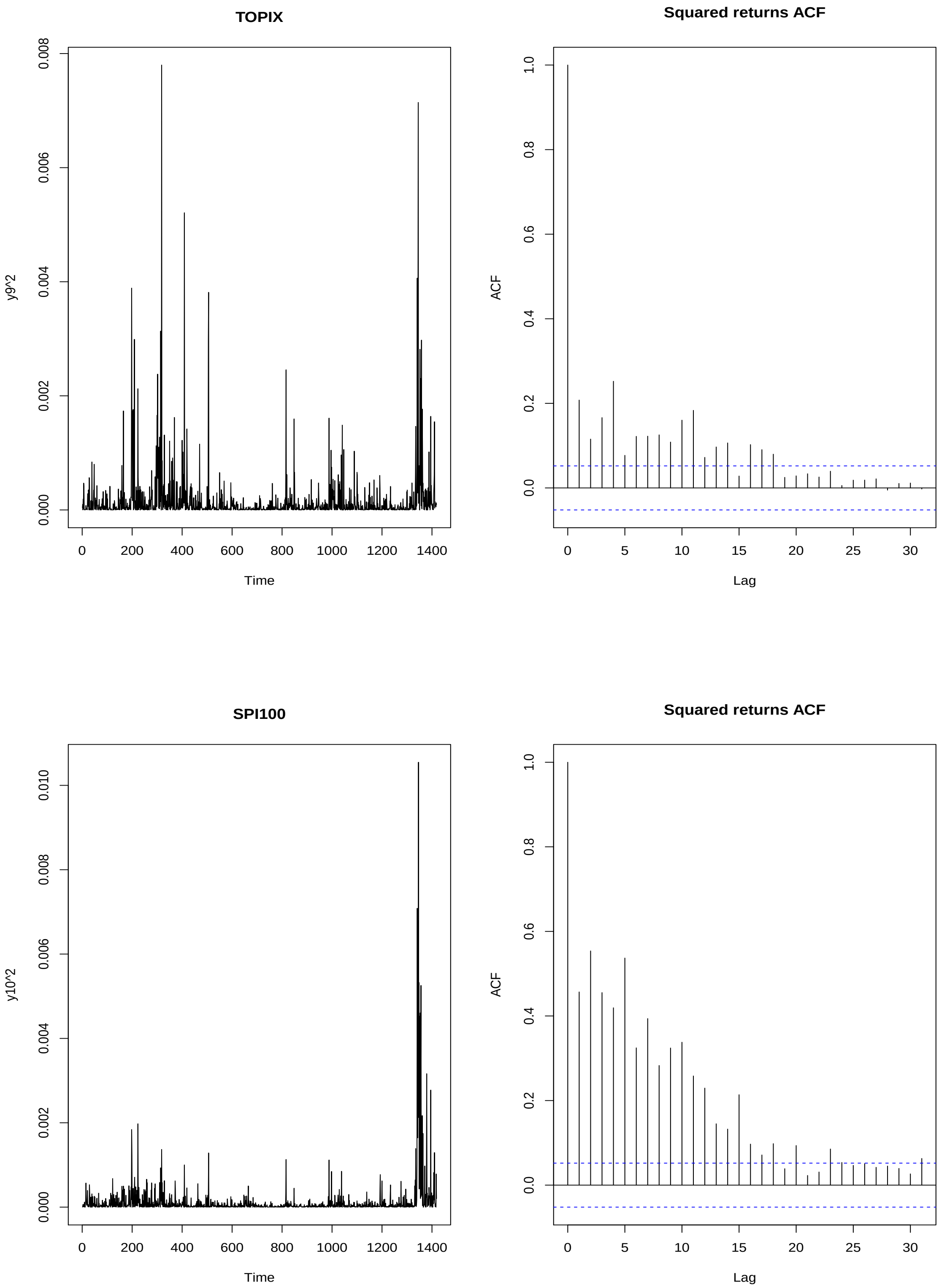



Figure 4: Squared Residuals

As we can show, the last period is influenced by the Covid-19 and the reaction of the time series were increase the volatility and then create a volatility clustering. The period analysed show highest evidence in the Covid period for many indices, excluding the HANG SENG index and the TOPIX index. We noted that the volatility increase during economical and political events like Brexit, USA elections and so on. The volatility of the HANG SENG is the opposite of the other indices showing almost always high volatility with a brief time of low volatility, given graphical evidence of volatility clustering. The TOPIX index has almost the same situation. Instead the CHINA A50 index shows high volatility during the first period and now, with the Covid crisis, is lower than before. The FTSE MIB has reached highest values of volatility during the last period due to Covid-19. In the right side, there are their respective correlogram showing that the lags, in all cases, decrease obtaining the autocorrelation of the squared residuals less than 0.05, the significance level. This is positive for the model because all time series shows volatility clustering and the squared residuals tends to zero.

## 3.6 The Heteroskedasticity

Another empirical analysis to evaluate the heteroskedasticity is using the Engle's ARCH test statistics. The test assesses the null hypothesis that a series of residuals exhibits no conditional heteroskedasticity (ARCH effects) against the alternative that an ARCH(p) model describes the series. The h is the hypothesis. The value 1 indicates rejection of the no ARCH effects null hypothesis in favour of the alternative. h=0 indicates failure to reject the no ARCH effects null hypothesis. We have used one lag. Then the output are pvalue, useful to consider the significance level, the test statistics, using a Lagrange multiplier test statistic and critical values, determined with the number of lags.

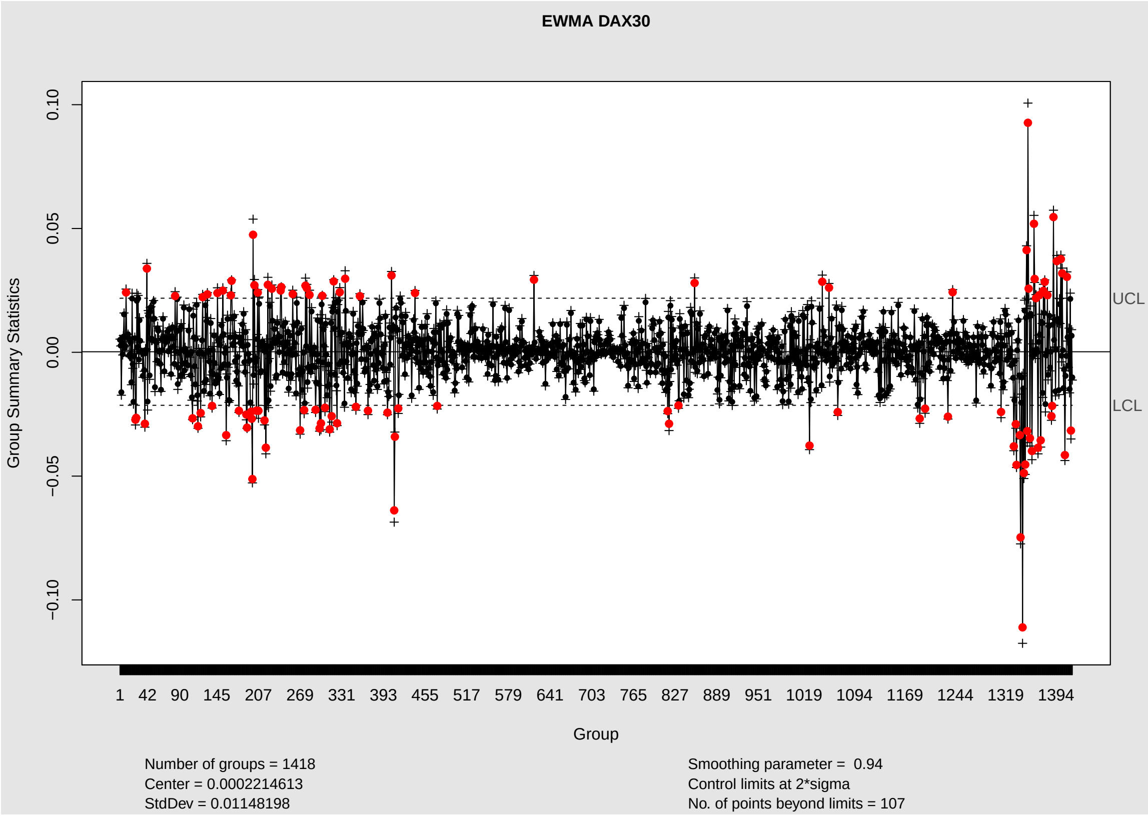
Table 5: Engle's ARCH test

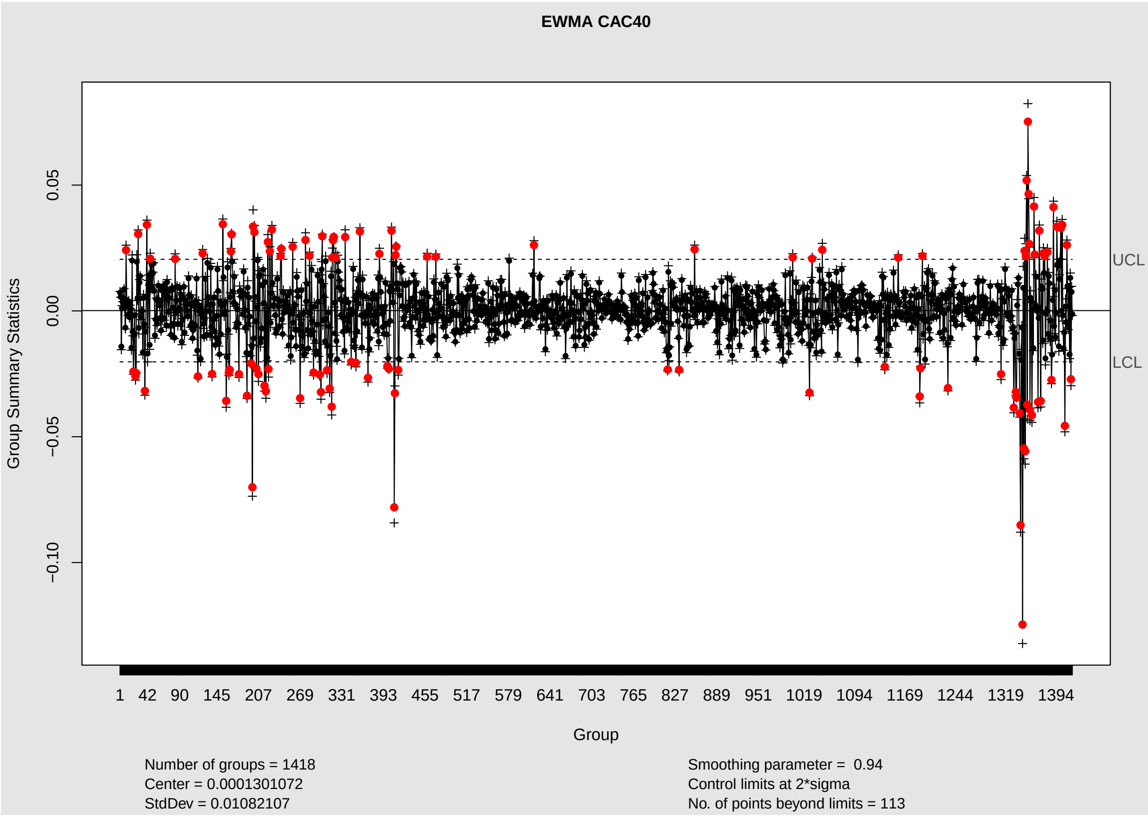
| | h hypothesis | pValue | stats | cValue |
|---|---|---|---|---|
| **DAX 30** | 1 | 0 | 13.1552 | 3.8415 |
| **CAC 40** | 1 | 0 | 22.6292 | 3.8415 |
| **FTSE 100** | 1 | 0 | 138.2454 | 3.8415 |
| **FTSE MIB** | 1 | 0 | 19.9905 | 3.8415 |
| **SP 500** | 1 | 0 | 295.8729 | 3.8415 |
| **NASDAQ** | 1 | 0 | 393.9357 | 3.8415 |
| **HANG SENG** | 1 | 0 | 122.7761 | 3.8415 |
| **CHINA 50** | 1 | 0 | 167.7209 | 3.8415 |
| **TOPIX** | 1 | 0 | 61.1238 | 3.8415 |
| **SPI 100** | 1 | 0 | 295.705 | 3.8415 |

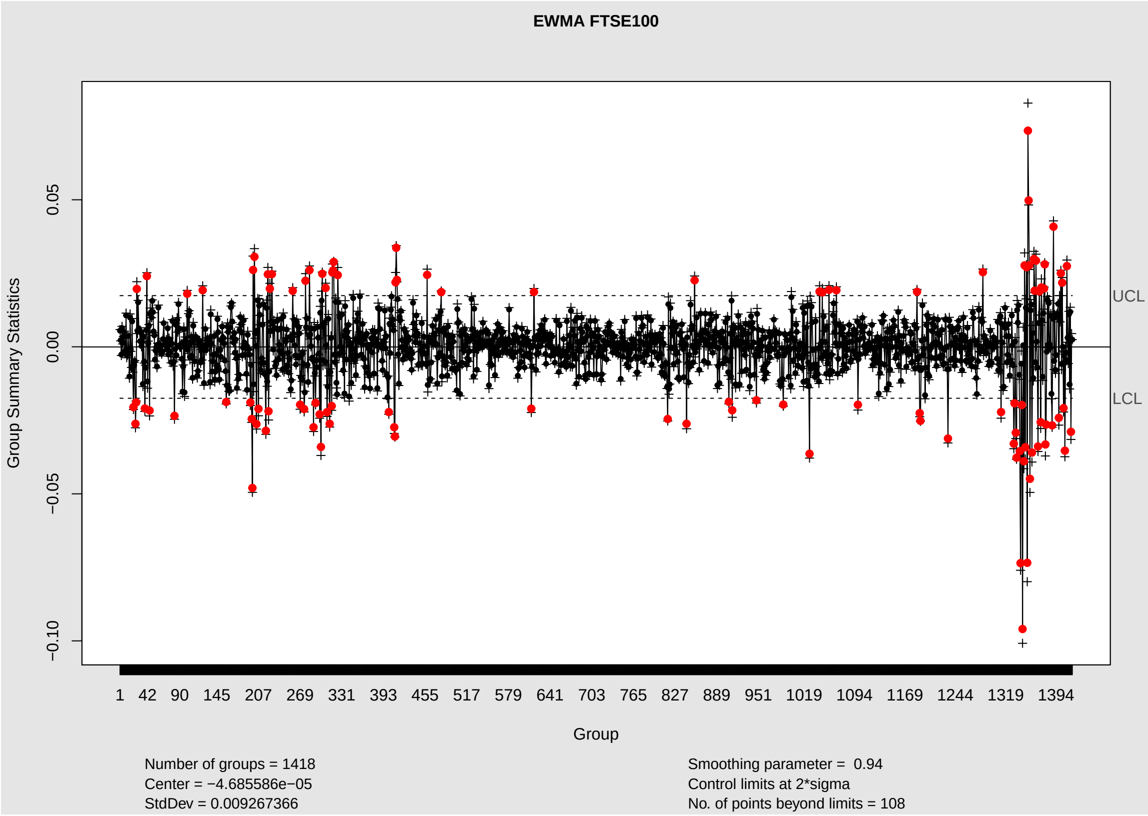
We immediately note all the time series rejects the null hypothesis in favour of the alternative. This in an additional evidence to fit a model that use the conditional variance. The time series statistics far exceeds the critical values. The p-value is 0, offering an further confirm of the rejection of the null Hypothesis. The analysis confirmed that an ARCH(1) model could be better than no consider any ARCH model.

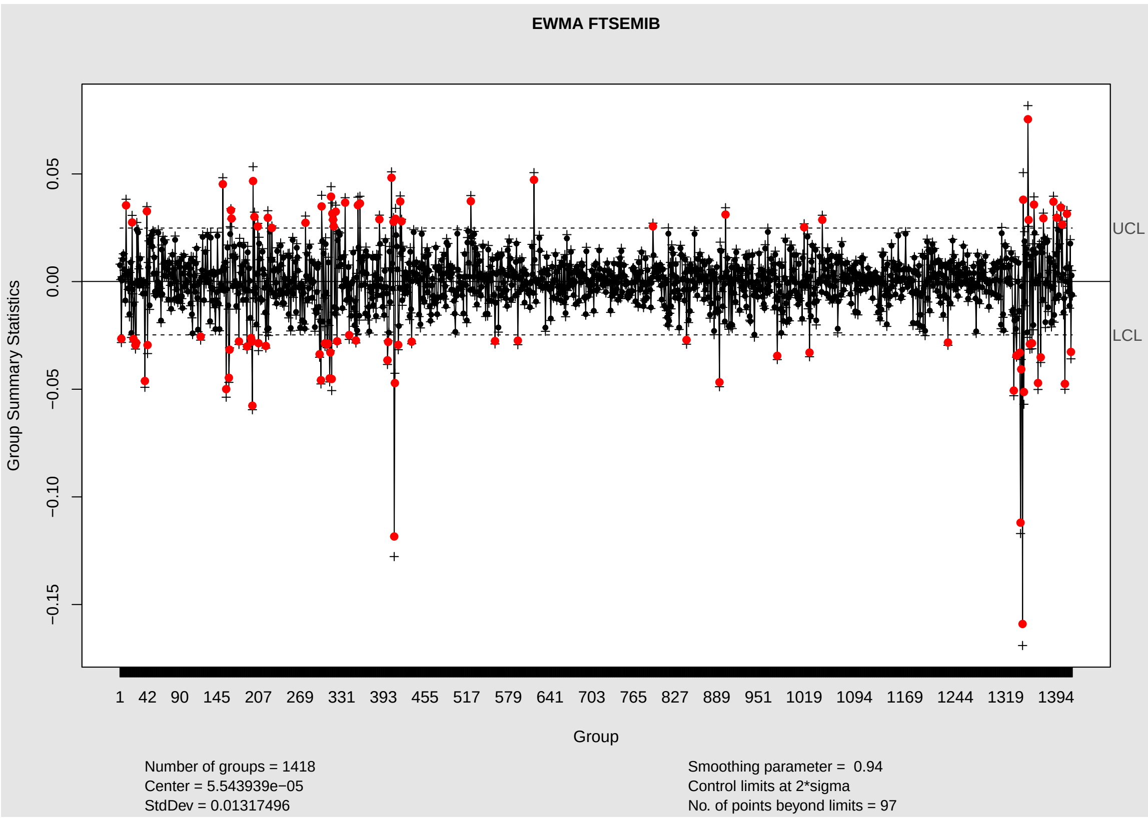
## 3.7 The empirical EWMA

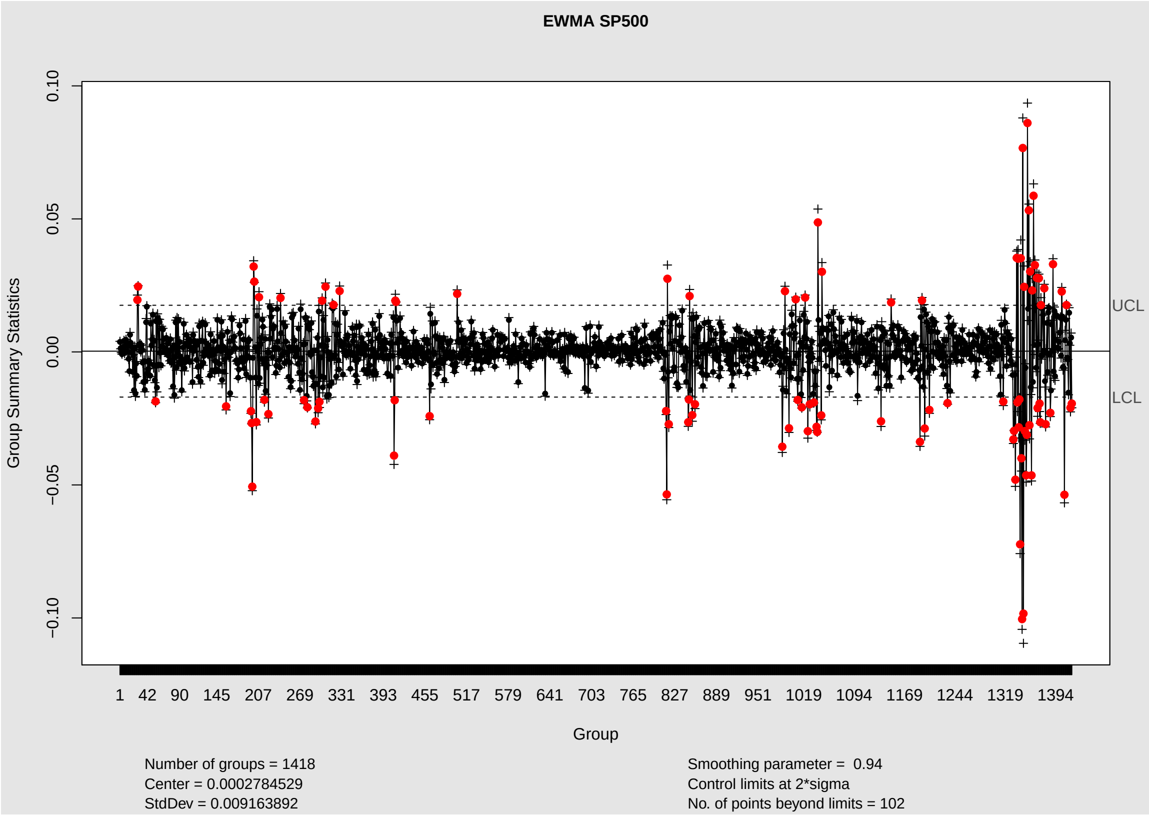
We already explained the structure of the EWMA model. That is simply a restricted integrated GARCH (iGARCH) model, with the restriction that the intercept $\omega$ is equal to zero, with the smoothing parameter ($\lambda$) equivalent to the autoregressive parameter $\beta$ in the GARCH equation. We can analyse every time series making a backtest and evaluate, using the VaR, how many times the historical returns exceed the confidence intervals. With the backtesting I determine the accuracy of the EWMA model and I can consider the model with the GARCH(1,1) and with the GARCHVIX(1,1,1). The $\lambda$ parameter takes a value between 0 and 1. We arbitrarily choose $\lambda = 0.94$, as in the RiskMetrics paper. We evaluate the EWMA model with the confidence interval of 95%, therefore the model fits on the time series when the outliers are exceed the confidence intervals for 5% of times. To compute the graph I used the package Stats by R. The black line is the log daily time series, the dashed lines are the confidence intervals, calculated with the VaR. The standard deviation is calculated with the EWMA formula, using lambda euqal to 0.94 and the constant equal to zero. The red points are the points beyond the confidence intervals.

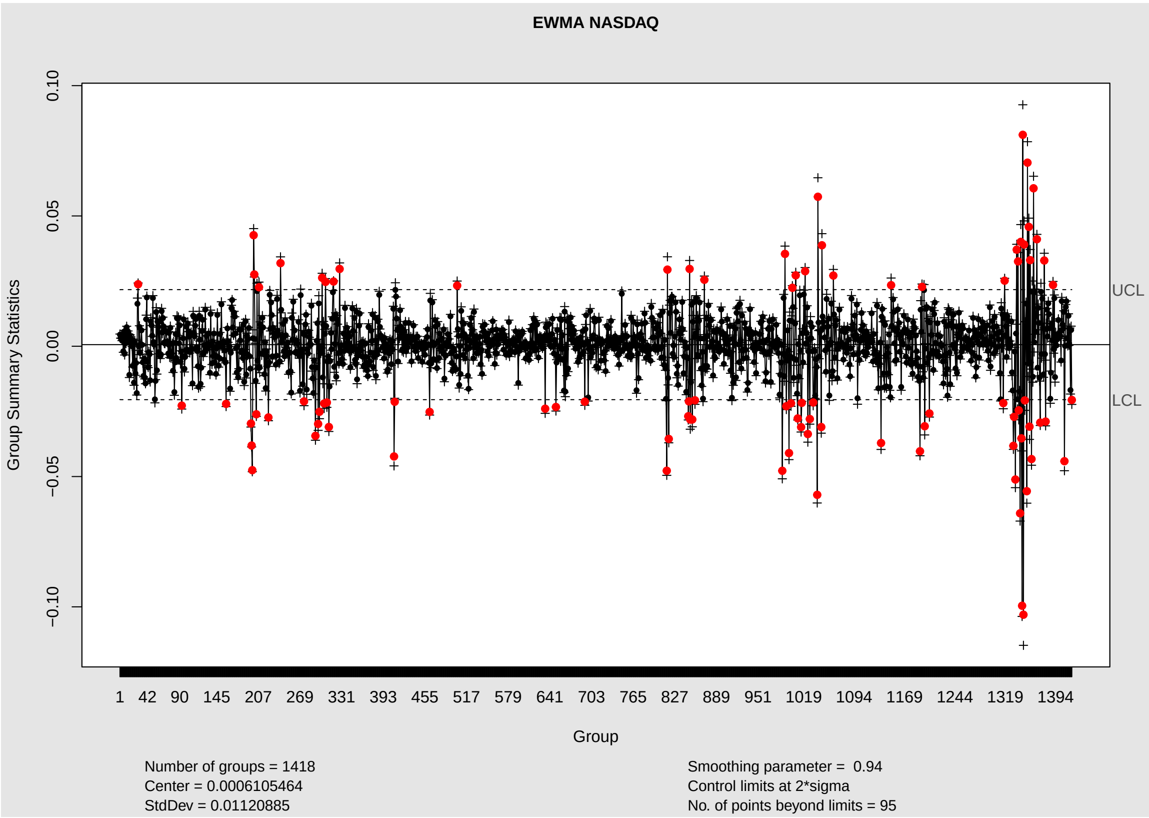

EWMA DAX30
Group Summary Statistics
0.10
0.05
0.00
−0.05
−0.10
UCL
LCL
1 42 90 145 207 269 331 393 455 517 579 641 703 765 827 889 951 1019 1094 1169 1244 1319 1394
Group
Number of groups = 1418
Center = 0.0002214613
StdDev = 0.01148198
Smoothing parameter = 0.94
Control limits at 2*sigma
No. of points beyond limits = 107

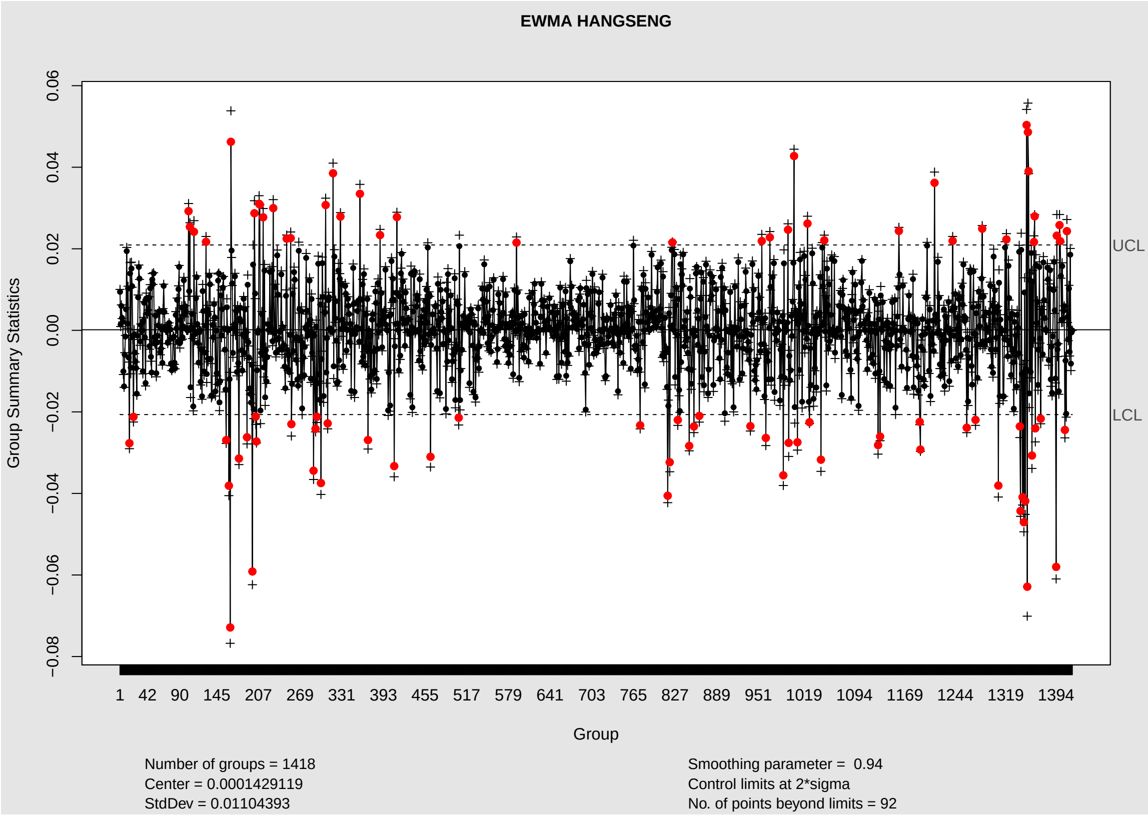

EWMA CAC40
Group Summary Statistics
0.05
0.00
−0.05
−0.10
UCL
LCL
1 42 90 145 207 269 331 393 455 517 579 641 703 765 827 889 951 1019 1094 1169 1244 1319 1394
Group
Number of groups = 1418
Center = 0.0001301072
StdDev = 0.01082107
Smoothing parameter = 0.94
Control limits at 2*sigma
No. of points beyond limits = 113

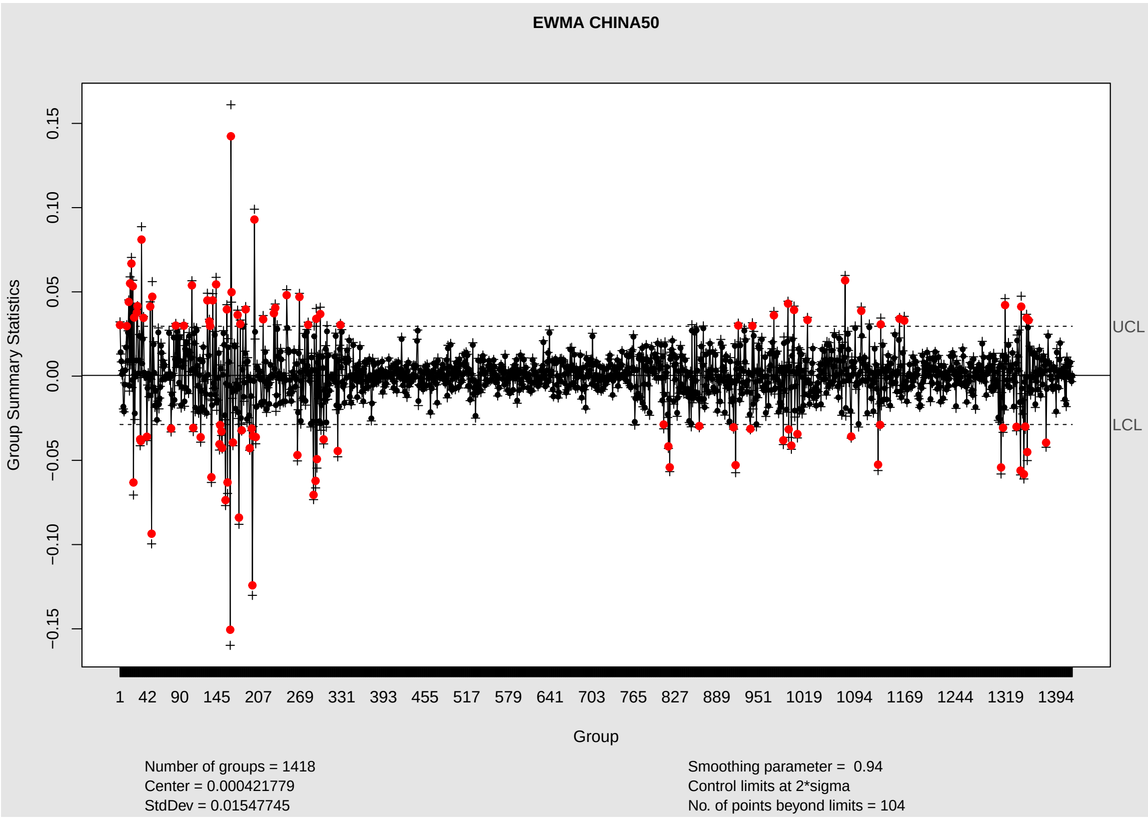

EWMA FTSE100
Group Summary Statistics
Group
UCL
LCL
Number of groups = 1418
Center = −4.685586e−05
StdDev = 0.009267366
Smoothing parameter = 0.94
Control limits at 2*sigma
No. of points beyond limits = 108

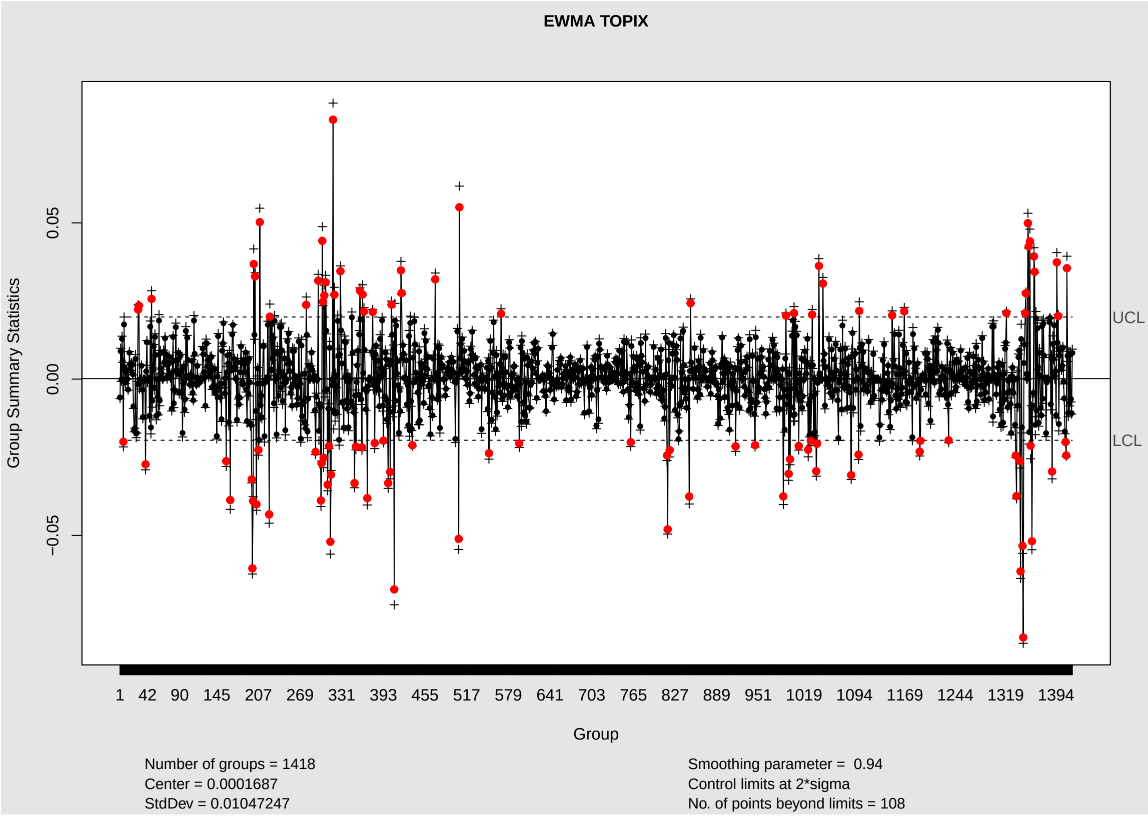

EWMA FTSEMIB
Group Summary Statistics
Group
UCL
LCL
Number of groups = 1418
Center = 5.543939e−05
StdDev = 0.01317496
Smoothing parameter = 0.94
Control limits at 2*sigma
No. of points beyond limits = 97


EWMA SP500
Group Summary Statistics
0.10
0.05
0.00
−0.05
−0.10
UCL
LCL
1 42 90 145 207 269 331 393 455 517 579 641 703 765 827 889 951 1019 1094 1169 1244 1319 1394
Group
Number of groups = 1418
Center = 0.0002784529
StdDev = 0.009163892
Smoothing parameter = 0.94
Control limits at 2*sigma
No. of points beyond limits = 102


EWMA NASDAQ
Group Summary Statistics
0.10
0.05
0.00
−0.05
−0.10
UCL
LCL
1 42 90 145 207 269 331 393 455 517 579 641 703 765 827 889 951 1019 1094 1169 1244 1319 1394
Group
Number of groups = 1418
Center = 0.0006105464
StdDev = 0.01120885
Smoothing parameter = 0.94
Control limits at 2*sigma
No. of points beyond limits = 95


EWMA HANGSENG
Group Summary Statistics
0.06
0.04
0.02
0.00
−0.02
−0.04
−0.06
−0.08
UCL
LCL
1 42 90 145 207 269 331 393 455 517 579 641 703 765 827 889 951 1019 1094 1169 1244 1319 1394
Group
Number of groups = 1418
Center = 0.0001429119
StdDev = 0.01104393
Smoothing parameter = 0.94
Control limits at 2*sigma
No. of points beyond limits = 92


EWMA CHINA50
Group Summary Statistics
0.15
0.10
0.05
0.00
−0.05
−0.10
−0.15
UCL
LCL
1 42 90 145 207 269 331 393 455 517 579 641 703 765 827 889 951 1019 1094 1169 1244 1319 1394
Group
Number of groups = 1418
Center = 0.000421779
StdDev = 0.01547745
Smoothing parameter = 0.94
Control limits at 2*sigma
No. of points beyond limits = 104



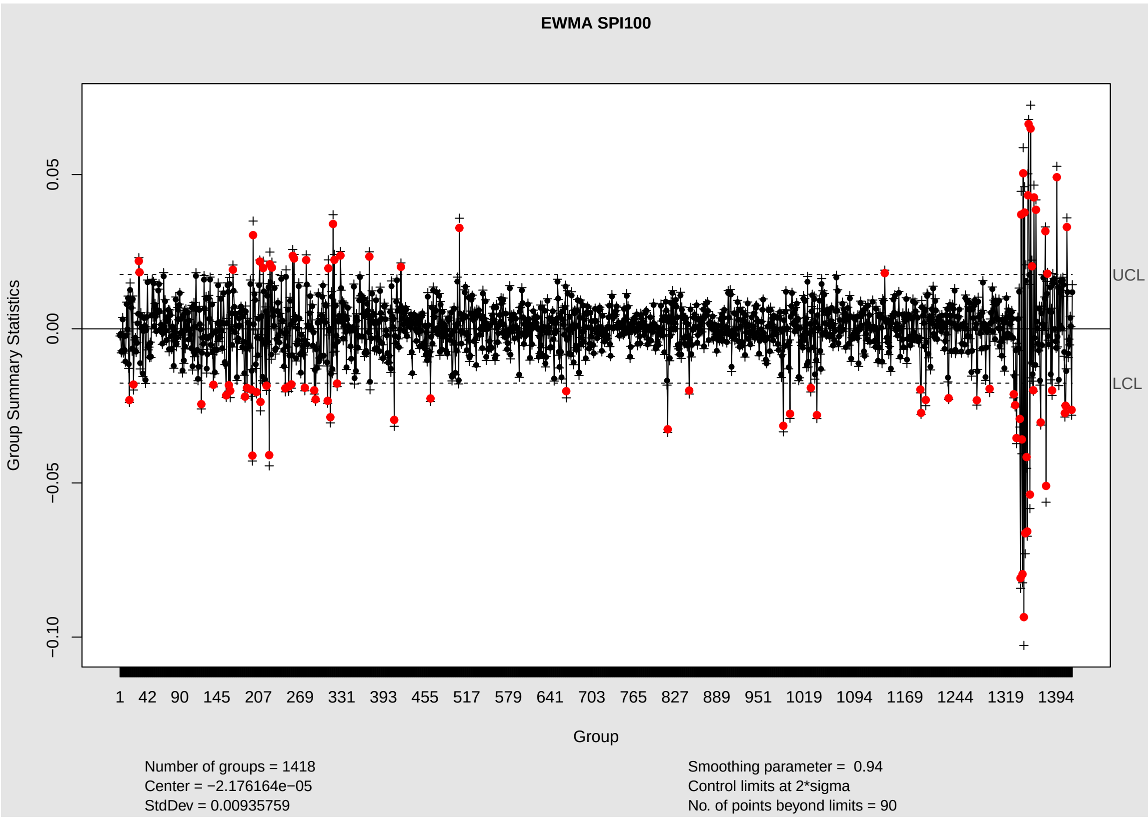




Figure 5: EWMA model with the confidence intervals

As I already explained, the EWMA model is a simple model because use only a parameter $\lambda$, suggested by J.P. Morgan to be set at 0.94 for daily returns. As we can see, in the graphs there are many points that go beyond the confidence intervals. For example, just the DAX 30 shows 18 points on the range of the last year. Every graph shows the unconditional standard deviation and the number of points beyond limits. in the DAX 30 index, the number of points beyond limits are 107, having 1418 observations means that the time series exceeds the 7.5% of times under evaluating the risk. The others indices shows the same thing, therefore this model is confirmed as a underestimation of the volatility in the empirical analysis. Among them, there are differences in volatility, range of volatility and number of points beyond the limits but in complexity they give us the information of the wrong model. The China 50, for example, has highest volatility with a standard deviation of 15%. The main disadvantage of the EWMA model is the fact that $\lambda$ is constant and identical for all assets. This implies that it is not optimal for any asset in the sense that the GARCH models discussed above are optimal. The EWMA model by definition gives inferior forecasts compared with GARCH models, even though the difference can be very small in many cases.

## 3.8 The structure of the GARCH VIX model

The majority of volatility forecast models in regular use belongs to the GARCH family of models. Our model is an implementation of the GARCH model using the VIX as an exogenous variable. Therefore, I use a conditional volatility model with the implied volatility determined with the variance swaps, explained above. Our approach is based on the fact that VIX approximates the 30-day variance swap rate on the S&P 500. The VIX can be interpreted to measure the risk-neutral expectation of integrated variance within the months, providing forward-looking parameter estimates during the risk-neutral measure.[22]

GARCH model parameters are often estimated by the Maximum Likelihood Estimation (MLE) method using return time series. The goal of the model is use information of the VIX index to estimate GARCH models and to improve their performance. Parameter estimates are found by maximizing the likelihood of returns and VIX. Therefore, I estimate the parameters $\omega, \alpha, \beta$ and $\delta$, useful to evaluate how the model fit. First of all, I explain the MLE and I estimate the parameters considering the GARCH model and the GARCHVIX model. Then, I compare to evaluate which one fit better than the other one.

[22] See, Kanniainen, Binghuan & Yang (2014).[21]

## 3.9 The Maximum Likelihood

This section introduces the maximum likelihood estimation useful for non-linear models instead of the regression methods such as ordinary least squared (OLS), useful in linear models. Bollerslev and Wooldridge (1992)[23] demonstrate that using the normal distribution in maximum likelihood estimation will give consistent parameter estimates if the sole aim is estimation of conditional variance, even if the true density is non-normal. This estimator is known as the quasi-maximum likelihood (QML) estimator. However, QML is not efficient unless the true density actually is normal.

So, I estimate the parameters with the Maximum Likelihood in the following way:

Let $Y_n = (Y_1, ......, Y_n)'$ be a random sample and let $\theta = (\theta_1, ....., \theta_p)'$ be a vector of parameters. Let $f(Y_n|\theta)$ be the conjoint density of $Y_n$, which depends on the parameters. If $Y_n$ is an iid sample, then:

$$f(Y_n|\theta) = \prod_{i=1}^{n} f(Y_i|\theta)$$

viewed as a function of $\theta$, is called the Likelihood function. The Maximum Likelihood Estimator (MLE) is the value of $\theta$ that maximizes the likelihood function.

The theta of the model GARCHVIX are omega, alpha, beta and delta taking into account the thresold of every parameter.

[23]See, Bollerslev, Wooldridge (1992).[18]

$$\theta = [\omega; \alpha; \beta; \delta]$$

Assume the errors, $\epsilon_t$ in a GARCH (p,q) VIX(s) model are standard normally distributed:

$$Y_t = \sigma_t \epsilon_t$$

$$\epsilon_t \sim N(0,1).$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i Y_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 + \sum_{s=1}^{s} \delta_s VIX_{t-s}^2$$

where $\alpha_i$, $\beta_j$ and $\delta_s$ are parameters of the model GARCHVIX(p,q,s). The presence of lagged returns means that the density function for $t = 1$ is unknown since we do not know $y_0$.

$$\prod_{i=1}^{T} f_{Y_i}(Y_i, \theta) = -\frac{T-1}{2} log(2\pi) - \frac{1}{2} \sum_{t=2}^{T} log(\sigma_t^2) - \frac{1}{2} \sum_{t=2}^{T} \frac{Y_t^2}{\sigma_t^2}$$

Therefore, substituting the $\sigma_t^2$ in the corresponding function of the maximum likelihood I obtain:

$$\prod_{i=1}^{T} f_{Y_i}(Y_i, \theta) = -\frac{T-1}{2} log(2\pi) - \frac{1}{2} \sum_{t=2}^{T} log(\omega + \sum_{i=1}^{p} \alpha_i Y_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 + \sum_{s=1}^{s} \delta_s VIX_{t-s}^2) \ldots$$

$$\cdots - \frac{1}{2}\sum_{t=2}^{T} \frac{Y_t^2}{\omega + \sum_{i=1}^{p} \alpha_i Y_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 + \sum_{s=1}^{s} \delta_s VIX_{t-s}^2}$$

The output is the GARCHVIX model derived from a normal distribution using the MLE. The parameter estimates are obtained by numerically maximizing the likelihood function with an algorithm called an optimizer. This can lead to numerical problems that adversely affect maximization or numerical instability. Anyway, I created the model in MATLAB and I estimate the parameters with the function fmincon.

### 3.10 Estimation

We can estimate the parameters using the maximum likelihood in MATLAB. I created a model computing the likelihood function of a normal distribution adding the VIX as a exogenous variable. Then I created a model, after I inserted the constraints to guarantee the positive variance. The output of the function fmincon give me the value of the maximum likelihood, estimate of the parameters and the hessian. I did the inverse of the hessian and I calculated the variance. Then I extrapolated the standard error of every parameter using the square root and then I calculated the test statistics of every parameter. We have used a GARCH(1,1) VIX(1) as a construction of the model and then I compare with the GARCH(1,1) model. Therefore,

$$\prod_{i=1}^{T} f_{Y_i}(y_i,\theta) = -\frac{T-1}{2} log(2\pi) - \frac{1}{2}\sum_t log(\omega + \alpha_1 Y_{t-1}^2 + \beta_1 \sigma_{t-1}^2 + \delta_1 VIX_{t-1}^2) - \ldots$$

$$\ldots - \frac{1}{2}\sum_t \frac{Y_t^2}{\omega + \alpha_1 Y_{t-1}^2 + \beta_1 \sigma_{t-1}^2 + \delta_1 VIX_{t-1}^2}$$

where the p,q and s becomes 1,1,1.

After a computation in MATLAB of the function maximum likelihood of the model, the management of the data and the constraints I estimated the parameter. The model created is therefore a conditional variance with time t using the MLE to estimate the parameters of the historical returns and of the historical conditional variance, therefore a GARCH adding the historical returns of the VIX. The parameters are estimated for every index and below the estimate of every parameter there is the t-statistics. When the t-statistics is below 1.96 is statistically significant reject the parameter with a normal distribution.

The output by MATLAB of the fmincon function are:

Table 6: Estimated parameters with the maximum likelihood

| | omega | alpha | beta | delta |
|---|---|---|---|---|
| **DAX 30** | 0.0001 (19.43) | 0.2065 (5.38) | 0.7915 (8.63) | 0.0138 (4.12) |
| **CAC 40** | 0.0001 (21.82) | 0.3108 (82.42) | 0.6891 (60.40) | 0.0157 (11.77) |
| **FTSE 100** | 0.0001 (19.53) | 0.5343 (10.46) | 0.4653 (0.44) | 0.0033 (3.74) |
| **FTSE MIB** | 0.0001 (20.93) | 0.1927 (4.53) | 0.7988 (1.62) | 0.0188 (7.08) |
| **S&P 500** | 0.0001 (21.54) | 0.6086 (7.42) | 0.3914 (3.86) | 0.0007 (0.92) |
| **NASDAQ** | 0.0001 (20.77) | 0.4799 (9.64) | 0.519 (2.06) | 0.0005 (0.47) |
| **HANG SENG** | 0.0001 (20.28) | 0.1216 (7.75) | 0.871 (6.65) | 0.0117 (6.24) |
| **CHINA 50** | 0.0002 (22.11) | 0.277 (6.74) | 0.7201 (0.25) | 0.0059 (2.40) |
| **TOPIX** | 0.0001 (17.74) | 0.1915 (5.00) | 0.8041 (1.31) | 0.0176 (7.81) |
| **SPI 100** | 0.0001 (17.37) | 0.424 (9.66) | 0.576 (2.03) | 0.0074 (6.78) |

In the table 6 we have the estimated parameters of every index. Below the parameter there is the test statistics. A test statistics contains information about the data that is relevant for deciding whether to reject the null hypothesis. As we can see we have the parameters of the S&P 500 and the NASDAQ where there is strong significance levels to reject the parameter $\delta$. Below the 0.05, therefore the 1.96 in a normal distribution, the parameter is irrelevant in the model. The correlation between these two indices is very high and the result follows the criteria that to analyze one of the other give me the same result. The others indices does not reject the $\delta$ parameter give us the information that the model fit better with the $\delta$ parameter. This is not true in the case of $\beta$ parameter where some indices reject it because is statistically significant.

### 3.11 Diagnosing volatility models

There are several statistics methods to compare models. We can use standard methods such as the t-test to see whether the parameters are statistically significantly different from zero or not. We start using the Likelihood Ratio test (LR test) that is used when one model nests inside another model. We considered the unrestricted model the GARCHVIX(1,1,1) and the restricted model the GARCH(1,1). The restricted log-likelihood minus the unrestricted log-likelihood, doubled, follows the chi-squared distribution, with the degrees of freedom equalling the number of restrictions. In this case we have only one degree of freedom an therefore is 3.84 with an $\alpha$ of 5%. I have also computed the Akaike information criterion (AIC) and the Bayesan information criterion (BIC), the information criteria test to select the model with the lowest value.

Table 7: Diagnostics table

| | MLE GARCH (1,1) | MLE GARCH VIX (1,1,1) | LR TEST | AIC GARCH | AIC GARCH VIX | BIC GARCH | BIC GARCH VIX |
|---|---|---|---|---|---|---|---|
| DAX 30 | 4179.2 | 4208.9 | 59.4 | -8354.4 | -8411.8 | -8343.5 | -8395.45 |
| CAC 40 | 4214.7 | 4265 | 100.6 | -8425.4 | -8524 | -8414.5 | -8507.65 |
| FTSE 100 | 4548.3 | 4556.6 | 16.6 | -9092.6 | -9107.2 | -9081.7 | -9090.85 |
| FTSE MIB | 3974.7 | 4016.2 | 83 | -7945.4 | -8026.4 | -7934.5 | -8010.05 |
| SP 500 | 4571.9 | 4572.2 | 0.6 | -9139.8 | -9138.4 | -9128.9 | -9122.05 |
| NASDAQ | 4303.8 | 4303.9 | 0.2 | -8603.6 | -8601.8 | -8592.7 | -8585.45 |
| HANG SENG | 4259 | 4289 | 60 | -8514 | -8572 | -8503.1 | -8555.65 |
| CHINA 50 | 3709.3 | 3713.3 | 8 | -7414.6 | -7420.6 | -7403.7 | -7404.25 |
| TOPIX | 4308.4 | 4366.4 | 116 | -8612.8 | -8726.8 | -8601.9 | -8710.45 |
| SPI 100 | 4626.5 | 4674.4 | 95.8 | -9249 | -9342.8 | -9238.1 | -9326.45 |

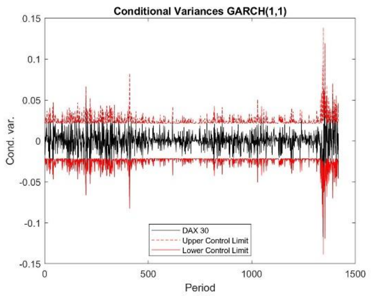
The table 7 shows the LR test and the information criteria. We can see that the LR test confirm a better model when we have a GARCHVIX model rather than the GARCH model in every index with two exception, the S%P 500 and the NASDAQ. This is what I determined before with the test statistics using the maximum likelihood. The results does not change and the AIC and BIC give us the same result.

## 3.12 Backtesting and VaR

We can diagnose individual models by testing for parameter significance or analyzing residuals, but these methods often do not properly address the risk-forecasting property of the models under consideration. We consider an important tool used in risk management, the Value-at-risk (VaR). It is a measure of risk exposure associated with a particular portfolio of assets. The VaR of a portfolio is defined as the maximum loss occurring within a specified time and with a give probability. The validity of the VaR measure is then backtested by comparing the number of expections. Backtesting is a procedure that can be used to compare the various risk models. It aims to take ex ante Value at risk (VaR) forecasts from a particular model and compare them with ex post realized return (historical observations). When ever losses exceed VaR, a VaR violation is said to have occurred. Models that do not perform well during backtesting should have their assumptions and parameter estimates questioned. With the backtesting we can consider if the model overestimate or underestimate the conditional variance. We evaluate the violation ratios, therefore every value that go beyond limits of $2{*}\sigma_t$ for a GARCH and for a GARCHVIX model. We show the table 8 where the percentile should be near 5% to be an accurate model in the time series. Then, I computed the graphs on the GARCH model on the left and the GARCHVIX model on the right.

Table 8: Backtesting

| | EWMA | GARCH(1,1) | GARCHVIX(1,1,1) |
|---|---|---|---|
| **DAX30** | 7.546% | 5.924% | 6.417% |
| **CAC40** | 7.969% | 5.501% | 5.853% |
| **FTSE100** | 7.616% | 5.501% | 5.853% |
| **FTSEMIB** | 6.841% | 4.654% | 5.148% |
| **SP500** | 7.193% | 5.642% | 5.501% |
| **NASDAQ** | 6.700% | 5.571% | 5.642% |
| **HANGSENG** | 6.488% | 5.783% | 5.571% |
| **CHINA50** | 7.334% | 5.501% | 5.289% |
| **TOPIX** | 7.616% | 5.078% | 5.571% |
| **SPI100** | 6.347% | 6.065% | 5.994% |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Cond. var.
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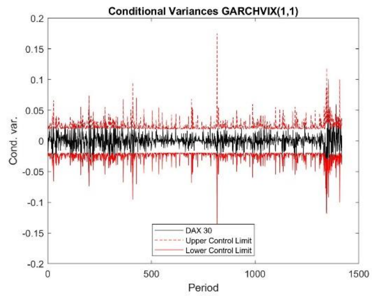
DAX 30
Upper Control Limit
Lower Control Limit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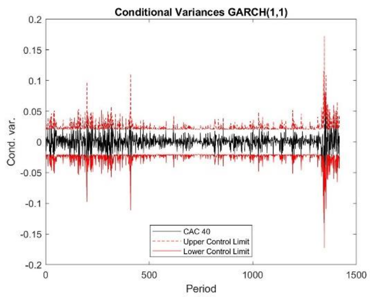
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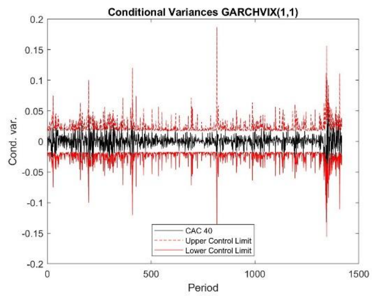
Lower Control Limit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Period
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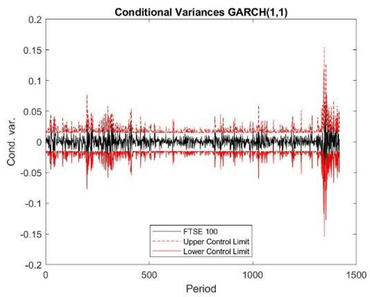
Upper Control Limit
Lower Control Limit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Period
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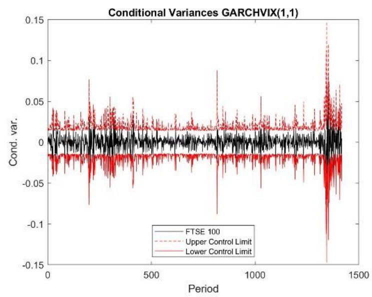
Upper Control Limit
Lower Control Limit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Cond. var.
Period
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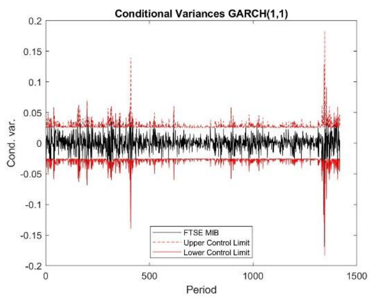
Upper Control Limit
Lower Control Limit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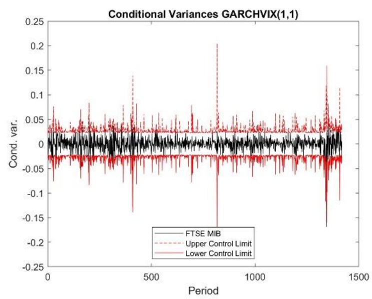
Cond. var.
Period
FTSE MIB
Upper Control Limit
Lower Control Limit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Period
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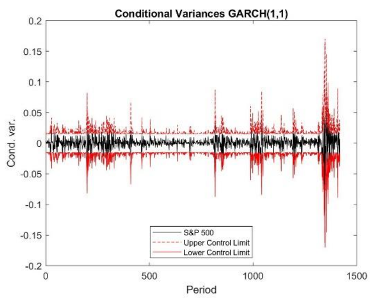
Upper Control Limit
Lower Control Limit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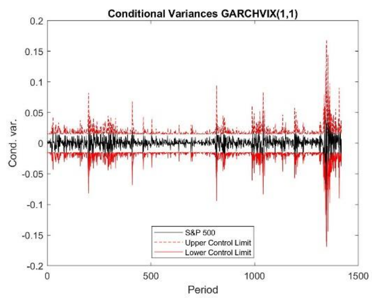
Period
S&P 500
Upper Control Limit
Lower Control Limit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Cond. var.
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S&P 500
Upper Control Limit
Lower Control Limit

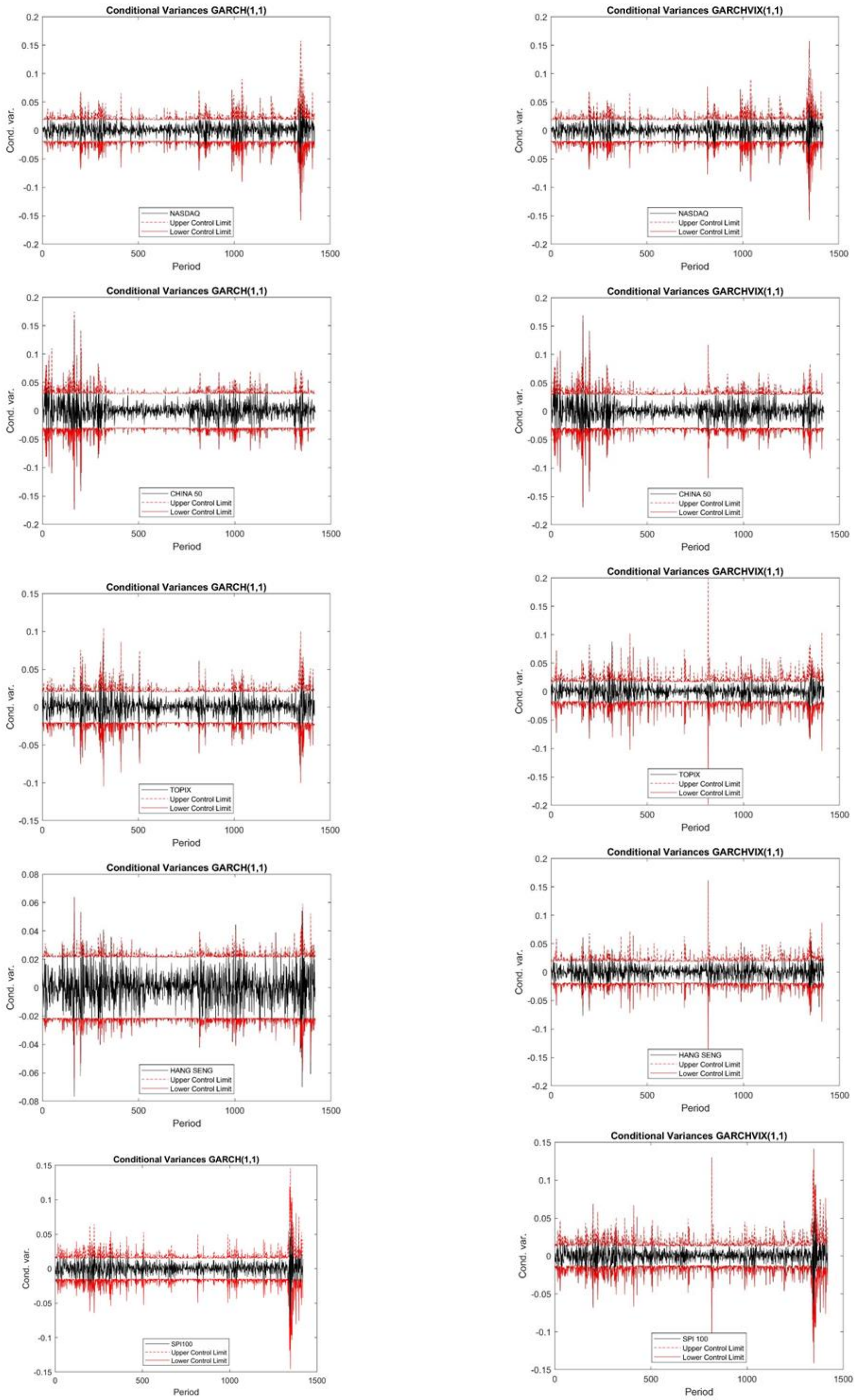


Figure 6: Conditional variance

Unfortunately, the GARCHVIX model is better only in some cases analyzing the time series. I have painted in yellow the best results and we do not have a single solution for every time series but depends on the time series. The results here, therefore, suggest that including VIX in the GARCH variance equation does not unequivocally increase the forecast power of the GARCH model. Another approach to evaluate the market risk and therefore the violation ratio we can consider the Expected Shortfall, more sensitive to the shape of the tail of the loss distribution. Expected shortfall (ES) is considered a more useful risk measure than VaR because it is a coherent, and moreover a spectral, measure of financial portfolio risk. It is calculated for a given quantile-level q and in our analysis we consider a 5%.

Table 9: Expected Shortfall

| EX SHORTFALL | GARCH (1,1) | GARCHVIX (1,1,1) |
|---|---|---|
| DAX30 | 5.078% | 5.289% |
| CAC40 | 4.654% | 5.078% |
| FTSE100 | 5.148% | 4.937% |
| FTSEMIB | 4.020% | 4.372% |
| SP500 | 4.866% | 4.795% |
| NASDAQ | 4.795% | 4.795% |
| HANGSENG | 4.584% | 4.937% |
| CHINA50 | 4.372% | 4.372% |
| TOPIX | 4.513% | 5.007% |
| SPI100 | 5.007% | 5.148% |

The table shows that an accurate evaluation could be useful to determine the best model. In most of the cases the GARCHVIX model is better than the GARCH model, considering that sometimes underperform or overperform the conditional variance. Some cases show an equal evaluation in the both models showing no difference of violation ratio.

# 4 Concluding remarks

In this thesis we obtained an univariate approach of the model, appropriate to contextualize the rigorous analysis of different volatility methodologies focusing mainly in the GARCH model implementing the VIX Index. With an empirical analysis we analysed the impact of the volatility clustering of the major global indices focusing of an appropriate model that explain the time series. Within on this research the author works on the dangerousness of volatility in the financial markets estimating the parameters useful to determine the conditional variance. The aim is forecasting a potential increase of volatility and measure the risk of the portfolios, anticipating huge variations of volatility with relative uncertainty. In the first chapter we studied the evaluation of the VIX, how it is computed and how is changed from the creation of the index to now. Essentially, we eliminated the common assumption that the VIX index is calculated using the Black-Scholes Merton (BSM) model, explained the structure of the VIX as a variance swaps. We confirmed that the new model model is more accurate than the previous one using the forward-looking plain vanilla options of the Standard & Poor's 500 index. The model that we created, a GARCH with an implementation of the VIX index as a exogenous variable helps to determine if the volatility index is useful for the accuracy of the model. Not only with the GARCH model we compared our model but also with the EWMA model, considering a 5% of confidence on the variance. In the first part of the empirical research the author examined the major indices using the descriptive statistics and some relative tests useful to determine and fix the time series. After being ensured that the volatility clustering on time series and therefore that the constructed model is useful in the real world, the author estimated the parameters with the maximum likelihood estimator looked out to constraints of the model. After an estimate of the parameters the model was created and the conditional variance estimated. Then, the model compared with the GARCH and the EWMA investigating the accuracy of the conditional variance on the models. The backtesting was useful to determine, with the Value at Risk and the expected shortfall, which one is the best model and be the more accurate one. However, not all assets that we considered are characterized by the same trend and thus the model perform differently for every time series, Hence, applying the model, we notice these differences and we cannot confirm if one model is better than the other one. Summarizing, both the GARCH (1,1) and the GARCH VIX (1,1,1) model are useful and accurate to the time series, showing a different accuracy within of different time series, not too distant of the 5% of confidence. The backtesting using the expected shortfall give an accurate consistence of the estimate of the conditional variance providing an good result on the evaluation of the model. An interesting citation by George Box that said “All models are wrong but some are useful” explains how we should fit the model on the time series considering the volatility clustering. Future research should determine and analyse the forecasts of the conditional variance using ARIMA models. Then, consider different distributions as the student distribution or taking different assets or different temporal series on the evaluation. Another implementation could be using the state space model, using the state variables to describe a system by a set of first-order differential or difference equations. The state-space model provides a flexible approach to time series analysis, especially for simplifying maximum-likelihood estimation and handling missing values. This provides a good starting point for discussion and further research.

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# A Appendix

## A.1 A Appendix

### A Appendix: tests

Jarque-Bera test

In statistics the Jarque-Bera test is a test where the sample data have the skewness and kurtosis that match a normal distribution. The test statistics give always a positive result. The skewness of a normal distribution is 0 and the kurtosis is 3, when is far from zero, it means that the data do not have a normal distribution.

The test statistic JB is defined as:

$$JB = \frac{n}{6}(S^2 + \frac{1}{4}(K-3)^2) \tag{50}$$

where n is the number of observations (or degrees of freedom in general); S is the sample skewness and K is the sample kurtosis. If the data comes from a normal distribution, the Jarque Bera statistics has a chi-squared distribution with 2 degrees of freedom, so the statistics can be used to test the hypothesis that the data are from a normal distribution. The null hypothesis is a joint hypothesis of the skewness being zero and the excess kurtosis being zero. Samples from a normal distribution have an expected skewness of 0 and an expected excess kurtosis of 0 (which is the same as a kurtosis of 3). As the definition of JB shows, any deviation from this increases the JB statistic.

Ljung-Box test

The Ljung-Box test is a type of statistical test of whether any of a group of

autocorrelations of a time series are different from zero. Instead of testing randomness at each distinct lag, it tests the "overall" randomness based on a number of lags, and is therefore a portmanteau test. The test statistic is:

$$Q = n(n+2)\sum_{k=1}^{h} \frac{\rho_k^2}{n-k} \tag{51}$$

where n is the sample size, $\rho_k$ is the sample autocorrelation at lag k, and h is the number of lags being tested. Under $H_0$ the statistic Q asymptotically follows a $\chi_h^2$. For significance level a, the critical region for rejection of the hypothesis of randomness is:

$$Q > \chi^2_{1-\alpha,h} \tag{52}$$

where $\chi^2_{1-\alpha,h}$ is the 1-a-quantile of the chi-squared distribution with h degrees of freedom.

## A.2 B Appendix

# B Appendix: table

### Table of the standard normal density function

$$\phi(z) = \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}$$

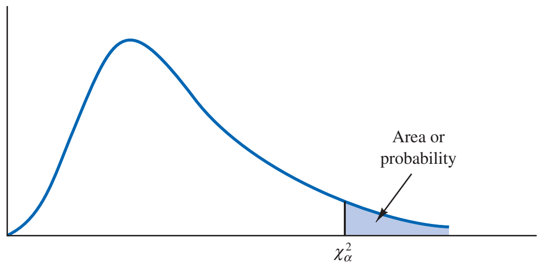
| | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.39894 | 0.39892 | 0.39886 | 0.39876 | 0.39862 | 0.39844 | 0.39822 | 0.39797 | 0.39767 | 0.39733 |
| 0.1 | 0.39695 | 0.39654 | 0.39608 | 0.39559 | 0.39505 | 0.39448 | 0.39387 | 0.39322 | 0.39253 | 0.39181 |
| 0.2 | 0.39104 | 0.39024 | 0.38940 | 0.38853 | 0.38762 | 0.38667 | 0.38568 | 0.38466 | 0.38361 | 0.38251 |
| 0.3 | 0.38139 | 0.38023 | 0.37903 | 0.37780 | 0.37654 | 0.37524 | 0.37391 | 0.37255 | 0.37115 | 0.36973 |
| 0.4 | 0.36827 | 0.36678 | 0.36526 | 0.36371 | 0.36213 | 0.36053 | 0.35889 | 0.35723 | 0.35553 | 0.35381 |
| 0.5 | 0.35207 | 0.35029 | 0.34849 | 0.34667 | 0.34482 | 0.34294 | 0.34105 | 0.33912 | 0.33718 | 0.33521 |
| 0.6 | 0.33322 | 0.33121 | 0.32918 | 0.32713 | 0.32506 | 0.32297 | 0.32086 | 0.31874 | 0.31659 | 0.31443 |
| 0.7 | 0.31225 | 0.31006 | 0.30785 | 0.30563 | 0.30339 | 0.30114 | 0.29887 | 0.29659 | 0.29431 | 0.29200 |
| 0.8 | 0.28969 | 0.28737 | 0.28504 | 0.28269 | 0.28034 | 0.27798 | 0.27562 | 0.27324 | 0.27086 | 0.26848 |
| 0.9 | 0.26609 | 0.26369 | 0.26129 | 0.25888 | 0.25647 | 0.25406 | 0.25164 | 0.24923 | 0.24681 | 0.24439 |
| 1.0 | 0.24197 | 0.23955 | 0.23713 | 0.23471 | 0.23230 | 0.22988 | 0.22747 | 0.22506 | 0.22265 | 0.22025 |
| 1.1 | 0.21785 | 0.21546 | 0.21307 | 0.21069 | 0.20831 | 0.20594 | 0.20357 | 0.20121 | 0.19886 | 0.19652 |
| 1.2 | 0.19419 | 0.19186 | 0.18954 | 0.18724 | 0.18494 | 0.18265 | 0.18037 | 0.17810 | 0.17585 | 0.17360 |
| 1.3 | 0.17137 | 0.16915 | 0.16694 | 0.16474 | 0.16256 | 0.16038 | 0.15822 | 0.15608 | 0.15395 | 0.15183 |
| 1.4 | 0.14973 | 0.14764 | 0.14556 | 0.14350 | 0.14146 | 0.13943 | 0.13742 | 0.13542 | 0.13344 | 0.13147 |
| 1.5 | 0.12952 | 0.12758 | 0.12566 | 0.12376 | 0.12188 | 0.12001 | 0.11816 | 0.11632 | 0.11450 | 0.11270 |
| 1.6 | 0.11092 | 0.10915 | 0.10741 | 0.10567 | 0.10396 | 0.10226 | 0.10059 | 0.09893 | 0.09728 | 0.09566 |
| 1.7 | 0.09405 | 0.09246 | 0.09089 | 0.08933 | 0.08780 | 0.08628 | 0.08478 | 0.08329 | 0.08183 | 0.08038 |
| 1.8 | 0.07895 | 0.07754 | 0.07614 | 0.07477 | 0.07341 | 0.07206 | 0.07074 | 0.06943 | 0.06814 | 0.06687 |
| 1.9 | 0.06562 | 0.06438 | 0.06316 | 0.06195 | 0.06077 | 0.05959 | 0.05844 | 0.05730 | 0.05618 | 0.05508 |
| 2.0 | 0.05399 | 0.05292 | 0.05186 | 0.05082 | 0.04980 | 0.04879 | 0.04780 | 0.04682 | 0.04586 | 0.04491 |
| 2.1 | 0.04398 | 0.04307 | 0.04217 | 0.04128 | 0.04041 | 0.03955 | 0.03871 | 0.03788 | 0.03706 | 0.03626 |
| 2.2 | 0.03547 | 0.03470 | 0.03394 | 0.03319 | 0.03246 | 0.03174 | 0.03103 | 0.03034 | 0.02965 | 0.02898 |
| 2.3 | 0.02833 | 0.02768 | 0.02705 | 0.02643 | 0.02582 | 0.02522 | 0.02463 | 0.02406 | 0.02349 | 0.02294 |
| 2.4 | 0.02239 | 0.02186 | 0.02134 | 0.02083 | 0.02033 | 0.01984 | 0.01936 | 0.01888 | 0.01842 | 0.01797 |
| 2.5 | 0.01753 | 0.01709 | 0.01667 | 0.01625 | 0.01585 | 0.01545 | 0.01506 | 0.01468 | 0.01431 | 0.01394 |
| 2.6 | 0.01358 | 0.01323 | 0.01289 | 0.01256 | 0.01223 | 0.01191 | 0.01160 | 0.01130 | 0.01100 | 0.01071 |
| 2.7 | 0.01042 | 0.01014 | 0.00987 | 0.00961 | 0.00935 | 0.00909 | 0.00885 | 0.00861 | 0.00837 | 0.00814 |
| 2.8 | 0.00792 | 0.00770 | 0.00748 | 0.00727 | 0.00707 | 0.00687 | 0.00668 | 0.00649 | 0.00631 | 0.00613 |
| 2.9 | 0.00595 | 0.00578 | 0.00562 | 0.00545 | 0.00530 | 0.00514 | 0.00499 | 0.00485 | 0.00470 | 0.00457 |
| 3.0 | 0.00443 | 0.00430 | 0.00417 | 0.00405 | 0.00393 | 0.00381 | 0.00370 | 0.00358 | 0.00348 | 0.00337 |

## Table of the standard normal distribution

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$

| | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.50000 | 0.50399 | 0.50798 | 0.51197 | 0.51595 | 0.51994 | 0.52392 | 0.52790 | 0.53188 | 0.53586 |
| 0.1 | 0.53983 | 0.54380 | 0.54776 | 0.55172 | 0.55567 | 0.55962 | 0.56356 | 0.56749 | 0.57142 | 0.57535 |
| 0.2 | 0.57926 | 0.58317 | 0.58706 | 0.59095 | 0.59483 | 0.59871 | 0.60257 | 0.60642 | 0.61026 | 0.61409 |
| 0.3 | 0.61791 | 0.62172 | 0.62552 | 0.62930 | 0.63307 | 0.63683 | 0.64058 | 0.64431 | 0.64803 | 0.65173 |
| 0.4 | 0.65542 | 0.65910 | 0.66276 | 0.66640 | 0.67003 | 0.67364 | 0.67724 | 0.68082 | 0.68439 | 0.68793 |
| 0.5 | 0.69146 | 0.69497 | 0.69847 | 0.70194 | 0.70540 | 0.70884 | 0.71226 | 0.71566 | 0.71904 | 0.72240 |
| 0.6 | 0.72575 | 0.72907 | 0.73237 | 0.73565 | 0.73891 | 0.74215 | 0.74537 | 0.74857 | 0.75175 | 0.75490 |
| 0.7 | 0.75804 | 0.76115 | 0.76424 | 0.76730 | 0.77035 | 0.77337 | 0.77637 | 0.77935 | 0.78230 | 0.78524 |
| 0.8 | 0.78814 | 0.79103 | 0.79389 | 0.79673 | 0.79955 | 0.80234 | 0.80511 | 0.80785 | 0.81057 | 0.81327 |
| 0.9 | 0.81594 | 0.81859 | 0.82121 | 0.82381 | 0.82639 | 0.82894 | 0.83147 | 0.83398 | 0.83646 | 0.83891 |
| 1.0 | 0.84134 | 0.84375 | 0.84614 | 0.84849 | 0.85083 | 0.85314 | 0.85543 | 0.85769 | 0.85993 | 0.86214 |
| 1.1 | 0.86433 | 0.86650 | 0.86864 | 0.87076 | 0.87286 | 0.87493 | 0.87698 | 0.87900 | 0.88100 | 0.88298 |
| 1.2 | 0.88493 | 0.88686 | 0.88877 | 0.89065 | 0.89251 | 0.89435 | 0.89617 | 0.89796 | 0.89973 | 0.90147 |
| 1.3 | 0.90320 | 0.90490 | 0.90658 | 0.90824 | 0.90988 | 0.91149 | 0.91309 | 0.91466 | 0.91621 | 0.91774 |
| 1.4 | 0.91924 | 0.92073 | 0.92220 | 0.92364 | 0.92507 | 0.92647 | 0.92785 | 0.92922 | 0.93056 | 0.93189 |
| 1.5 | 0.93319 | 0.93448 | 0.93574 | 0.93699 | 0.93822 | 0.93943 | 0.94062 | 0.94179 | 0.94295 | 0.94408 |
| 1.6 | 0.94520 | 0.94630 | 0.94738 | 0.94845 | 0.94950 | 0.95053 | 0.95154 | 0.95254 | 0.95352 | 0.95449 |
| 1.7 | 0.95543 | 0.95637 | 0.95728 | 0.95818 | 0.95907 | 0.95994 | 0.96080 | 0.96164 | 0.96246 | 0.96327 |
| 1.8 | 0.96407 | 0.96485 | 0.96562 | 0.96638 | 0.96712 | 0.96784 | 0.96856 | 0.96926 | 0.96995 | 0.97062 |
| 1.9 | 0.97128 | 0.97193 | 0.97257 | 0.97320 | 0.97381 | 0.97441 | 0.97500 | 0.97558 | 0.97615 | 0.97670 |
| 2.0 | 0.97725 | 0.97778 | 0.97831 | 0.97882 | 0.97932 | 0.97982 | 0.98030 | 0.98077 | 0.98124 | 0.98169 |
| 2.1 | 0.98214 | 0.98257 | 0.98300 | 0.98341 | 0.98382 | 0.98422 | 0.98461 | 0.98500 | 0.98537 | 0.98574 |
| 2.2 | 0.98610 | 0.98645 | 0.98679 | 0.98713 | 0.98745 | 0.98778 | 0.98809 | 0.98840 | 0.98870 | 0.98899 |
| 2.3 | 0.98928 | 0.98956 | 0.98983 | 0.99010 | 0.99036 | 0.99061 | 0.99086 | 0.99111 | 0.99134 | 0.99158 |
| 2.4 | 0.99180 | 0.99202 | 0.99224 | 0.99245 | 0.99266 | 0.99286 | 0.99305 | 0.99324 | 0.99343 | 0.99361 |
| 2.5 | 0.99379 | 0.99396 | 0.99413 | 0.99430 | 0.99446 | 0.99461 | 0.99477 | 0.99492 | 0.99506 | 0.99520 |
| 2.6 | 0.99534 | 0.99547 | 0.99560 | 0.99573 | 0.99585 | 0.99598 | 0.99609 | 0.99621 | 0.99632 | 0.99643 |
| 2.7 | 0.99653 | 0.99664 | 0.99674 | 0.99683 | 0.99693 | 0.99702 | 0.99711 | 0.99720 | 0.99728 | 0.99736 |
| 2.8 | 0.99744 | 0.99752 | 0.99760 | 0.99767 | 0.99774 | 0.99781 | 0.99788 | 0.99795 | 0.99801 | 0.99807 |
| 2.9 | 0.99813 | 0.99819 | 0.99825 | 0.99831 | 0.99836 | 0.99841 | 0.99846 | 0.99851 | 0.99856 | 0.99861 |
| 3.0 | 0.99865 | 0.99869 | 0.99874 | 0.99878 | 0.99882 | 0.99886 | 0.99889 | 0.99893 | 0.99896 | 0.99900 |

**TABLE 3** CHI-SQUARE DISTRIBUTION




Entries in the table give $\chi^2_\alpha$ values, where $\alpha$ is the area or probability in the upper tail of the chi-square distribution. For example, with 10 degrees of freedom and a .01 area in the upper tail, $\chi^2_{.01} = 23.209$.

| Degrees of Freedom | Area in Upper Tail | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | **.995** | **.99** | **.975** | **.95** | **.90** | **.10** | **.05** | **.025** | **.01** | **.005** |
| 1 | .000 | .000 | .001 | .004 | .016 | 2.706 | 3.841 | 5.024 | 6.635 | 7.879 |
| 2 | .010 | .020 | .051 | .103 | .211 | 4.605 | 5.991 | 7.378 | 9.210 | 10.597 |
| 3 | .072 | .115 | .216 | .352 | .584 | 6.251 | 7.815 | 9.348 | 11.345 | 12.838 |
| 4 | .207 | .297 | .484 | .711 | 1.064 | 7.779 | 9.488 | 11.143 | 13.277 | 14.860 |
| 5 | .412 | .554 | .831 | 1.145 | 1.610 | 9.236 | 11.070 | 12.832 | 15.086 | 16.750 |
| 6 | .676 | .872 | 1.237 | 1.635 | 2.204 | 10.645 | 12.592 | 14.449 | 16.812 | 18.548 |
| 7 | .989 | 1.239 | 1.690 | 2.167 | 2.833 | 12.017 | 14.067 | 16.013 | 18.475 | 20.278 |
| 8 | 1.344 | 1.647 | 2.180 | 2.733 | 3.490 | 13.362 | 15.507 | 17.535 | 20.090 | 21.955 |
| 9 | 1.735 | 2.088 | 2.700 | 3.325 | 4.168 | 14.684 | 16.919 | 19.023 | 21.666 | 23.589 |
| 10 | 2.156 | 2.558 | 3.247 | 3.940 | 4.865 | 15.987 | 18.307 | 20.483 | 23.209 | 25.188 |
| 11 | 2.603 | 3.053 | 3.816 | 4.575 | 5.578 | 17.275 | 19.675 | 21.920 | 24.725 | 26.757 |
| 12 | 3.074 | 3.571 | 4.404 | 5.226 | 6.304 | 18.549 | 21.026 | 23.337 | 26.217 | 28.300 |
| 13 | 3.565 | 4.107 | 5.009 | 5.892 | 7.041 | 19.812 | 22.362 | 24.736 | 27.688 | 29.819 |
| 14 | 4.075 | 4.660 | 5.629 | 6.571 | 7.790 | 21.064 | 23.685 | 26.119 | 29.141 | 31.319 |
| 15 | 4.601 | 5.229 | 6.262 | 7.261 | 8.547 | 22.307 | 24.996 | 27.488 | 30.578 | 32.801 |
| 16 | 5.142 | 5.812 | 6.908 | 7.962 | 9.312 | 23.542 | 26.296 | 28.845 | 32.000 | 34.267 |
| 17 | 5.697 | 6.408 | 7.564 | 8.672 | 10.085 | 24.769 | 27.587 | 30.191 | 33.409 | 35.718 |
| 18 | 6.265 | 7.015 | 8.231 | 9.390 | 10.865 | 25.989 | 28.869 | 31.526 | 34.805 | 37.156 |
| 19 | 6.844 | 7.633 | 8.907 | 10.117 | 11.651 | 27.204 | 30.144 | 32.852 | 36.191 | 38.582 |
| 20 | 7.434 | 8.260 | 9.591 | 10.851 | 12.443 | 28.412 | 31.410 | 34.170 | 37.566 | 39.997 |
| 21 | 8.034 | 8.897 | 10.283 | 11.591 | 13.240 | 29.615 | 32.671 | 35.479 | 38.932 | 41.401 |
| 22 | 8.643 | 9.542 | 10.982 | 12.338 | 14.041 | 30.813 | 33.924 | 36.781 | 40.289 | 42.796 |
| 23 | 9.260 | 10.196 | 11.689 | 13.091 | 14.848 | 32.007 | 35.172 | 38.076 | 41.638 | 44.181 |
| 24 | 9.886 | 10.856 | 12.401 | 13.848 | 15.659 | 33.196 | 36.415 | 39.364 | 42.980 | 45.558 |
| 25 | 10.520 | 11.524 | 13.120 | 14.611 | 16.473 | 34.382 | 37.652 | 40.646 | 44.314 | 46.928 |
| 26 | 11.160 | 12.198 | 13.844 | 15.379 | 17.292 | 35.563 | 38.885 | 41.923 | 45.642 | 48.290 |
| 27 | 11.808 | 12.878 | 14.573 | 16.151 | 18.114 | 36.741 | 40.113 | 43.195 | 46.963 | 49.645 |
| 28 | 12.461 | 13.565 | 15.308 | 16.928 | 18.939 | 37.916 | 41.337 | 44.461 | 48.278 | 50.994 |
| 29 | 13.121 | 14.256 | 16.047 | 17.708 | 19.768 | 39.087 | 42.557 | 45.722 | 49.588 | 52.335 |