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The Fractal Market Theory and its application in a trading system

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Chapter 1

Introduction

Since the birth of financial markets, both analysts and investors have tried to find a model that can efficiently explain (and possibly predict) the path the markets are following. Knowing such model, investors can try to speculate on price changes both in the short- or in the long-term period through different investment strategies.
Among all the theories that has been developed over the years, one of the most important is the *Efficient Market Hypothesis*, proposed by Eugene Fama in 1970. Many other scholars gave their contribution, either directly or indirectly, to the EMH that is still studied and referred to as the basis of modern risk-based theories of asset prices.

However, despite the success this theory had over the years, some scholars claim that EMH does not explain well the behaviour of financial markets. Indeed, it has been demonstrated and proved that this theory has many fallacies and does not efficiently represent the evolution of prices in real markets. Then many alternative theories has been proposed, and in this dissertation we are going to analyse the *Fractal Market Hypothesis* proposed by Edgar E. Peters in 1991. This theory solves all the inconsistencies that were present in the *Efficient Market Hypothesis* and analyses markets from a different perspective, allowing the existence of long-term memory in market prices and offering new possibilities for investors.

As the name suggest, the FMH recalls some concepts from fractals. Although there is not a rigorous definition, in mathematics a fractal is a particular figure which, when zoomed in, exhibits local patterns that are similar to the whole figure. This property is called *self-similarity*, and it is one of the feature of the model proposed by Peters. To better understand the concepts that we are going to use in this dissertation, the main characteristics of fractals and fractal geometry will be deeply analysed and described with examples before introducing the *Fractal Market Hypothesis*.

Furthermore, fractals are present also in the trading strategy we decided to apply to some time series of market prices. The strategy, which is merely speculative and do not have any hedging purpose, is developed on the same assumptions of the FMH and applies some concepts of fractal geometry to a new technical indicator used to take investment decisions. The time series to which we will test this trading strategy are six highly-traded American futures traded at the Chicago Mercantile Exchange, and are four indices and two currency couples.
Before moving to the technical application, some important concepts about trading are explained in-depth in order to better understand the investment

strategy that will be adopted later. Hence, we provide a brief introduction to trading and investing techniques, presenting the various types of investors and describing the tools used in a trading strategy such as candlestick charts and indicators. Also, we briefly present the paper produced by Petr Kroha and Miroslav Škoula, the two Czech scholars that developed the new technical indicator we are going to use in our application.

The strategy will be computed in five different ways for each instrument. Each method uses a different procedure to estimate the value of the parameter used by the technical indicator, and so different results are obtained. In addition, for each instrument a similar strategy using a traditional indicator (the Moving Average Convergence/Divergence) will be applied and used as a benchmark.
Then, performances of the trading strategy will be evaluated for all the instruments and compared. At last, after a few clarifications on some trading aspects, conclusions will be drawn along with some suggestions for further analyses.

Chapter 2

The traditional theory

2.1 Louis Bachelier and the *Théorie de la spéculation*

The first important contribution towards the theory that will become the *Efficient Market Hypothesis* has been given by Louis Bachelier in his *Théorie de la spéculation*, published in 1900. In this dissertation the French mathematician models the fluctuations in stock prices using a random walk process. By stating so, he argues that the small fluctuations in price seen over a short time interval should be independent of the current value of the price. Furthermore, he also assumes them to be independent of past behaviour of the process. Combining these assumptions with the Central Limit Theorem he came to the conclusion that increments of the process are also independent and normally distributed. Indeed, by taking all these steps he just obtained a Brownian motion process as the diffusion limit of random walk. Hence, according to Bachelier, price changes cannot be predicted due to the countless variables that affect their evolution, but it can be assigned a probability to each future possible event in order to evaluate their chance of happening.

2.2 Eugene Fama and the *Efficient Market Hypothesis*

In the first years after the publication of the *Théorie de la spéculation*, Bachelier's work did not received particular attention by scholars, also due to the use of French language. But then from '30s to half of '60s many papers on similar topics started to be published, and a book published in 1912 by Bachelier in which he detailed his work was cited many times by various mathematicians such as Andrey Kolmogorov and William Feller, making more academics discover the work of the French mathematician. Finally in 1965 Eugene Fama, studying the work of Paul Samuelson and all the previous papers written by the already cited and other scholars, introduced in his doctoral thesis the *Efficient Market Hypothesis*, which added a more statistical approach to better evaluate market behaviour.

First of all it is necessary to specify that in this theory investors are considered to be rational. An investor is rational (or better, has rational expectations) if he implements a decision-making process that is based on making choices that result in the optimal level of benefit for him. In simple terms, an investor has rational expectations if he efficiently uses the information he knows in order to maximise profit or minimise losses.

Actually Fama structured his *Efficient Market Hypothesis* (EMH) in three different hypothesis:

- **Weak-form efficiency**: market prices reflect all the available information of the historical prices;

- **Semi-strong-form efficiency**: market prices reflect all the available information of the historical prices, plus all the available public information;

- **Strong-form efficiency**: market prices reflect all the available information of the historical prices, plus all the available information both public and private.

Obviously, to have strong-form efficiency it is necessary to have semi-strong-form efficiency, that in turn needs the weak-form efficiency.
The first implication of this theory is that future market prices always depend only on the most recent known price, since it is the most complete set of information available. In statistics, this characteristic is called *martingale*: it is a stochastic process X_t in which the expected value of X_t conditioned to X_s, with $s \leq t$, is equal to X_s. This leads to the conclusion the price instantly changes as soon as a new information becomes available: the new information gets immediately incorporated by the price, resulting in an appropriate positive or negative variation. Consequently, outperforming the market is not possible since any sort of information is reflected on the market in no time, leaving no possibility of sure gains for investors.

The stochastic process that better resemble this behaviour is the *random walk*.

2.3 *Random walk* and *Geometric Brownian Motion*

As just anticipated, the behaviour of markets described by Fama can be well represented by the stochastic process known as *random walk*, since the forecast of a future price is based only on the most recent known price, which evolution is almost random.
A random walk is indeed a stochastic process with independent increments, and it is defined as

$$Y_t = Y_{t-1} + \eta_t$$
$$Y_0 = y_0$$

where ε_t is a White Noise process, defined as

$$\eta_t \sim WN(0, \sigma^2).$$

This is a simple random walk process. Indeed, it is better to consider a slightly different process with an additional component, that is the *random walk with drift*, defined as

$$Y_t = \alpha + Y_{t-1} + \varepsilon_t$$
$$Y_0 = y_0.$$

It is possible to see the addition of the deterministic component α, which is called the *drift* term of the process. Furthermore, the stochastic component ε_t, called *diffusion* term, is usually a Gaussian White Noise defined as

$$\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$$

or, alternatively,

$$\varepsilon_t \sim GWN(0, \sigma^2).$$

On the basis of such premises, a random walk stochastic process with drift and independent increments able to represent the variations of a financial price could be the following one:

$$P_{t_{i+1}} = P_{t_i} + \mu P_{t_i}(t_{i+1} - t_i) + \sigma P_{t_i}\sqrt{(t_{i+1} - t_i)}\varepsilon_{t_i} \tag{2.1}$$

where, considering an interval $[0, T]$ divided in n intervals $[t_i,\ t_{i+1}]$ of same length Δt with $i = 1, ..., n$, we have then

$$t_i = i \cdot \left(\frac{T - 0}{n}\right) = i \cdot \Delta t,\ \ \forall\, i = 0, 1, ..., n$$

and for which

$$t_{i+1} - t_i = \frac{T}{n} = \Delta t,\ \ \forall\, i = 0, 1, ..., n.$$

Furthermore, we also have that

$$\mu \in \mathbb{R}$$

$$\sigma \in \mathbb{R}_+$$

$$\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2).$$

where, under a financial point of view, μ is the expected value of the process and σ is the standard deviation of the process (hence it can only be positive).

Also, knowing that $t_i = t_{i-1} + \Delta t$, we can rewrite equation (2.1) as

$$P_{t_{i+1}} = P_{t_i} + \mu P_{t_i}\Delta t + \sigma P_{t_i}\sqrt{\Delta t}\varepsilon_{t_i} \tag{2.2}$$

and then we get

$$P_{t_{i+1}} - P_{t_i} = \mu P_{t_i} \Delta t + \sigma P_{t_i} \sqrt{\Delta t} \varepsilon_{t_i}. \tag{2.3}$$

This model is defined in a discrete time interval. If we release this construction assumption and extend the model to a continue time interval, letting then the number of intervals $n \to \infty$, and consequently $\Delta t \to 0$, under certain conditions it is possible to prove[1] that we get

$$dP(t) = \mu P(t) dt + \sigma P(t) \sqrt{dt} \varepsilon_t. \tag{2.4}$$

Equations like this are part of a specific family of stochastic differential equations whose solution provides the *Geometric Brownian Motion*. In particular, a GBM is the stochastic process described by the following SDE (Stochastic Differential Equation):

$$\begin{cases} dX(t) = \alpha X(t) dt + \sigma X(t) dW(t) \\ X(0) = x_0. \end{cases} \tag{2.5}$$

Equation (2.4) is very similar to the function described above, but it is necessary to model the diffusion term of the equation as a Wiener process (also called Brownian Motion)[2].

It is possible to prove that the just described process has the following explicit solution:

$$X(t) = x_0 \cdot e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma W(t)}. \tag{2.6}$$

[1] See S.E. Shreve - "Stochastic Calculus for Finance II: Continuous-Time Models", Springer (2004)

[2] A stochastic process W(t)is called a Wiener process if the following conditions hold: W(0) = 0, W(t) has independent increments, for s < t the stochastic variable W(t) - W(s) has the Gaussian distribution N(0, t-s), W(t) has continuous trajectories. See T. Bjork - "Arbitrage Theory in Continuous Time", Oxford University Press (2009)

The distribution of X(t) is log-normal, meaning that $logX(t)$ is normally distributed. Moreover, we have that

$$\mathbb{E}[X(t)] = x_0 \cdot e^{\alpha t}. \tag{2.7}$$

Coming back to our model described in (2.4), from (2.7) we know that

$$\mathbb{E}[P_{t_i}] = P_{t_0} \cdot e^{\mu t_i} \tag{2.8}$$

from which

$$\mathbb{E}[P_{t_{i+1}}|P_{t_i}] = P_{t_i} \cdot e^{\mu \Delta t}. \tag{2.9}$$

It is now clear why the expected value of the price in a future time instant depends only on the most recent known price of that time series. This means that future price changes will reflect only future news and, since news is by definition unpredictable, therefore price changes must be unpredictable and random, leaving no possibility for investors to achieve a sure return.

2.4 Critics on the EMH

Although the Efficient Market Hypothesis had been widely accepted, claiming that future prices rely only on the most recent known price and not on the entire time series exposed Fama and his theory to various critics in years, coming from scholars and investors that analyse markets from different points of view.

First critics came from investors and scholars that analyse stock markets through the tools offered by the technical analysis.

Technical analysis is a particular analysis methodology used to analyse and eventually forecast financial markets which is based on the study of graphs and plots and on the use of technical indicators. Some indicators are based on past market data, such as highs, lows, volume, . . . , while others are built through greater or lesser complex mathematical functions such as Exponential Moving Average, Commodity Channel Index, and Relative Strength Index.
Hence technical analysts base their actions and their investment decisions on the belief that market trends actually can in part be forecast using mathematical instruments, so they reject Fama's thesis of unpredictability and randomness of future prices.

Other critics came from fundamental analysts, which study price behaviour relying on information available on the financial statements of the inspected firm, along with competitors' ones and a general market analysis. These information are usually business's assets, liabilities, and earnings, but many other parameters can be considered by a fundamental analyst.
So, fundamental analysts too state that market trends can be somehow predicted, at least in part, through a correct evaluation of company's financial data, thus they too reject the implications of Fama's theory.

But the critics that better relate with this dissertation are the ones coming from mathematicians and statisticians, and in general from whoever analyses market data under the statistical point of view.
In the Efficient Market Hypothesis, the density distribution of returns is

assumed to be normally distributed. The fact is that returns much often appears not to be distributed in such a way. In particular, tails appear to be fatter, and with a higher peak of the mean with respect to the Normal distribution. In normal market conditions, these differences slightly influence the effect on an investor's account, but having fatter tails means that extreme events (both positive and negative) occurs often then expected in the Normal case, so these events could become very important and could hugely affect the investor's account.

Furthermore, there is a difference also in the behaviour of the volatility of returns. In the Normal distribution, having a daily variance equals to σ implies that in t days the variance becomes t times the daily variance. In terms of standard deviation (which measures the volatility), the formula then becomes

$$\sigma_t = \sigma \cdot \sqrt{t} \tag{2.10}$$

where σ_t is the standard deviation observed in t days. *"This practice is derived from Einstein's*[3] *observation that the distance that a particle in brownian motion covers increases with the square root of time used to measure it."* (Peters, 1994)

Many scholars[4] yet have observed that often returns' volatility does not behave as expected in case of Normal distribution, and in particular have

[3]See A. Einstein - "Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen" (English: "On the movement of small particles suspended in a stationary liquid demanded by the molecular-kinetic theory of heat") (1905)

[4]See R. J. Shiller - "Market Volatility", The MIT Press (1989), A. L. Turner and E. J. Weigel - "An Analysis of Stock Market Volatility", Russell Research Commentaries Press (1990), E. E. Peters - "Chaos and Order in the Capital Markets: A New View of Cycles, Prices, and Market Volatility", John Wiley & Sons (1991), E. E. Peters - "Fractal Market Analysis: Applying Chaos Theory to Investment and Economics", John Wiley & Sons (1994)

observed that standard deviation scales at a faster rate than the square root of time.

To overcome all these problems that affect the efficiency of the Efficient Market Hypothesis, many theories have been proposed. The one that is going to be analysed in details is the *Fractal Market Hypothesis* (FMH) proposed by Edgard E. Peters.

Chapter 3

Fractals: characteristics and examples

3.1 Fractal geometry

Before presenting the *Fractal Market Hypothesis*, it is necessary to first introduce and describe what is a fractal and which are its characteristics. The term “fractal” comes from the Latin *fractus*, which means “broken” or “fractured”, but formal definition of “fractal” is still missing, as there is some disagreement among mathematicians about how it should be defined. In 1982 Mandelbrot, the first one to use this term in 1975[1], stated that “A fractal is by definition a set for which the Hausdorff-Besicovitch dimension strictly exceeds the topological dimension”, and then later changed the definition stating that “A fractal is a shape made of parts similar to the whole in some way”. We will come back later on the definitions of Hausdorff-Besicovitch and topological dimensions.

[1] See B. Mandelbrot - "Les object Fractals: Forme, Hazards et Dimension" (English: "Fractals: Form, Chance and Dimension"), W.H.Freeman & Co Ltd (1977)

One of the first examples of fractals, although at that time it was not defined in such a way, is the *Weierstraß function*:

$$f(x) = \sum_{n=0}^{\infty} a^n \cdot \cos(b^n \pi x) \tag{3.1}$$

where $0 < a < 1$ and b is an odd integer number such that $ab > 1 + \frac{3}{2}\pi$.

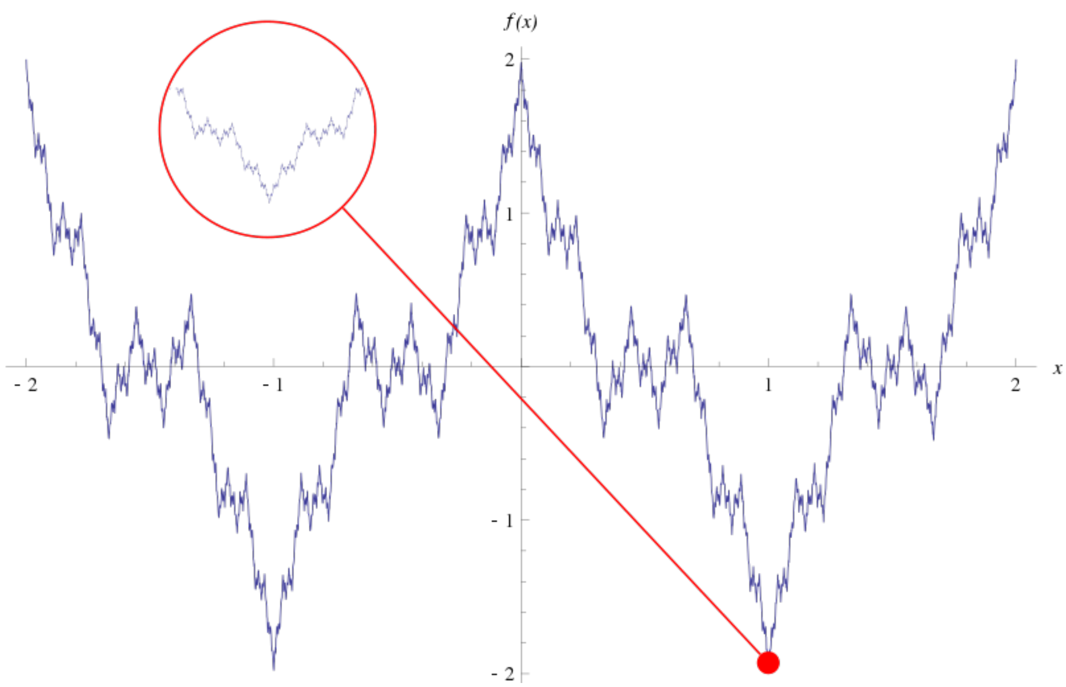

Figure 3.1: Weierstraß Function in (-2, 2)

It is possible to see that in the highlighted red circle in which the function is zoomed in the path of the function in some way resemble the path of the entire function. This property is called *self-similarity*. Furthermore, the Weierstraß function, as well as fractals, is everywhere continuous but nowhere differentiable.

To better explain this last aspect, consider now one of the most known fractal as well as one of the earliest ones to have been described[2], known as *Koch snowflake*:

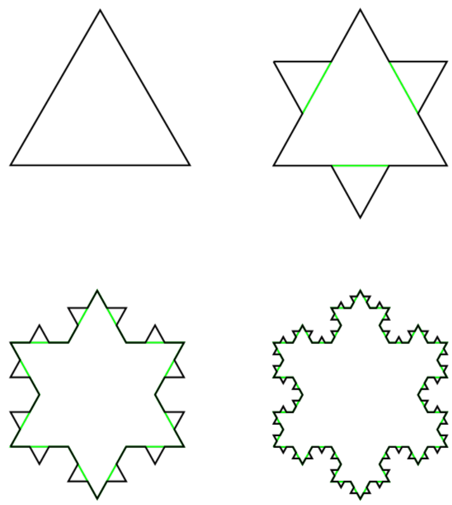

Figure 3.2: The first four iterations of the Koch snowflake

This fractal is constructed starting from a simple equilateral triangle. Then, divide each side of the triangle in three equal parts and build a smaller equilateral triangle outward on the middle one, remove the segment used as the base of the new equilateral triangle and repeat the procedure at each successive stage.

It is a very simple procedure, yet it has some interesting properties. Indeed, it is easy to demonstrate that, as the number of iteration increases, the area of the snowflakes converges to a specific value (which is $\frac{8}{5}$ of the original

[2]See H. von Koch - "Sur une courbe continue sans tangente, obtenue par une construction géométrique élémentaire" (English: "On a Continuous Curve Without Tangents, Constructible from Elementary Geometry") (1904)

triangle), while the perimeter increases without bound, having then a finite area and an infinite perimeter. But now we focus on its shape, that it makes its border everywhere continuous but nowhere differentiable, the same as the Weierstrass function cited before.

In order to better analyse this property we now consider the Koch curve, which is built in the same way of the Koch snowflake, but considering only one side of the triangle instead of three. This means that three Koch curves make a Koch snowflake.

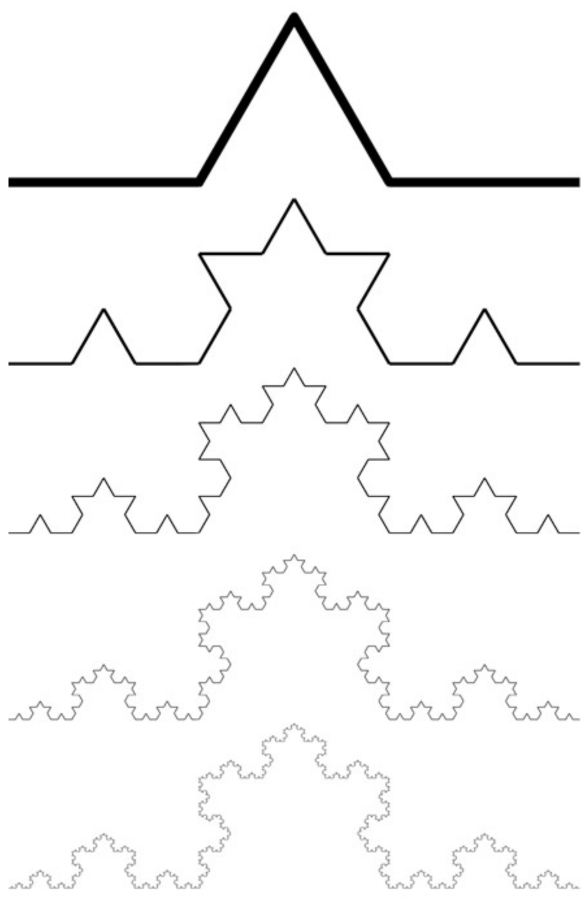

Figure 3.3: The first five iterations of the Koch curve

First of all, the self-similarity of this fractal is now very clear to see: indeed, one can easily notice that the upper part in the middle of each figure is exactly the entire curve at the previous step.
Also, the Koch curve is nowhere differentiable: indeed when a generic point x_0 on a differentiable curve is considered, the curve on the neighbourhood of x_0 tends to resemble a straight line when zoomed. Instead the Koch curve due to its self-replicating structure is identical to itself at each scale.

3.2 Fractal dimension

Jagged patterns are a good starting point to introduce fractal dimension, often called Hausdorff dimension in honour of the German mathematician Felix Hausdorff who firstly introduced it in 1918.
Usually in nature objects are known for having three dimensions at most: a point has zero dimensions, a line segment has one, a square has two, and a cube has three. Sometimes in physics dimensions are considered to be four, if we include time, or even more up to eleven dimensions in strings theory. But the common denominator of all these numbers is that they are natural numbers. Fractals allow objects to have a real number of dimensions.

Mandelbrot did not coin the term “fractal” until 1975, but in one of his first papers on the topic[3] he analysed this interesting aspect starting from measuring the length of Great Britain coastline. He stated that the measured length of a stretch of coastline depends on the scale of measurement. In other

[3]See B. Mandelbrot - "How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension" (1967)

words, the smaller the increment of measurement, the longer the measured length becomes: imagining of walking along the coastline and measuring it using the number of steps, the measure done by a kid would be significantly greater (and also more precise) than the one done by ad adult.

Figure 3.4: Great Britain coastline. Left: unit = 200 km, length = ~2400 km. Right: unit 50 km, length = ~3400 km

Letting the measurement scale decreasing towards zero implies that the measure length increases without limit. The point that Mandelbrot wanted to highlight was that it is meaningless to talk about the length of a coastline, and that some other means of quantifying coastlines are needed. This led the author to anaylise an empirical law discovered by Lewis Fry Richardson which links the measured length of various geographic borders with the measurement scale used to measure it.

The main idea is: if the coastline is smooth, then it has 1 dimension (since it is a simple line), while if it becomes more jagged and irregular the number of dimensions its dimension tend to 2, since the coastline itself is more than a line but less than a plane (which has 2 dimensions). In particular, the dimension of the Great Britain coastline is known to be 1.25.

This lead towards the definition of *fractal dimension*. In simple words, the fractal dimension indicates *how much a curve "fills" the space*. Thinking about the coastline of Great Britain, as said before it is more than a line and less then a plane, so its fractal dimension is between 1 and 2 because it has some "voids". The same could be said for many other things in nature, starting from the Earth itself: our planet is not a perfect sphere since it has mountains and valleys, and it is also flattened at the poles, so its fractal dimension is a non-integer value between 2 and 3, since it is more than a plane and less than a solid. To be precise, a fractal dimension of 2.3 is found to be a common value in describing the relief on the Earth.

There are more definitions of fractal dimension, and in particular two of the most important are:

- the Hausdorff-Besicovitch dimension
- the Minkovski-Bouligand dimension (also known as box-counting dimension).

3.2.1 The Hausdorff-Besicovitch dimension

One of the first methods to estimate the fractal dimension of a curve involves covering the curve with circles of radius r. Then, the number of circles needed to cover the curve is counted, and the radius is decreased. The number of circles is known to scale as the following equation:

$$N \cdot (2 \cdot r)^d = 1 \tag{3.2}$$

where N is the number of circles, r is the radius, and d is the fractal dimension. Solving for d, it turns out that

$$d = \frac{\ln N}{\ln\left(\frac{1}{2 \cdot r}\right)}. \tag{3.3}$$

As previously said, a straight line has a fractal dimension of 1. Instead a random walk, which has a 50% probability of going either up or down, has then a fractal dimension equal to 1.50 since it is halfway between a line and a plane.

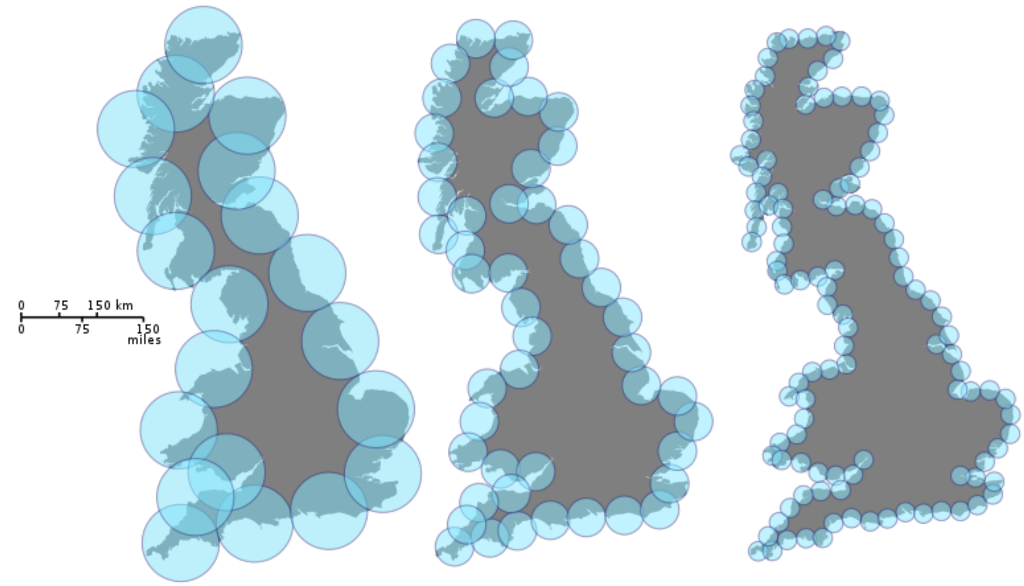

Figure 3.5: Estimating the Hausdorff dimension of the coast of Great Britain

A more rigorous definition of the Hausdorff dimension needs using some concepts of the earlier Lebesgue measure from measure theory, and using the diameter of the circles rather than the radius.

In measure theory, a measure is defined using a collection of parallelepipedons as cover of a generic set A, but Hausdorff replaced them with balls[4] and to define the Hausdorff measure he used their diameters, defined as following:

[4]In mathematics, a ball (or hyperball) is the space bounded by a sphere. A ball in n dimensions is called n-ball and is bounded by an (n-1)-sphere. For example, a ball in Euclidean 3-space is taken to be the volume bounded by a 2-dimensional sphere.

Definition: diameter

Let $E \subseteq \mathbb{R}^n$ not empty, the diameter of E is then

$$diam(E) = \sup\{||x - y||,\ \forall\, x, y \in E\} \tag{3.4}$$

where $||x - y|| = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$ is the Euclidean distance between the two points x and y. ■

It is now necessary to define the $\delta - cover$ of a given set:

Definition: δ-cover of E

Let $E \subseteq \mathbb{R}^n$ not empty and $\delta > 0$. A δ-cover of E *is a countable family $\{E_i\}$ of not empty subsets E_i such that*

$$E \subseteq \bigcup_i E_i \quad \text{and} \quad 0 < diam(E_i) \leq \delta \quad \forall i. \blacksquare \tag{3.5}$$

Then, the Hausdorff s-dimensional measure is defined as:

Definition: Hausdorff s-dimensional measure of scale δ for a set E

Let $E \subseteq \mathbb{R}^n$ not empty and $\delta > 0, s \leq 0$. We define

$$\mathcal{H}^s_\delta(E) = \inf\left\{\sum_i (diam(E_i))^s \mid \{E_i\} \text{ is a } \delta\text{-cover of } E\right\} \tag{3.6}$$

and $\mathcal{H}^s_\delta(\emptyset) = 0.$■

Hence we can apply the Hausdorff s-dimensional measure to a subset of $\mathbb{R}^n$:

> ***Definition: Hausdorff s-dimensional measure for a subset*** $E \subseteq \mathbb{R}^n$
>
> *We define*
>
> $$\mathcal{H}^s(E) = \sup_{\delta>0} \mathcal{H}^s_\delta(E) = \lim_{\delta\to 0^+} \mathcal{H}^s_\delta(E). \blacksquare \tag{3.7}$$

It is obvious then that, as δ decreases, the number of sets of $\{E_i\}$ increases, and tend to infinite as $\delta \to 0$.

Thus, when $\delta \to 0$ the measure $\mathcal{H}^s(E)$ becomes a sum of infinite subsets with diameter zero, which is the undetermined form $[\infty \cdot 0]$. The value assumed by the exponent *s* helps us to solve this undetermined form. Indeed, naming $s_0 = D_{\mathcal{H}}(E)$, we obtain that that $\mathcal{H}^s(E) \to +\infty$ as $s \to 0$ and $\mathcal{H}^s(E) \to 0$ as $s \to +\infty$. The Hausdorff dimension is then the value of s that makes $\mathcal{H}^s(E)$ jumps from 0 to $+\infty$. The formal definition is the following:

> ***Definition: Hausdorff dimension for a subset*** $E \subseteq \mathbb{R}^n$
>
> *We define*
>
> $$D_{\mathcal{H}}(E) = \inf\{s \geq 0 : \mathcal{H}^s(E) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(E) = +\infty\}. \blacksquare \tag{3.8}$$

3.2.2 The Minkowski-Bouligand dimension

The Minkowski-Bouligand dimension, also known as *box-counting* dimension is a variation of the approach to compute fractal dimension using circles (or spheres, for objects with more than 2 dimensions). Now, as the name suggests, squares (cubes) are used instead of circles (spheres). It is a relatively easier approach, since squares are easier to manipulate rather than circles, but the procedure is identical. Starting from a square with a side length of ε, the number of boxes N_ε necessary to completely cover the curve is counted, and then the procedure is repeated reducing the value of ε. Then, the Minkowski-Bouligand dimension is defined as

$$D_{\mathcal{M}} = \lim_{\varepsilon \to 0} \frac{\ln N_\varepsilon}{\ln\left(\frac{1}{\varepsilon}\right)}. \tag{3.9}$$

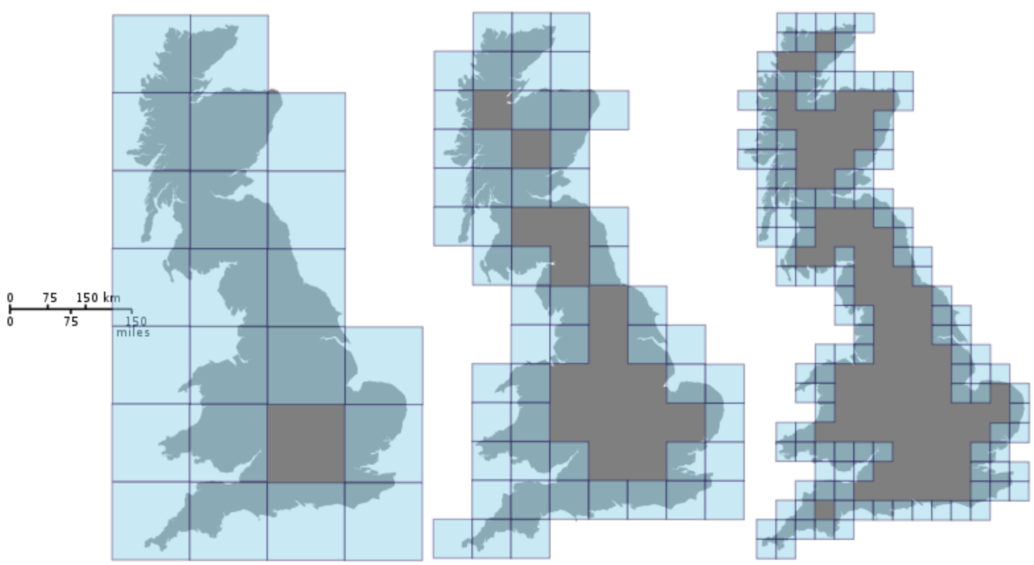

Figure 3.6: Estimating the box-counting dimension of the coast of Great Britain

Furthermore, it has been proven[5] that

$$D_{\mathcal{H}} \leq D_{\mathcal{M}}. \tag{3.10}$$

[5] See K. Falconer - "Fractal Geometry - Mathematical Foundations & Applications", John Wiley & Sons (1990)

Thinking about the Koch snowflake described before, we are now able to understand its fractal dimension and even compute it using a even simpler method.

After each iteration, all original line segments are replaced with $N = 4$, where each self-similar copy is $1/S = 1/3$ as long as the original. So, to calculate the dimension D we must solve

$$N = S^D \tag{3.11}$$

for D, that is

$$D = \frac{\ln N}{\ln S} = \frac{\ln 4}{\ln 3} \approx 1.262.$$

So, the Koch snowflake has a fractal dimension of approximately 1.262.

3.3 Examples of fractals

Fractals can be classified in various categories depending on their construction process:

- IFS fractals (Iterated Function System);
- LS fractals (Lindenmayer System).

An IFS fractal is a generated by a series of affine transformations[6]. These transformations, which are originated by simple equations such as rotations, traslations, reflections, and homotheties[7], let the iterative process tend to

[6] In geometry, an affine transformation is a function which preserves points, straight lines, and planes, though it does not necessarily preserve angles between lines or distances between points. For further details, see M. Berger - "Geometry I", Springer (1987)

[7] In geometry, a homothety is a particular geometric transformation of an affine space that dilates or shrinks the objects, leaving unchanged the angles.

a specific *attractor*[8] independent from starting conditions. This mean that continuing the iteration process for an infinite number of steps will lead to obtain always the same final figure.
The most famous fractal of this type is probably the Sierpiński triangle. The procedure to obtain it is the following:

1. The starting figure is an equilateral triangle with side length ε;
2. Four new triangles are generated by connecting the midpoints of the three sides of the original triangle, with the middle one upside down, all with side equals to $\frac{\varepsilon}{2}$;
3. Midpoints of all the new not-upside down triangles are now connected obtaining then 9 further not-upside down triangles with side $\frac{\varepsilon}{4}$;
4. When the number of iterations tends to infinity, the limit is the Sierpiński triangle. At the n-th iteration there are 3^n not-upside down triangles of side length $2^{-n}\varepsilon$.

Figure 3.7: First five iterations of the Sierpiński triangle

To compute the fractal dimension of the Sierpiński triangle, remember that the scaling factor S is 2 (since each new triangle's side is half the previous

[8]In mathematics, an attractor is a set of numerical values toward which a system tends to evolve, after a sufficiently long time, for a wide variety of starting conditions of the system.

one's) and that the new triangles to which the procedure is always the triple of the number of triangles at the previous step, thus N = 3. Hence having $N = S^D$ it is easy to solve for D: $D = \frac{\ln 3}{\ln 2} \approx 1.585$.

On the other hand, LS fractals (sometimes called also L-system) are complex images originated by successive iterations consisting in the substitution of part of the object at the previous step, and in each step the rewriting procedure is always the same.
We have already analysed a fractal of this kind, which is the Koch snowflake, and then obviously the Koch curve, but there are many more.

One of the most interesting fractals is also a three-dimensional (meant in the common sense) object. That's the Menger sponge.
The procedure to obtain a Menger sponge is quite easy:

1. The starting figure is a cube of side length ε;
2. The cube is then divided in 27 smaller cubes of side length $\frac{\varepsilon}{3}$ (similar to a Rubik's cube);
3. The central cubes of each face are removed as well as the central one on the inside;
4. The procedure is repeated from step 2) for each remaining cube.

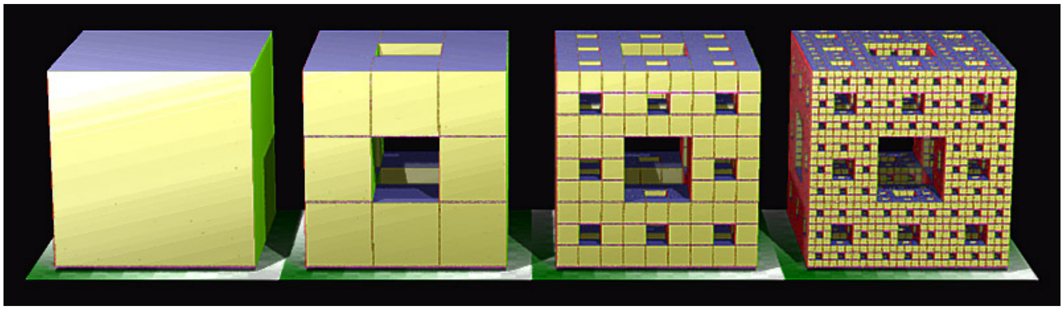

Figure 3.8: First four iterations of the Menger sponge

Now we want to calculate the fractal dimension of the Menger sponge. We know that the scale factor is $S = 3$, since each new cube has a side length of one third of the cube at the previous iteration. Furthermore, at each iterations 27-6-1 = 20 new cubes are generated, which leads to $N = 20$. Hence, solving $N = S^D$ we $D = \frac{\ln 20}{\ln 3} \approx 2.727$.

Chapter 4

Fractals and markets

4.1 Presence of fractals in markets trends

After this brief introduction to fractals and their characteristics, it is necessary to link them to stock markets and market analysis. Hence, it is necessary to find some evidence of the presence of fractals in price trends.

Fractals are regulated by strict mathematical laws and equations, while the evolution of market prices is not, or would not seem. According to the EMH, the only factor that affect the price and can lead to a rise or a drop of the value of the security is the release of some new information. However, we already discussed about the fallacy of the EMH, and actually a change in the price of a security can happen due to many reasons. Indeed for example, as pointed out by behavioural economists, not every investor is fully rational or has rational expectations, and this can lead to strange changes in market prices.

But, even if markets are not directly math-regulated, some regularities can be found observing the behaviour of stock markets, and in particular the presence of fractal patterns. A fractal pattern is a section of a time series which is repeated multiple times on the graph, analogously to the *self-similarity* property of fractals. Consider now the price history of the S&P 500 index[1] illustrated below:

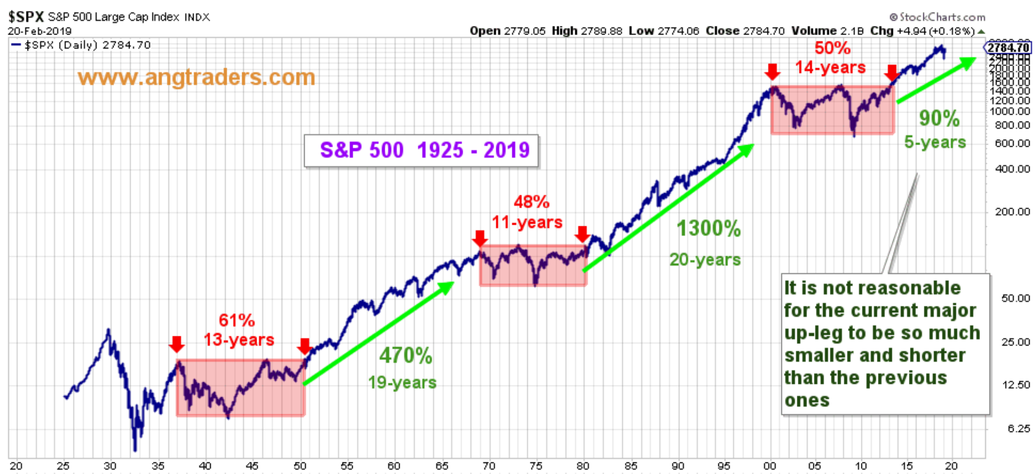

Figure 4.1: S&P 500 price history from 1925 (from ANG Traders)

It is possible to identify a recurring pattern over the years, as highlighted with red rectangles and green arrows. A *range-trade* sections colored in red, delimited by two red arrows, start when a new maximum is reached and end when that high is surpassed. It defines a portion of the graph in which the price oscillates within a bounded range and without reaching new higher values. On the other hand, *trending patterns* are the up-legs between two

[1]The Standard & Poor 500, known as S&P 500, is a market index created by Standard & Poor's in 1957 and it reflects the trend of the 500 American companies with greater market capitalization. Values before that date are recreated considering the best companies at the evaluated year.

range-trade sections, and they are indicated by the green upwarding arrows. These patterns determine a section of the graph in which the price is rising with greater or lesser steepness.
The three range-trade sections are very similar in duration, and are respectively 13-, 11-, and 14-years long. Likewise, the first two up-legs are of similar length, respectively 19- and 20- years long. The repetitiveness in behaviour suggests that also the third leg will be of similar length.
The variances within the range-trade patterns, expressed as percent of the maximum reached in those sections, are quite similar too: they are respectively 61%, 48%, and 50%.

However, it is interesting to notice that these patterns can be spotted also in a shorter time interval, as shown in the following chart:

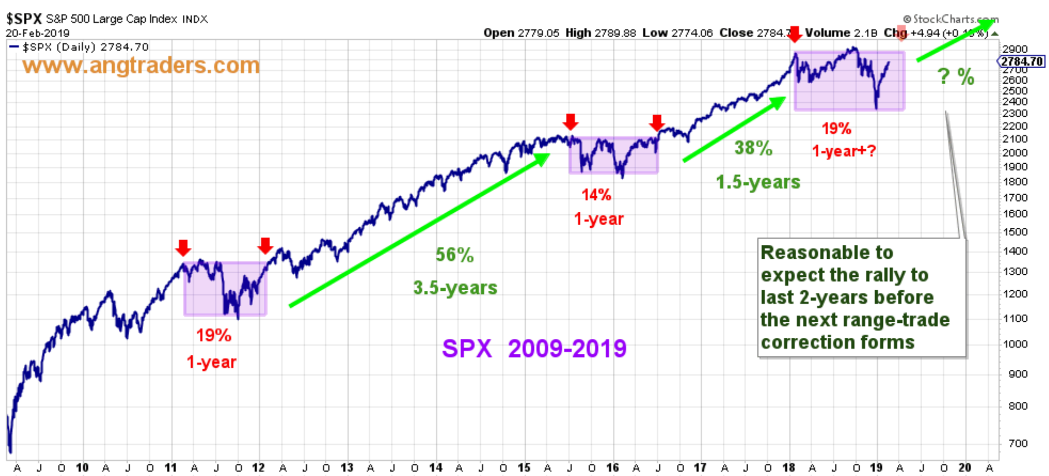

Figure 4.2: S&P 500 price history from 2009 (from ANG Traders)

Here the time interval is much smaller than before, starting from the beginning of 2009 instead that from 1925, as in the previous figure. However, similar range-trade sections and up-legs are highlighted. Furthermore, the S&P 500 index is currently in the up-leg phase, since it reached and surpassed the maximum value of the latest range-trade phase in April 2019, and now in January 2020 its value is currently more than 3300.

4.1.1 Elliott Wave Theory

At the end of ’30s, taking the cue from the Dow theory on stock price movement developed by Charles H. Dow, a new form of technical analysis has been developed by Ralph Nelson Elliott. In this theory, defined in his book “The Wave Principle” published in 1938[2], the American accountant described a recurrent pattern that can be spotted observing market price time series. Elliott posited this theory thinking that crowd psychology moves between optimism and pessimism in natural sequence, but he did not mean to link his theory to fractals. Indeed, they were not even defined yet. However, one of the biggest evidence of fractals in markets can be found in the *Elliott Wave Theory*. Indeed, the specific pattern of price evolution that can be recognised after the first upward and downward movements is recurrent in market price time series, at various time interval between two consecutive observations and at multiple resolution scales. The pattern described by the author is the following:

[2]See R. N . Elliott - "The Wave Principle" (1938), and R. R. Prechter - "The Basics of the Elliott Wave Principle", New Classics Library (1995)

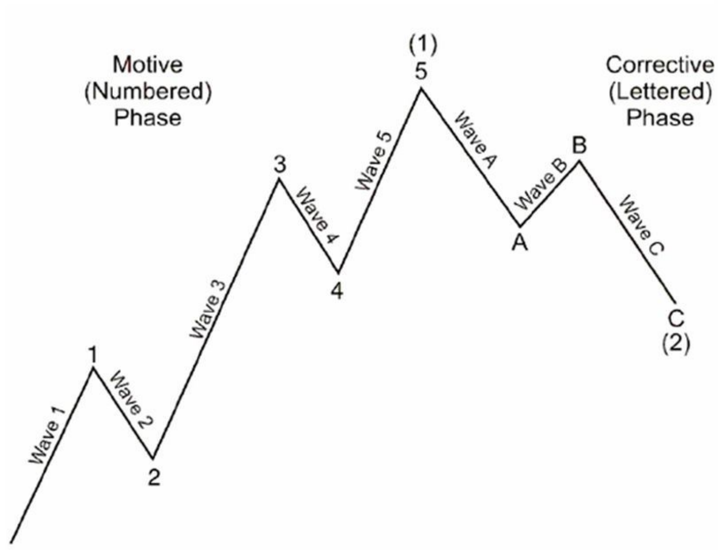

Figure 4.3: Pattern described in the Elliot Wave Theory

In his theory, Elliott defined two wave types: impulse waves and corrective waves. The former consists of five sub-waves that together make the price moves upward, while the latter consists of three sub-waves that correct the upward movement through a slight decrease in market price.
There are some simple rules that help to spot the presence of this kind of structure:

- The impulse wave is formed by three motive waves and two corrective waves, all labeled by numbers (from 1 to 5). The rules that define its formation are unbreakable, and if one of these is violated then the structure is not an impulse wave. In particular, wave 2 cannot retrace more than the entire wave 1, and wave 3 can never be shorter than wave 1 and wave 5;

- The corrective wave is formed by two corrective waves and one motive wave, all labeled by letters (from A to C). Corrective waves are typically harder to identify than impulse moves. Rules state that wave B cannot retrace more than the entire wave A, and that wave C is typically at least as large as wave A.

The entire structure just described is the so-called Bullish Elliott Wave Cycle, but also the opposite pattern can be spotted and it would describe a Bearish Elliott Wave Cycle.[3]
Elliott suggested also that often inside one cycle of waves other small cycles can be found, describing then a structure that also meet the common definition of a fractal, since it consists in self-similar patterns appearing at every degree of trend.

[3] In finance, the market is described as *bullish* when prices are moving upward and *bearish* when the overall movement is heading downwards.

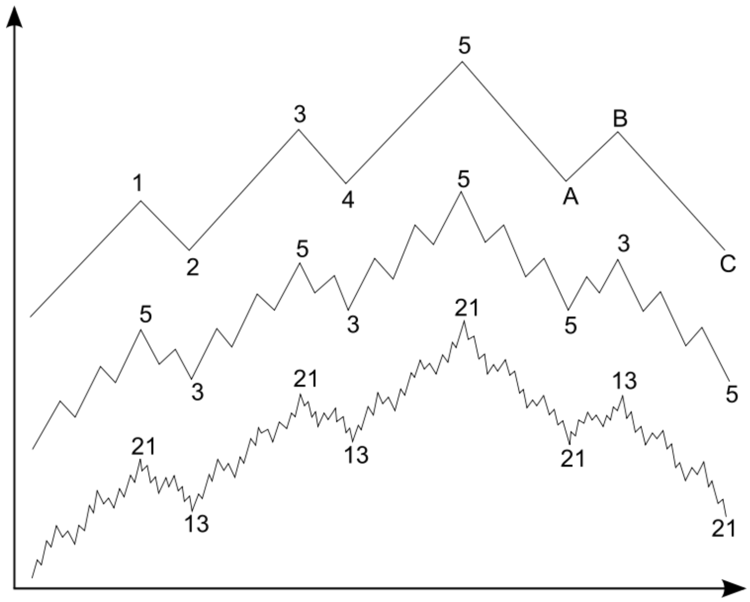

Figure 4.4: Cycle and sub-cycles of a Bullish Elliott Wave Cycle

Nowadays Elliott Wave Theory is still used as technical instrument by some market analysts, but many others thinks that the application to modern financial markets is no more possible due to changes in economies and social systems, making then the Elliot Principle obsolete and outdated. As stated by market analyst Glenn Neely[4]:

> *"Elliott wave was an incredible discovery for its time. But, as technologies, governments, economies, and social systems have changed, the behavior of people has also. These changes have affected the wave patterns R.N. Elliott discovered. Consequently, strict application of orthodox Elliott wave concepts to current day markets skews forecasting accuracy. Markets have evolved, but Elliott has not."*

[4]See G. Neely - "Picking Up the Elliott Wave Pieces" (1996)

At last, also Benoit Mandelbrot questioned about the efficacy of Elliot waves[5]:

> *"But Wave prediction is a very uncertain business. It is an art to which the subjective judgement of the chartists matters more than the objective, replicable verdict of the numbers. The record of this, as of most technical analysis, is at best mixed."*

4.1.2 *Fractals* indicator

In technical analysis there is also another indicator that take advantage of repetitive patterns in price movements, which is the self-similarity property of fractals previously described. This technical indicator, which has been developed by Bill Williams, is called *fractals* precisely and it consists in a precise sequence of price movements that can be spotted only in graphs that use the Japanese candlestick data visualisation. The necessity of using this chart type is due to the construction of the *fractals* indicator that needs the price information represented by candlesticks.

[5] See B. Mandelbrot, D.L. Hudson - "The (mis)Behavior of Markets", Basic Books (2004)

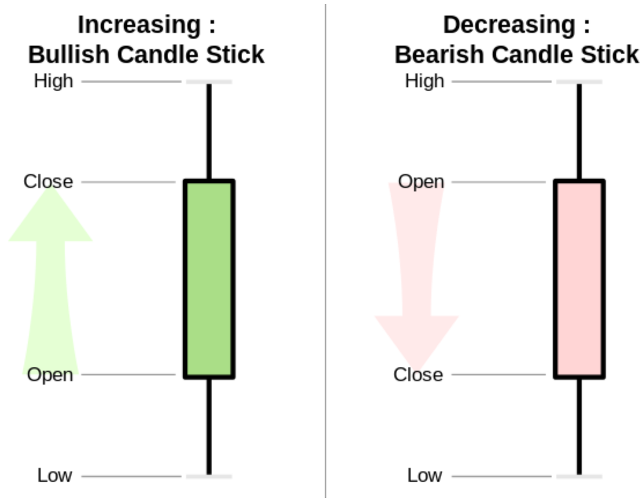

Figure 4.5: Examples of bullish and bearish candelsticks

Candlestick representation is widely used due to its clarity in reading, along with the presence of all the relevant price information (open, high, low, close) in a single figure. The area between open and close is the *real body*, while price excursions delimited by high and low above and below the real body are the *shadows*. If the market moved upward in the time interval to which the candlestick refers to, close would be higher than open and the body of the candle would be green (or white), otherwise if the market moved downward close would be lower than open and the body would be red (or black).

The *fractals* indicator is used to identify change in price trends, both from bearish to bullish and vice versa. In particular, a *bearish fractal* is drawn on the candlestick chart over a candle when the high of that candle is greater than the highs of both the two previous and the two successive candles. On the other hand, a *bullish fractal* is drawn when the low is lesser then the lows of both the two previous and the two successive candles. As illustrated in the following figure, bearish fractals are represented by an upward arrow (resembling a peak), while bullish fractals are represented by a downward arrow.

These kind of structures appears quite frequently when observing candlestick charts and are widely used, together with other technical indicators, to identify investment opportunities.

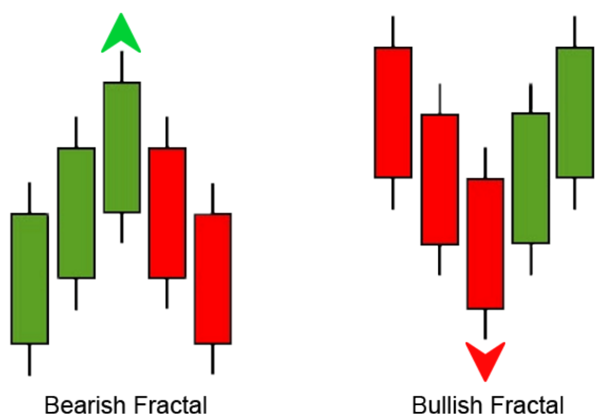

Figure 4.6: Examples of bearish and bullish fractals

4.1.3 Failure of the EMH

But fractals does not necessarily imply the presence of clearly visible repetitive patterns. As already stated before, the term fractal comes from the Latin *fractus* which means “broken” or “fractured”, and in mathematics is somewhat similar to “fractional”. Indeed, the presence of fractals in markets can be found also analysing the time series of prices. For example, when considering the volatility of prices with respect to time, the EMH supposed it should follow the so-called *T to the one-half rule* previously described when illustrating the Brownian Motion. The interesting thing is that this rule is often not valid.

Furthermore, we already showed that returns might be non-normally distributed, so two important pillars of the EMH are violated. Many scholars[6] showed that the Normal distribution (and consequently the Geometric Brownian Motion) is not the best possible choice to evaluate time series of market prices, along with the presence of non-Markovian memory[7] in many historical prices, so a different solution needs to be found.

4.2 Fractional Brownian Motion

The presence of non-Markovian memory allows the possibility of long-range dependence (LRD) in price values. A phenomenon is usually considered to have long-range dependence if the dependence between two observations at lag τ, with $\tau = 1, 2, ..., n$, decays slowly and has significant values for many lags[8]. We need a model that can include the possibility of long-range dependency in price time series, since we already showed that the Geometric Brownian Motion is often not suitable due to not-independent values of

[6] See, among the others, E. Fama - "The Behavior of Stock Market Prices" (1965), M. F. M. Osborne - "Brownian Motion in the Stock Market" (1959), W. F. Sharpe - "Portfolio theory and capital markets", McGraw-Hill (1970), and E. E. Peters - "Fractal Market Analysis - Applying Chaos Theory to Investment and Economics", John Wiley & Sons (1994)

[7] A Markov chain is a stochastic model describing a sequence of possible events in which the probability of each event depends only on the state attained in the previous event. Thus, the presence of non-Markovian memory is index of a stochastic process in which the state of the random variable is dependent not only on the last observation but also from previous ones.

[8] To be precise, a phenomenon is considered to have LRD if the dependence decays more slowly than an exponential decay, typically a power-like decay. For further details, see exponential and power decay in mathematics and physics.

returns. It is then necessary to generalise the Brownian Motion by adding an important feature that can be useful to better evaluate time series.

Definition: fractional Brownian Motion (fBM)

A stochastic process $\left\{X_t^H\right\}_{t\geq 0} \mapsto \mathbb{R}$ *is a fractional Brownian Motion if:*

- $X_0^H = 0$
- X_t^H is continuous $\forall t \in \mathbb{R}^+$
- $\mathbb{E}\left[X_t^H\right] = 0 \quad \forall t \in \mathbb{R}^+$
- $$\begin{aligned}\gamma_{\Delta t} &= \mathbb{E}\left[X_t^H X_{t+\Delta t}^H\right] \\ &= \tfrac{1}{2}\left(t^{2H} + (t+\Delta t)^{2H} - |t-(t+\Delta t)|^{2H}\right) \quad \forall t, \Delta t \in \mathbb{R}^+\end{aligned}$$

where H is a real number in (0, 1).■

When the exponent H is set equal to $\frac{1}{2}$, the fractional Brownian Motion becomes a simple Brownian Motion. Although every fractional Brownian Motion has stationary increments, they are independent only in the Brownian Motion case. Here instead we have three different cases:

- $H = \frac{1}{2}$: as just said, in this case we get $\gamma_{\Delta t} = 0$ and thus the simple Brownian Motion;
- $H < \frac{1}{2}$: in this case we get $\gamma_{\Delta t} < 0$ and consequently increments of the process are negatively correlated;
- $H > \frac{1}{2}$: in this case we get $\gamma_{\Delta t} > 0$ and consequently increments of the process are positively correlated.

The following image shows three different fractional Brownian Motions, highlighting the difference between the three possible values of exponent H. Left plot represents a fBM process with $H > \frac{1}{2}$, right plot a fBM process with $H < \frac{1}{2}$, and center plot, having $H = \frac{1}{2}$, represents the simple Brownian Motion.

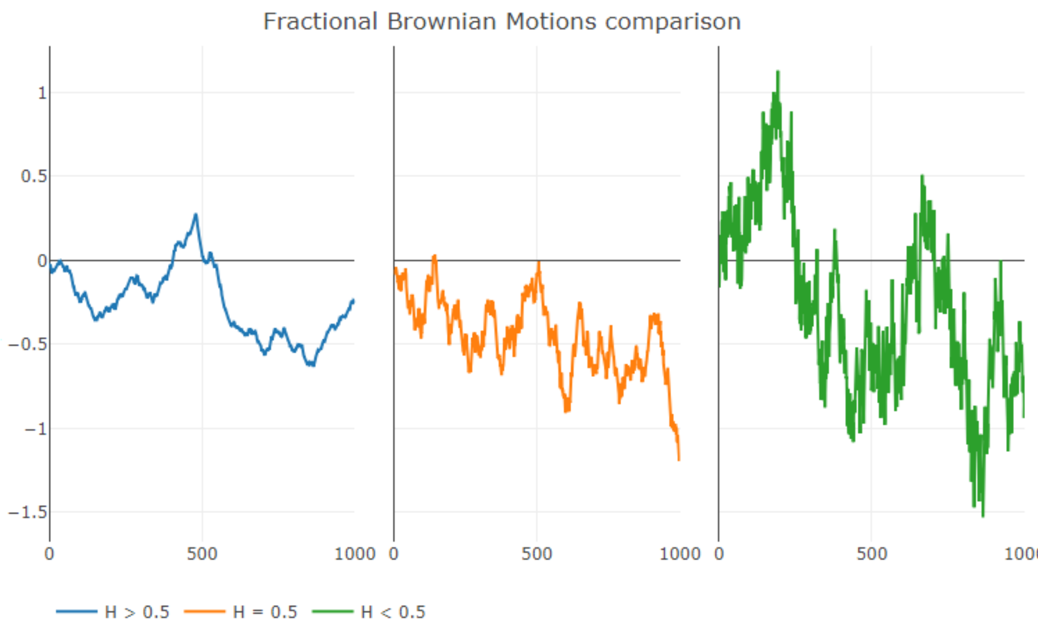

Figure 4.7: Comparison of fractional Brownian Motions

Furthermore, we know that increments, as in the simple Brownian Motion, are Normally distributed with mean 0 but with variance Δt^{2H}. Again, it is simple to recognise that if $H = \frac{1}{2}$ the fBM get reduced to the simple Brownian Motion with variance Δt.

It is now necessary to define the coefficient H, which regulates the entire path followed by the fractional Brownian Motion process.

4.3 Hurst exponent and R/S analysis

The coefficient H, namely the Hurst exponent, has been called in this way in honour of the British hydrologist Harold Edwin Hurst, who was the inventor of the *rescaled range* analysis used to estimate it.
Hurst from 1907 to the beginning of '50s worked in Egypt as hydrologist, and his task was defining a mathematical model to simulate the problem of monitoring the amount of Nile water supply. As suggested by other colleagues, the initial hypothesis he made was that Nile floods were random and unpredictable in entity, meaning that the flood range should follow the *T to the one half rule* already mentioned. Hurst analyses however seemed to lead to different conclusions, and this convinced him to do more analyses. In particular, Hurst noticed the existence of a persistent behaviour, suggesting that great overflows are generally followed by other significantly serious floods, and when the trend inverted small floods are usually followed by other relatively scarce overflows.

Hurst then created a tool, the *rescaled range analysis*[9] to evaluate the magnitude of the periodicity of a cycle. Starting from a time series X_t with $t = 1, \ldots,$ N observations, he divided it into 2^{k} (with k = 0, 1, ...) sub-periods of same length n so that $2^k \cdot n = N$, obtaining 1 sub-period of length N, 2 sub-periods of length N/2, 4 sub-periods of length N/4, etc. Then he computed the following operations for each sub-period.
First of all, he computed the arithmetic mean $\bar{X}$ and the standard deviation

[9] Also called *R/S analysis*.

which is given by

$$s_n = \sqrt{\frac{1}{n}\sum_{t=1}^{n}\left(X_t - \bar{X}\right)^2}. \tag{4.1}$$

Then, he calculated the cumulated sum of deviations from the mean as

$$Y_t = \sum_{m=1}^{t}\left(X_m - \bar{X}\right), \quad t = 1, ..., n \tag{4.2}$$

where X_m is the observation in year *m*. Please note that, by definition, the last value of Y (namely Y_n) will always be zero since it is computed as the total sum of the *n* observations minus *n*-times the mean.
The range R_n of Y is given by the difference between the greatest and the smallest values of Y, namely

$$R_n = \max(Y_1, Y_2, ..., Y_n) - \min(Y_1, Y_2, ..., Y_n).$$

In order to get a standardised measure and a pure number, Hurst divided the range by the standard deviation of the observation, obtaining then

$$\frac{R_n}{s_n} = \frac{\max(Y_1, Y_2, ..., Y_n) - \min(Y_1, Y_2, ..., Y_n)}{\sqrt{\frac{1}{n}\sum_{t=1}^{n}\left(X_t - \bar{X}\right)^2}}. \tag{4.3}$$

The relationship that link the rescaled range to the Hurst exponent has been found[10] to be

$$\frac{R_n}{s_n} = c \cdot n^H \tag{4.4}$$

[10] Actually, Hurst found the relationship to be $\frac{R_n}{s_n} = \left(\frac{n}{2}\right)^H$, and then later Mandelbrot proposed the form expressed in equation (4.4).

where c is a constant, n is the number of observations, and H is the Hurst exponent. Then, the Hurst exponent can be easily computed taking logarithms of both the left and the right hand sides of the previous equation, obtaining

$$\ln(R_n/s_n) = \ln(c) + H \cdot \ln(n). \tag{4.5}$$

Finally, the value of H is obtained computing simple linear regression for all the sub-periods and estimating the angular coefficient of the regression.

The exponent H can assume any real value between 0 and 1, but three different important scenarios can be defined, and they are the ones described also in the fBM. However, knowing the mathematical meaning of the exponent H, some further details can be added:

- $H = \frac{1}{2}$: there is no dependence between observed values, which then follow a *random walk*;
- $H < \frac{1}{2}$: there is negative dependence, making the series *anti-persistent* and thus the trend is often reverted: this means that a positive trend is usually followed by a negative one, creating a *mean-reverting* series;
- $H > \frac{1}{2}$: there is positive dependence, making the series *persistent* and thus a positive trend is supposed to be followed by further observations in the same direction.

4.3.1 Relationship with fractal dimension

Although it does not seem obvious the link, Mandelbrot[11] explained the relationship between the Hurst exponent and the fractal dimension. In

[11] See Mandelbrot - "The Fractal Geometry of Nature", Times Books (1982).

particular, it has been proven that for a time series described by a fractional Brownian Motion, the fractal dimension of the fBM graph can be described by

$$D_{\mathcal{H}}(G_X) = D_{\mathcal{M}}(G_X) = 2 - H \tag{4.6}$$

where

$$G_X = \Big\{(t, Y) \in \mathbb{R}^2 \mid Y = X(t) \ \wedge \ t \in [0, \infty)\Big\}$$

is the graph of a fractional Brownian Motion *X(t)* with Hurst exponent $0 < H < 1$.

Hence, having that the Hurst exponent can take any value between 0 and 1, this leads to the conclusion that the fractal dimension of the graph of a fractional Brownian Motion is a value between 1 and 2, meaning that its pattern is more than a line and less than a plane. In formulas:

$$1 < D_{\mathcal{H}}(G_X) = D_{\mathcal{M}}(G_X) < 2.$$

Chapter 5

Fractal Market Hypothesis

5.1 Premises and introduction

We already synthetically described the *Efficient Market Hypothesis* and we also discussed about the fallacies that make it not adherent to the reality. It is now necessary to provide a suitable alternative model to better evaluate the behaviour of markets and overcome all the problems highlighted while describing EMH.
Probably the most relevant issue described was the necessity for the time series to be a *martingale*, meaning that observations had to be independent or, at best, had to have a short-term memory. This implies that current change in prices could not be inferred from previous changes.
Furthermore, according to EMH the frequency of price changes should be well-represented by the Normal distribution. However, it has been widely showed that returns do not follow this rule since there are too many differences with respect to the Gaussian distribution. Actually, these differences are usually labeled as "anomalies" and the distribution is said to be "approx-

imately Normal". Many alternatives had been proposed, such as the stable Paretian distribution[1], but they are less used due to the major complexity of using standard statistical analysis.

The basis principle of EMH is that the current price reflects all the available information related to it. Then it can be said that the price of a security is always "fair" since every data that concerns the price is incorporated in the price itself, no matter what happens. Some scholars do not agree with this statement, and among them Edgar E. Peters suggested an interesting alternative model. The American scholar indeed suggested that the fairness of the price is directly linked with the liquidity of the security: if there is not enough liquidity the price is not fair, and an investor that wanted to complete a trade at any cost would be forced to accept any price to close the position, no matter it was fair or not. To use Peter's words[2]:

> *"A stable market is not the same as an"efficient" market, as defined by the EMH. A stable market is a liquid market. If the market is liquid, then the price can be considered close to "fair". However, markets are not always liquid. When lack of liquidity strikes, participating investors are willing to take any price they can, fair or not."*

Furthermore, Peters considered that the presence of liquidity is linked to

[1]The stable distribution, also called stable Pareto-Lévy distribution after Vilfrido Pareto and Paul Lévy, is family of distributions depending on four variable parameters that allows to vary not only mean and variance of the distribution but also its skewness and kurtosis. The Normal distribution is a particular case of stable Pareto-Lévy distribution. For further details, see J. P. Nolan - "Stable Distributions - Models for Heavy Tailed Data", American University Press (2009)

[2]For this and next quotes, see E. E. Peters - "Fractal Market Analysis - Applying Chaos Theory to Investment and Economics", John Wiley & Sons (1994)

investors' investment horizon. The reasoning that led him to this conclusion starts considering the impact of information on investors. Consider now that some new information on a security come out: if the news had the same impact on all investors, there would be no liquidity, since all investors would be executing the same trade trying to get the same price.
This means that investors are not homogeneous, and in particular investors do not react in the same way to news. The difference that Peters suggest is linked to investors' investment horizon. Consider for example two different investors holding the same security: a day trader, and a pension fund. The former is surely more reactive to news than the latter, which has a much more longer investment horizon and so he is less affected by new information coming out. To sum up, quoting Peters:

> *"All of the investors trading in the market simultaneously have different investment horizons. We can also say that the information that is important at each investment horizon is different. Thus, the source of liquidity is investors with different investment horizons, different information sets, and consequently, different concepts of"fair price".*

5.2 Definition of FMH

These are the main keys that Peters formulated to introduce his *Fractal Market Hypothesis*. Peters named it *Fractal* due to the self-similar statistical structure, just like the self-similar structure of fractals. The presence of self-similarity is highlighted by Peters looking at returns at different investment horizons: if a day-trader experiences a significant price change, an investor with a long-term investment horizon steps in and stabilise the market since for him the price change is not so relevant.

> *"As long as another investor has a longer trading horizon than the investor in crisis, the market will stabilize itself. For this reason, investors must share the same risk levels (once an adjustment is made for the scale of the investment horizon), and the shared risk explains why the frequency distribution of returns looks the same at different investment horizons."*

The next point to discuss is what happens when market crashes. According to the American scholar, markets become unstable when the fractal structure breaks down. This occurs when long-term horizon investors stop acting as usual, either by behaving as short-term investors or by not participating in the market at all. This usually happens in case of crises, political or economic (or both), and long-term forecasts and information lose reliability. Hence long-term investors who usually base their strategies on fundamental analysis of companies and economy, panic and lose focus. This translates in stopping considering a long-term investment horizon since they become less confident about the future of the economy, beginning then to act as short-term investors and taking investment decisions mainly relying on their perceptions rather than mathematical and economic evidence. This leads to high levels of short-term volatility and eventually to significant market shocks in a limited amount of time.

If the reason that caused the shock is important but does not affect too much the economy (in contrast to, for example, a war), investors with a long-term horizon return to behave normally usually after some days and market returns stable.

Hence, according to Peters, the stability of the market is assured by the presence of long-term investors who "absorb" sudden price changes that could worry short-term traders. This is possible because long-term investors are less concerned about new information since news usually affect market prices in the short-term, while these investors mostly rely on fundamental analysis on the long-term. Indeed, quoting Peters:

> *"As the investment horizon grows, technical analysis gradually gives way to fundamental and economic factors. Prices, as a result, reflect this relationship and rise and fall as earnings expectations rise and fall. [...] If the market has no relationship with the economic cycle, or if that relationship is very weak, then trading activity and liquidity continue their importance, even at long horizons."*

It is then clear that stocks of healthy companies in solid economies have quite stable returns on the long-term, since stock prices are linked also to the economic cycle. Since the latter is less volatile than trading activity, long-term returns of stocks are less volatile as well. The same reasoning could be applied to bonds, while it is not valid for currencies since these are a trading market only.

Summing up all the features of the *Fractal Market Hypothesis*, we know that:

1. The market is stable when investors have many different investment

horizons, assuring then enough liquidity for traders.;

2. Information is valued according to the investment horizon of the investor, implying that at any one time prices may not reflect all available information but only the information important to that investment horizon. Short-term investors rely more on technical analysis and factors, while long-term investors prefer fundamental analysis and economic information;
3. Markets become unstable when fundamental information become questionable and long-term investors lose faith and stop behaving as such: if this occurs, there is no one offering liquidity to short-term investors;
4. Securities with no tie to the economic cycle have no long-term trend, and trading and short-term information will dominate.

No assumptions are done about memory of prices, unlike in the EMH. Then, not only short-term memory but also long-term memory can exist, and can be spotted using adequate tools such the Rescaled Range Analysis.

5.2.1 An example of market reaction

A concrete example of this previously described situation of uncertainty can be found observing the futures[3] on the two main indices of United States, the Dow Jones Industrial Average and the S&P 500 already mentioned. In particular, at the end of November 2019 they were both quite stable and

[3] A future contract is an agreement between two parties to buy or sell an asset at a certain time in the future for a certain price. These instruments are derivative contracts, and are normally traded on an exchange. To make trading possible, the exchange specifies certain standardised features of the contract. For further details, see J. C. Hull - "Options, Futures, and Other Derivatives", Prentice Hall (2000).

actually they seemed to be in a bullish phase. Then, on the first days of December, President of U.S.A. Donald Trump published a tweet[4] in which he announced the restoration of all the tariffs on steel and aluminium imported in U.S.A. from Argentina and Brazil. Furthermore, the next day he also declared that the negotiation of tariffs on imports from China will probably be delayed after the Presidential Elections that will hold at the end of 2020. Financial markets instantly reacted to Trump's declarations, carrying out a sharp fall in both the two days: long-term investors probably panicked and immediately started selling, and consequently both the indices dropped significantly. Then, after some days the situation came back to normality, and the two indices returned to pre-announcements levels.

Figure 5.1: Effect of President Donald Trump declarations on DJIA and S&P 500 indices futures

It is possible to see from the above image that both the drops on the Dow

[4]See President Donald Trump's post on Twitter.

Jones and the S&P 500 were significant, as they were of almost 1000 and 100 points respectively. In terms of percentage, in two days both the futures lost more than 3%. Then markets realised they overreacted to some news that were important but not so tragic, and in a few days both the futures came back to their original levels.

This is a concrete example of market reaction to a bad news: the thought of possible future problems in world economy instilled fear on both short- and long-term investors who immediately started selling, making the price rapidly drop. Then, after some days of uncertainty, market became stable again and both the futures corrected the loss.

Chapter 6

A fractal trading strategy

6.1 Introduction to trading

Before outlining the trading strategy we are going to use, it is necessary to do a brief introduction to describe what trading is and which are the tools necessary to trade efficiently.
We already highlighted the difference between fundamental analysis and technical analysis, and we know they are both used to have a general overview of a company or a security. However, this is not the only feature that creates a distinction between different traders, indeed there are many more.

6.1.1 Different markets for different investments

First of all, the kind of market, and consequently the kind of security. There are many different markets in which a trader could choose to invest, but the main classification is between *exchanges* and *OTC* markets:

- Exchanges are **organised** markets with a specific real location where stocks, bonds, commodities[1], futures, and options are traded with a regulation For example, English stocks are traded in the London Stocks Exchange, and American futures are traded in the Chicago Mercantile Exchange. Products traded on the exchange must be well standardised, with pre-defined quantities, quality, and identity.

- OTC (short for *Over The Counter*) markets are online regulated markets, in which two parties trade directly without the supervision of an exchange. OTC trading, as well as exchange trading, occurs with commodities, financial instrument, and derivatives of such products, in particular interest rates, *FOREX* (short for FOReign EXchange), stocks, and commodities. OTC markets do not have limitations on quantity and quality.

It is then clear that investors have to choose the markets in which they want to operate depending on what they want to trade, since not every market trades everything. Furthermore, there are some additional limitations for some specific markets. For example, a private trader cannot simply decide to trade on futures since there are some specific requirements that one needs to meet in order to be allowed to trade futures. Hence a private trader who wants to invest on futures can either go to his bank and ask to execute the trade for him, or he can trade on a CFD (Contract For Difference) on

[1]In economics, a commodity is an economic good or service that has full or substantial fungibility, meaning that the market has no regard to who produced the commodity but treats instances of the good as equivalent. Most commodities are raw materials, agricultural, or mining products, such as sugar, rice, wheat, or iron ore. For further details, see J. C. Hull - "Options, Futures, and Other Derivatives", Prentice Hall (2000).

the desired future[2]. In the latter case, the trader is no longer investing on a regulated future but has a contract with a private broker in a Over The Counter market.

6.1.2 Investment horizon and indicators

Another fundamental characteristic that make a distinction between various investors is the investment horizon, as already outlined beforehand. Depending on what kind of investment one wants to do, some instruments are better indicated than others. For example, a person who wants to **invest** in a company because he wants to financially support that company and believes its value will grow in the future should think of using stocks of that firm. On the other hand, a private day-trader who simply wants to speculate and take profits from price changes could consider to trade CFDs.

As already outlined when explaining Peters' FMH, time horizon of investment affect not only the kind of financial instrument used by the investor but also the analysis he uses to evaluate investment opportunities. Indeed, traders with short-term investment horizon would prefer technical analysis, while investors with longer perspectives would prefer fundamental analysis. This distinction leads to another important feature that characterise both kind of investors: indicators.

Indicators are mathematical tools used to assess a company (or a security) under a financial and statistical point of view. Indicators can be essentially

[2] In finance, a Contract For Difference (CFD) is a contract between two parties, typically described as "buyer" and "seller", in which the buyer pays an interest rate to the seller in exchange for the return of the underlying.

divided in two categories: economic indicators and technical indicators.

- Economic indicators: these are statistics about an economic activity, and allow analysis of economic performance and predictions of future performance. Economic indicators include various indices, earning reports, and economic summaries. Thus, it is obvious that this kind of indicators is mainly used by fundamental analysts. Examples of economic indicators are the consumer price index and the gross domestic product.
- Technical indicators: these are mathematical calculation based on historic markets data such as price and volume, and their aim is to forecast financial markets direction. These indicators are mainly used by technical analysts who want to identify patterns that the price will probably trace on the chart. There are many technical indicators that had been developed over the years, and new variants continue to be developed by traders with the aim of getting better results. Examples of technical indicators are the Average directional movement index (ADX) developed by J. Welles Wilder in 1978 and the Bollinger Bands developed by John Bollinger in the '80s.

6.2 The trading strategy

The trading strategy we are going to define later has a short-term investment horizon since its aim is to speculate on price movements, and the time interval between two consecutive price observation is half-an-hour long. It is clear then we are going to use technical analysis to take investment decisions, and

in particular we are going to use the MH (Moving Hurst), a new indicator developed by two Czech scholars which will be described in detail later. Simultaneously, we will show profit results that could be obtained using three further trading strategies: the first one using the MACD (Moving Average Convergence/Divergence) indicator, the second one using two different moving averages, and the third one using the CCI (Commodity Channel Index) indicator. All these three strategies will be used as benchmarks to compare the performance of the Moving Hurst with trading strategies using more traditional indicators.
We are now going to describe the mathematical construction of MACD, CCI, and MH indicators.

6.2.1 Moving Average Convergence/Divergence

To correctly define the MACD indicator, it is necessary to first introduce the EMA (Exponential Moving Average). As the name suggests, it is a moving average calculated on the last n observations, where n is chosen, but it is slightly different from the SMA (Simple Moving Average). While the SMA is computed as the mathematical average of the last n, so simply as

$$SMA_t = \frac{1}{n} \sum_{i=0}^{n-1} P_{t-i} \tag{6.1}$$

the EMA (also known as EWMA, Exponentially Weighted Moving Average) applies weighting factors which decrease exponentially, giving more weight to recent observations and less weight to obvservations far in time. EMA can be calculated recursively using the following formula:

$$EMA_t = \left(P_t \cdot \frac{2}{1+n}\right) + EMA_{t-1} \cdot \left(1 - \frac{2}{1+n}\right) \tag{6.2}$$

where $\frac{2}{1+n}$ is the *smoothing factor*. So, since to compute the current EMA the previous EMA is needed, after the first *n* observation the $\mathrm{SMA_n}$ is computed, therefore having $\mathrm{EMA_n} = \mathrm{SMA_n}$. Then, from the following observation it will be used (6.2) to compute $\mathrm{EMA_{n+1}}$, $\mathrm{EMA_{n+2}}$ and so on.

Now it is possible to define the MACD indicator. First of all, it is necessary to compute two EMAs, usually on closing price: a "fast" EMA on 12 observation, and a "slow" EMA on 26 observation. The difference of these two series originates the MACD line proper. Then, the EMA of the MACD series is computed, and it originates the so called "signal line". Finally, the difference between the MACD line and the signal line originates the divergence series, that is the only one represented with bars in the chart, while others are simple lines.

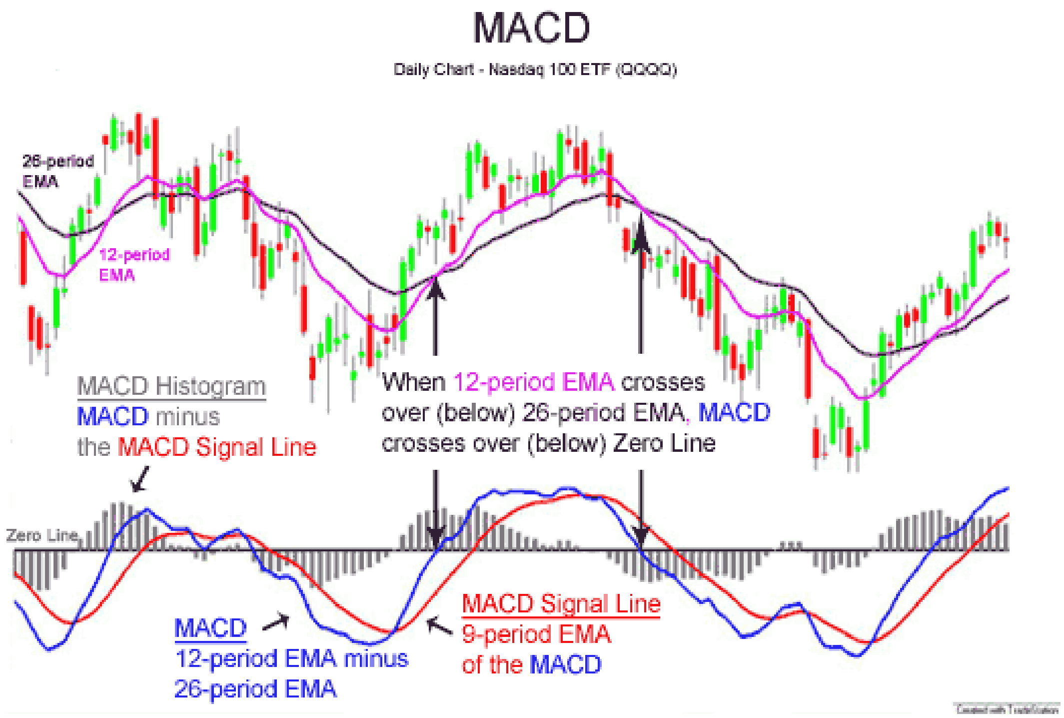

Figure 6.1: Example of MACD construction

The lengths of the moving average windows we have employed are the most commonly used values, since when having daily observations and the old working week used to be 6-days these numbers represent respectively two weeks, one month, and one and a half week. Now the working week is 5-days long and many investors changed the default numbers to more reactive values such as *MACD(5, 35, 5)*, but the classic *MACD(12, 26, 9)* is still widely used.

The technical interpretation of the MACD indicator is pretty simple: it generates a buy-signal when the MACD line crosses up through the signal line (bullish crossover, also called *golden cross*), and a sell-signal when the

MACD line crosses down through the signal line (bearish crossover, also called *devil cross*). Some traders also attribute special significance to the MACD line crossing the zero axis, but we will not consider in a particular way this event.

The MACD indicator is usually used together with other indicators like the RSI (Relative Strength Index)[3], but it could also be used alone, and we will opt for this alternative.

6.2.2 Commodity Channel Index

The Commodity Channel Index (CCI) is an indicator developed by Donald Lambert in 1980. It has been originally developed to identify cyclical trends in commodities' markets, but its popularity grew over the years and now it is widely used in many different markets.

The calculation is pretty simple: it is computed as the difference between the typical price and its simple moving average, divided by the mean absolute deviation of the typical price times a constant:

$$CCI_t = \frac{p_{typ_t} - SMA_n(p_{typ})}{0.015 \cdot MAD_n(p_{typ})} \tag{6.3}$$

where the typical price at time t is computed as

$$p_{typ_t} = \frac{p_{high} + p_{low} + p_{close}}{3} \tag{6.4}$$

and $SMA_n(p_{typ})$ is the Simple Moving Average of typical prices computed on

[3] The Relative Strength Index (RSI) is a technical indicator developed by J. Welles Wilder intended to chart the current and historical strength or weakness of a stock or market based on the closing prices of a recent trading period. For further details, see J. Welles Wilder - "New Concepts in Technical Trading Systems", Trend Research, Pristine (1978).

a period of n days. Furthermore, the mean absolute deviation $MAD_n(p_{typ})$ is computed as

$$MAD_n(p_{typ}) = \frac{1}{n} \sum_{i=0}^{n-1} |p_{t-i} - SMA_n(p_{typ})|. \tag{6.5}$$

The constant in formula (6.3) had been set to 0.015 from Lambert to ensure that approximately 70 to 80 percent of CCI values would fall between -100 and +100, even though this percentage depends also on the number of period used.

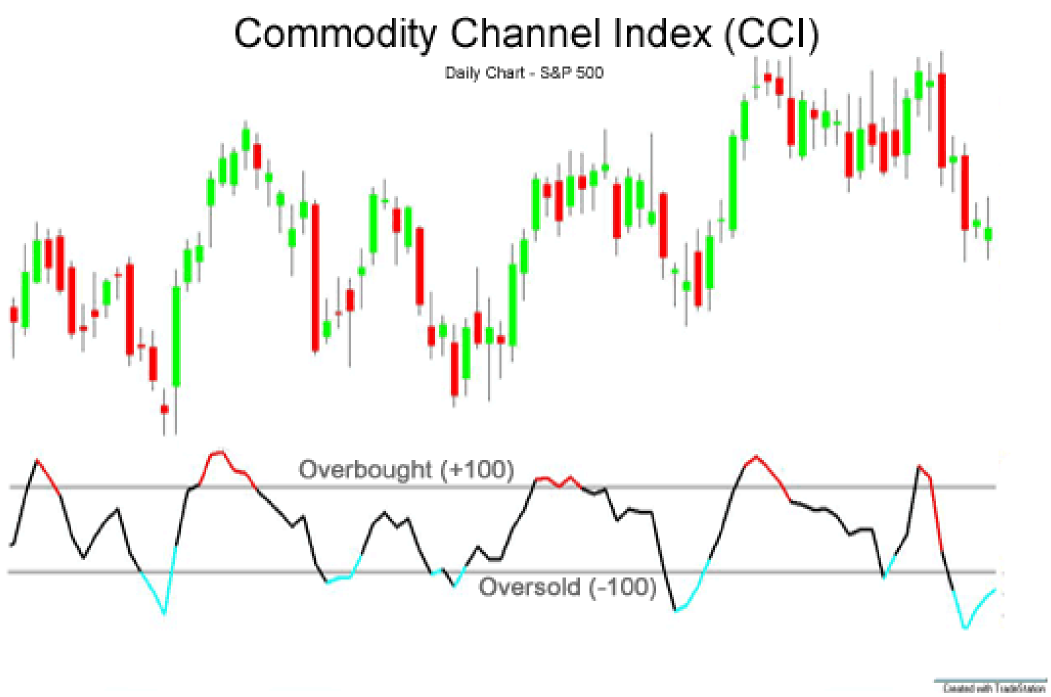

Figure 6.2: Example of CCI construction

It is really simple to technically interpretate the value of the CCI indicator: when it produces a value greater than 100 it means the security has reached an overbought level, while when it produces a value smaller than -100 it means the security has reached a oversold level. Thus, the trading strategy consists in buying the security when the CCI exceed 100 and sell it when it returns lower than this value, and on the other hand short-selling[4] the security when the CCI decreases below -100 and buy it back when it returns higher than this level.

The most commonly used value for the CCI parameter is 20, but if used alone this indicator generates too many signals, so we decided to smooth the resulting plot of CCI setting the parameter to 50 and consequently generating less investment signals.

6.2.3 Moving Hurst

Many scholars and many traders over the years tried to develop trading strategies based on the Hurst exponent and on the concept of long memory in financial markets, but not many concrete and valid application were found. Among the few ones that came to light, the one developed by the Czech scholars Petr Kroha and Miroslav Škoula seems quite interesting. While the majority of other scholars used the Hurst exponent in its classical interpretation (a tool to identify the presence of long-term dependence between observations), Kroha and Škoula took advantage of the relationship it has with the fractal dimension. In simple terms, they linked the variation of fractal dimension to the change in direction of the price trend, and use this

[4] The definition of *short-selling* will be provided later.

connection to predict and identify trend inversions. In particular[5]:

> *"The main idea of our approach is that changes in fractal dimension of a time series, which describe the history of prices, invoke changes in behavior of investors and traders. They buy or sell, and the feedback can be either negative, i.e., the fluctuation of prices decreases (a trend appears or continues), or positive, i.e., the fluctuation of prices increases."*

To better understand the process that led them to develop this technical indicator, it is appropriate to first describe their point of view on financial markets and price changes.

They explicitly declare they follow ideas of Peters' Fractal Market Hypothesis, though they also delineate their point of view on markets. In particular, they state that financial markets can be seen as a complex mixture of deterministic chaotic systems and stochastic non-linear systems. Chaotic systems are deterministic because, although being extremely dependent on the initial conditions, it is possible to predict their final state knowing the equations that regulate the entire system. The term "chaotic" refers to the unknown equations that describe the system, but it does not affect the fact that it is totally predictable. On the other hand, stochastic systems are affected not only by small differences of input parameters but also by unpredictable random external events having unpredictable impact on system behaviour. These systems are indeterministic because their rules of behaviour involve probabilities.

[5] For this and next quotes, see P. Kroha, M. Škoula - "Hurst Exponent and Trading Signals Derived from Market Time Series" (2018)

> *"Processes behind markets have their weak deterministic component [...], but they have a strong built-in randomness component, because the main changes are reactions on unpredictable, random events in the world, e.g. volcano eruption, terrorist attack on World Trade Center, floods in Thailand, some political decision.*
> *Additionally, compared with deterministic chaotic systems in physics [...], markets are nonlinear feedback systems, because they contain a component including psychology of human investors called behavior finance. This component brings reflexivity into the system, i.e., circular relationship between cause and effect. For example, when we would predict weather very exactly, weather were not change because of it. On the other hand, a well-known, precise market prediction would change markets completely."*

The stochastic component cannot be predicted, so their work focuses on the chaotic part of financial markets. In particular, they suppose that chaotic properties can react before prices change, meaning that a signal of a future reverse in the trend followed by the price of the security can be spotted *before* it actually happens. They focused on predicting changes on trends looking at changes in fractal dimension as previously stated. To do so, they developed a new indicator they called *MH* (short for *Moving Hurst*) which simply consists in the difference between two Hurst exponents, one “slow” and one “fast”, computed on closing prices. When this indicator performs a golden cross on the zero line, it generates a buy-signal, while when it performs a devil cross it generates a sell-signal.

After many optimisations, they found that the best profit results were obtained using 16 observations for the "fast" Hurst exponent and 32 observations for the "slow" one. The trading strategy is then so defined:

$$\begin{cases} (H_{32} - H_{16})_{t-1} < 0 & \wedge \quad (H_{32} - H_{16})_t \geq 0 \Rightarrow \text{buy-signal} \\ (H_{32} - H_{16})_{t-1} \geq 0 & \wedge \quad (H_{32} - H_{16})_t < 0 \Rightarrow \text{sell-signal.} \end{cases}$$

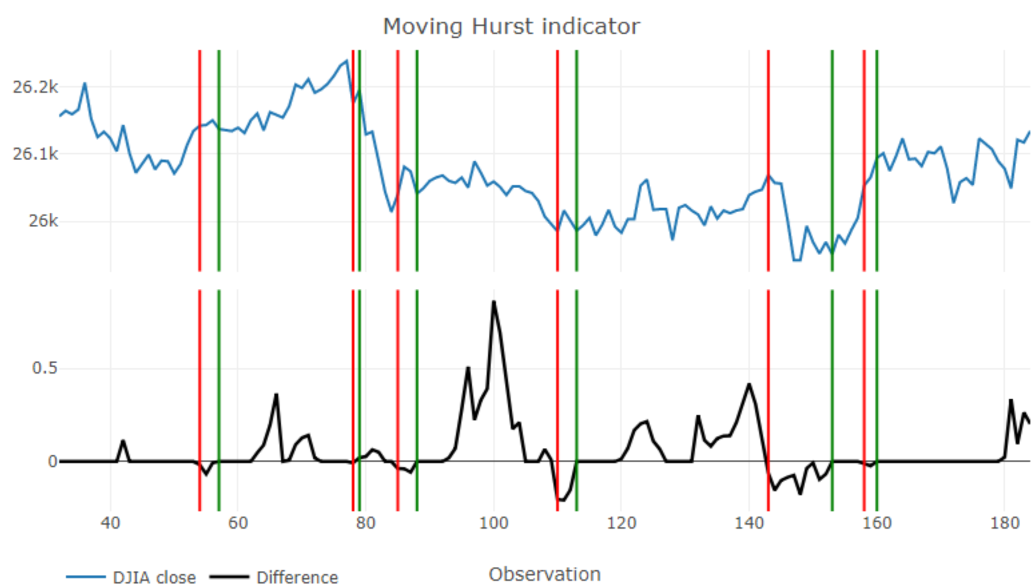

Figure 6.3: Application of Moving Hurst on DJIA. Red lines are sell-signals, green lines are buy-signals.

It is important to remind that this indicator, altough being computed calculating the Hurst exponent which measures long-term memory, is not based on the classical meaning of H. Indeed, it does not try to find cycles in the past observations of the time series, but instead uses the relationship between the Hurst exponent and the fractal dimension to try to predict changes in trend before they happen. Hence for this reason Kroha and Škoula applied this indicator also to short-term trading even though it uses parameter that are usually calculated to identify cycles in the long-term.

This indicator, as well as the MACD and the CCI, could be used together with other indicators to obtain better results. However, in our analysis it will be employed alone for the sake of simplicity.

Chapter 7

Technical application

7.1 Data input

We have described the indicators we are going to use and briefly introduced the trading strategy we will adopt, so it is now necessary to apply them to real past data. The data we are going to use are closing prices of futures on four american indices and from two currencies couples[1]. Observations are taken every thirty minutes from 00:00 to 24:00 for the following futures:

- Dow Jones Industrial Average (YM)
- S&P 500 (ES)
- Nasdaq 100 (NQ)
- S&P 400 MidCap (MC)
- Euro FX (6E)
- British Pound (6B)

The two currencies are coupled with the American dollar, and the symbols

[1]In trading, a *currency couple* refers to the security on the exchange rate between two currencies.

for these futures reflects the ones used in the Chicago Mercantile Exchange, one of the main futures markets. Observations for every future starts on the beginning of 2009 (except for the British Pound, which starts on 19th November 2010 due to missing data), and end on 16th October 2019, so they cover a period of a little more than ten years.

	From	To	Observations
Dow Jones Industrial Average	2009-01-02	2019-10-16	128483
S&P 500	2009-01-02	2019-10-16	128588
Nasdaq 100	2009-01-02	2019-10-16	128536
S&P 400 MidCap	2009-01-02	2019-10-16	111304
Euro FX	2009-01-02	2019-10-16	128720
British Pound	2010-11-19	2019-10-16	105993

Table 7.1: Summary of analysed time series

The main trading strategy more in detail will consists in a continuous sequence of opening and closure of positions depending on the value of the Moving Hurst. So, after identifying the first cross, the strategy opens a long position in case a golden cross happened, or a short position in case a devil cross happened[2]. Then, when a golden (devil) cross happen, the open short (long) position is closed and a new long (short) position is opened, both at the current closing price. Eventually, any open position is closed at the last observation.

To compute the MH indicator, five different methods are adopted in order to highlight the performance differences between the various alternatives:

[2]In trading, opening a *long* position means buying the instrument object of the trade, while opening a *short* position means *short-selling* the instrument. Short-selling is a trading practise that allows the investor to sell an instrument he does not own with the obligation of buying it back later.

- Fractal dimension (Fd): this is the only method that compute the fractal dimension and uses equation (4.6) to get the value of Hurst exponent. Among the possible methods to compute the fractal dimension of the given time series, the *box-counting* (or Minkovski-Bouligand) method has been chosen;
- Simplified Hurst (Hs): this method uses a simpler version of the R/S Analysis in which no sub-periods are computed and the Hurst exponent is computed simply as $H = \ln(R/S)/\ln(N)$;
- Corrected Hurst (Hrs): the classical R/S analysis to compute the Hurst exponent;
- Empirical Hurst (He): a corrected version of the classical R/S analysis which calculates all the exact divisors of the number of observations and computes a more precise Hurst exponent;
- Corrected Empirical Hurst (Hal): the most precise way to compute the Hurst exponent, derived from Anis and Lloyd work, which starts from the Empirical Hurst and considers also the expected R/S using the following formula:

$$Hal = \frac{He \cdot \sqrt{\frac{d \cdot \pi}{2}}}{\mathbb{E}[R_n/S_n]} \tag{7.1}$$

where *d* is the number of divisors for the evaluated number of observations, and $\mathbb{E}(R_n/S_n)$ is given by

$$\mathbb{E}[R_n/S_n] = \left(\frac{n \cdot \pi}{2}\right)^{-\frac{1}{2}} \cdot \sum_{r=1}^{n-1} \sqrt{\frac{n-r}{r}} \tag{7.2}$$

if $n > 340$, otherwise by

$$\mathbb{E}[R_n/S_n] = \frac{\Gamma\left(\frac{1}{2}(n-1)\right)}{\sqrt{\pi}\cdot\Gamma\left(\frac{1}{2}n\right)}\cdot\sum_{r=1}^{n-1}\sqrt{\frac{n-r}{r}}. \tag{7.3}$$

where $\Gamma(x)$ is the *Euler gamma function* defined as

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx, \quad \Re(z) > 0 \tag{7.4}$$

with $\Re(x)$ meaning the real part of the complex number x.

The one just described is the main strategy, which operates using the Moving Hurst indicator only. Additionally, to every time series of future prices two similar strategies of continuous buy and sell signals will be applied as a benchmark.

The first one relies on the MACD indicator previously described using standard values for its parameters.

The second one uses two different Exponential Moving Averages, one fast and one slow considering a window of 12 and 26 observations respectively, similarly to the MACD. The technical interpretation of these two EMAs can be compared to the one regarding the MACD line and the signal line: when the fast EMA performs a golden gross on the slow EMA the strategy generates a buy-signal, while vice versa when the fast EMA performs a devil cross on the slow EMA it generates a sell-signal. Lastly, an additional strategy considering the CCI indicator will be used as a benchmark. This strategy does not continuously alternate long and short positions since it is possible to have periods without open positions if the CCI oscillates in the [-100, 100] range, as well as two or more consecutive long or short operations.

7.2 Implementation and results

The following graphs show the evolution of profits for each instrument, applying the previously described trading strategy for each different way of computing the Hurst exponent, plus the benchmark strategies using the standard MACD indicator *MACD(12, 26, 9)*, the two moving averages *EMA(12)* and *EMA(26)*, and the *CCI(50)*. Profits are expressed simply as price variation between opening and closure of the position, so values shown in the graphs are “absolute” and do not depends on the amount invested in operations. For the two currencies, profits are multiplied by 1000 in order to get non-decimal price variations.

7.2.1 Performance of Dow Jones Industrial Average

	Net profit	# trades	% wins	Avg win	Avg loss	Avg trd
Fd	13983.92	54988	48.55	19.24	-18.92	0.25
Hs	17698.08	7484	49.73	51.53	-48.04	2.36
Hrs	19395	15922	49.47	36.46	-34.61	1.22
He	5770	16924	49.28	36.52	-36.03	0.34
Hal	6773	18714	49.07	35.05	-34.32	0.36
MACD	5816	9566	33.24	74.11	-36.39	0.61
MA	959	4307	28.37	122.62	-48.51	0.22
CCI	2418.04	7006	29.17	62.76	-25.99	0.35

Table 7.2: Trading results for the Dow Jones Industrial Average

Profits achieved by the trading strategy applied to the Dow Jones Industrial Average are positive for every different way to compute the Hurst exponent. Furthermore, also the benchmark strategies lead to a gain at the end of the considered time period, although they all had negative profits most of the time. In particular, the MACD indicator seems to provide the better results among the three alternatives, ending in a profit that is comparable to profits obtained using Empirical Hurst and Corrected Empirical Hurst, while strategies using Moving Averages and CCI lead to positive but much smaller gains.

Considering only the five main strategies, the best performing strategy is the one using the classical R/S analysis, while the worst one is the one based on the Empirical Hurst.

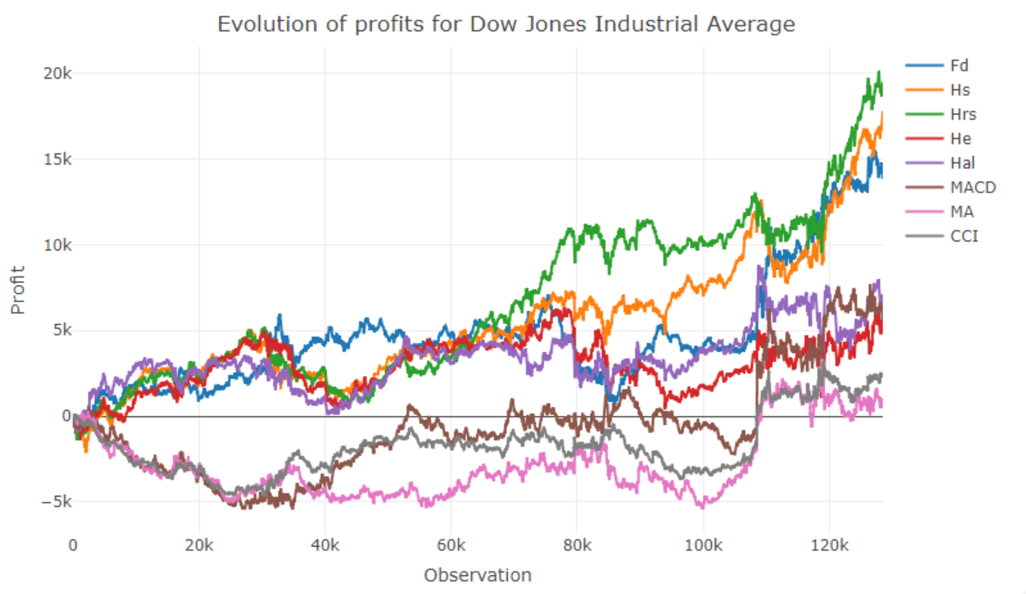

Figure 7.1: Profit on the Dow Jones Industrial Average

7.2.2 Performance of S&P 500

	Net profit	# trades	% wins	Avg win	Avg loss	Avg trd
Fd	1151.65	55231	46.52	2.33	-2.28	0.02
Hs	1562.25	7314	49.12	6.04	-5.75	0.21
Hrs	2283.3	15758	48.93	4.3	-4.17	0.14
He	-1085.55	16834	47.71	4.29	-4.31	-0.06
Hal	-225.15	18768	47.66	4.13	-4.08	-0.01
MACD	36	9584	32.99	8.58	-4.31	0
MA	-433.75	4391	27.99	13.92	-5.6	-0.1
CCI	287.8	6995	28.23	7.44	-3.01	0.04

Table 7.3: Trading results for the S&P 500

Profits on the S&P 500 are quite various, ranging from -1000 to more than 2000 depending on the chosen computation algorithm. On the other hand the benchmark strategies realises restrained losses most of the time, and at the end the ones using the MACD and the CCI manage to achieve marginal gains, while the one using Moving Averages ends in a discrete loss.

Again, the higher profit is obtained using the strategy with H computed through the classical R/S analysis, while the most serious loss is originated by the strategy which uses the Empirical Hurst exponent.

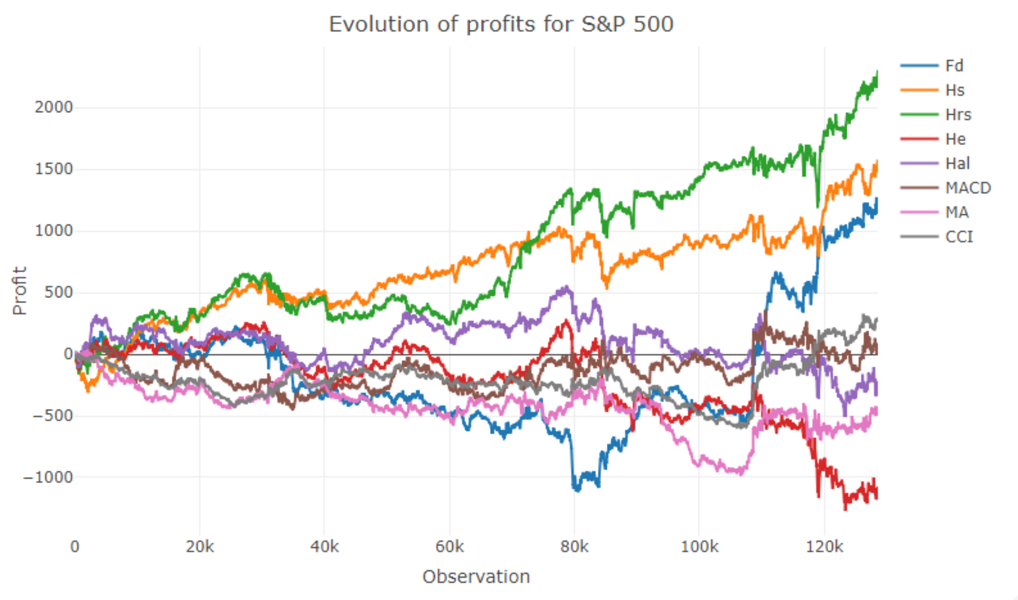

Figure 7.2: Profit on the S&P 500

7.2.3 Performance of Nasdaq 100

	Net profit	# trades	% wins	Avg win	Avg loss	Avg trd
Fd	-448.5	54972	48.38	5.57	-5.6	-0.01
Hs	4414	7818	49.78	15.2	-14.39	0.56
Hrs	2006.5	16062	49.38	10.59	-10.52	0.12
He	-2404.25	17080	49.04	10.65	-10.89	-0.14
Hal	-2403.75	18812	49.3	10.12	-10.44	-0.13
MACD	-119	9512	32.18	22.88	-11.01	-0.01
MA	-825	4352	27.94	37.06	-14.68	-0.19
CCI	1722	6911	28.72	19.66	-7.75	0.25

Table 7.4: Trading results for the Nasdaq 100

For the Nasdaq 100 the range between the best and the worst performing strategies is even wider than before, varying from almost -2500 to slightly less than 4500. Benchmark strategies, after being in constant loss for almost all the time, in the last years exhibit a significant rise in profits, managing to almost break even the starting value and even ending in a substantial gain in the case of CCI.

Here the best performing strategy is the one using the Simplified Hurst, while the Corrected Hurst only realises less than half of Simplified Hurst strategy's profit. On the other hand, Empirical Hurst is the worst performing strategy, along with the Corrected Empirical Hurst.

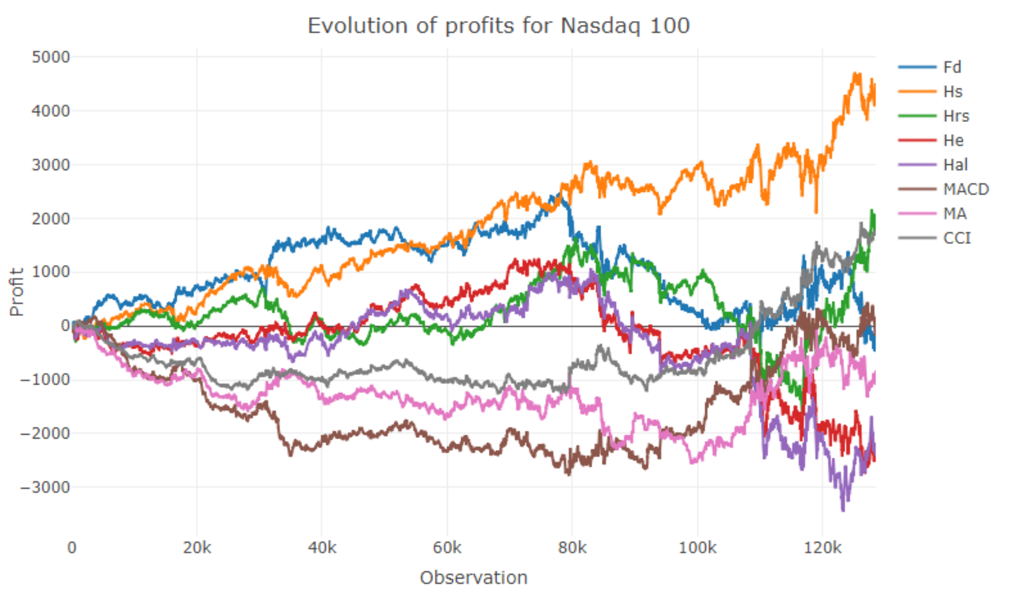

Figure 7.3: Profit on the Nasdaq 100

7.2.4 Performance of S&P 400 MidCap

The only strategies which realise profits on the S&P 400 MidCap are the ones using the Simplified Hurst and the Corrected Hurst. All the other strategies end up with significant losses, as well as all the benchmark strategies, with

	Net profit	# trades	% wins	Avg win	Avg loss	Avg trd
Fd	-1133.7	47895	47.88	1.97	-1.98	-0.02
Hs	781.2	6306	50.03	5.33	-5.22	0.12
Hrs	542.5	13765	49.07	3.72	-3.63	0.04
He	-888.5	14369	49.05	3.73	-3.82	-0.06
Hal	-872.3	16136	48.85	3.56	-3.61	-0.05
MACD	-895.4	8425	33.07	7.42	-3.86	-0.11
MA	-340.2	3833	28.18	12.27	-4.96	-0.09
CCI	-266.6	6381	29.9	5.68	-2.53	-0.04

Table 7.5: Trading results for the S&P 400 MidCap

the MA and CCI ones performing much better than the MACD one.

Here the best strategy is the *Hs* one, gaining a little more than the *Hrs* one, while the worst one is the one using the fractal dimension, which loses slightly more than the *He* and the *Hal* strategies.

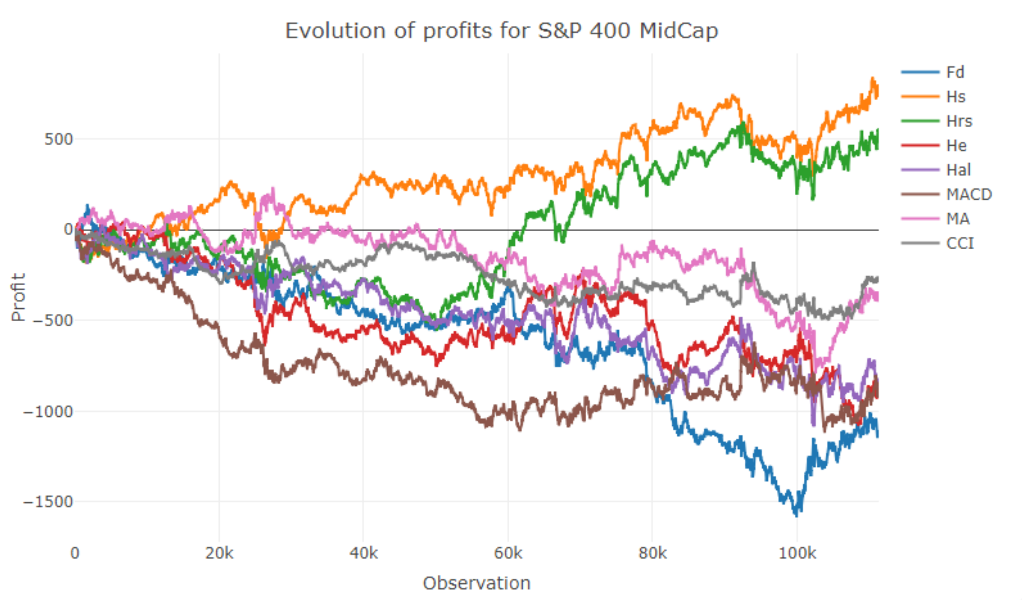

Figure 7.4: Profit on the S&P 400 MidCap

7.2.5 Performance of Euro FX

	Net profit	# trades	% wins	Avg win	Avg loss	Avg trd
Fd	-136.6	55390	47.67	1.06	-1.06	0
Hs	-317.9	7297	49.77	2.67	-2.85	-0.04
Hrs	-323.3	15326	48.97	1.92	-1.98	-0.02
He	52.8	16521	48.8	1.99	-1.98	0
Hal	255.4	18517	48.89	1.91	-1.89	0.01
MACD	-325.8	9816	33.65	3.8	-2	-0.03
MA	-1.4	4367	29.88	6.11	-2.62	0
CCI	-648.05	7311	27.83	3.08	-1.35	-0.09

Table 7.6: Trading results for the Euro FX

For the Euro FX strategies' behaviour is totally the opposite compared to previous indices. Here the *Hs* and *Hrs* strategies, along with the *Fd* one and the benchmark strategy using the MACD, leads to moderate losses, while the Empirical Hurst and above all the Corrected Empirical Hurst result in reasonable profits. The benchmark strategy using the MA ends with a profit of almost zero, while the CCI one is the worst strategy at all, having negative profits since the beginning.
Furthermore, it is possible to notice that all these series have much more jagged paths rather than the previous analysed ones.

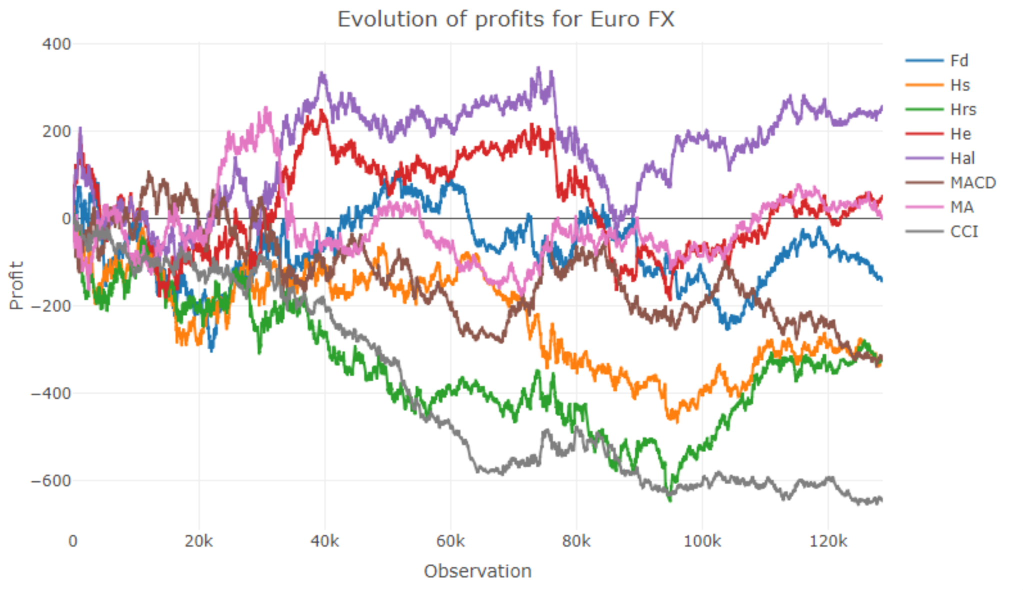

Figure 7.5: Profit on the Euro FX

7.2.6 Performance of British Pound

	Net profit	# trades	% wins	Avg win	Avg loss	Avg trd
Fd	430.2	45175	47.8	1.1	-1.1	0.01
Hs	-160.6	6098	48.18	2.87	-2.85	-0.03
Hrs	-689.2	13179	47.61	1.99	-2.02	-0.05
He	160.9	13743	48.29	2.09	-2.03	0.01
Hal	163.1	15241	48.13	2.01	-1.95	0.01
MACD	-721.8	8045	32.67	4.01	-2.11	-0.09
MA	147.6	3552	28.86	6.67	-2.67	0.04
CCI	-412.9	5934	27.67	3.37	-1.43	-0.07

Table 7.7: Trading results for the British Pound

Finally, for the British Pound the behaviour is similar to the Euro FX. Here again both the Simplified and Corrected Hurst do not realise profits, as well as the MACD and the CCI strategies which are in continuous and constant loss. On the other hand, the strategy using the Fractal Dimension is the only one which almost always has a positive profit, as well as being the best performing one. The MA strategy ends with a positive profit too, but for more than half the considered time interval had discrete losses, recovered only in the last years.

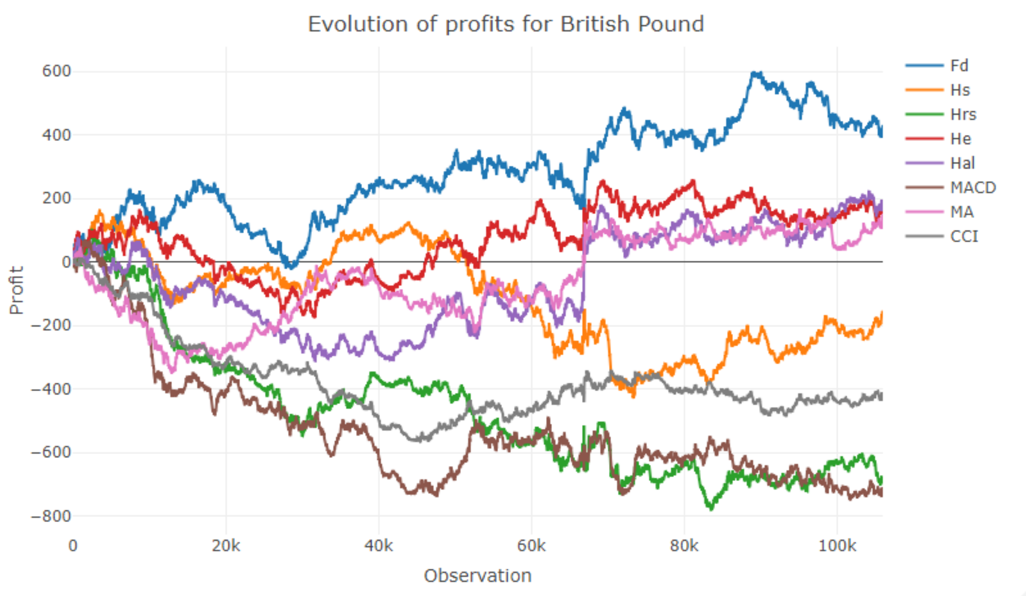

Figure 7.6: Profit on the British Pound

7.3 Analysis of results

Talking about the four indices, it is interesting to notice that the best performing computation methods are the ones using either the Simplified or the Corrected Hurst exponent, while the others do not perform very well.
In particular, the two specified methods are the only one which always generate a profit at the end of the considered time period. On the other hand,

the strategies using the Empirical Hurst and the Corrected Empirical Hurst always leads to substantial losses but in the case of the Dow Jones Industrial Average, in which they are the worst performing ones anyway. The strategy using the Fractal Dimension is somehow halfway in performance, ending with a significant profit when applied to the Dow Jones Industrial Average and the S&P 500, and in a moderate or considerable loss when applied to the Nasdaq 100 and the S&P 400 MidCap respectively.

The matter is different for what concerns the two analysed currency couples. Here the Simplified Hurst and the Corrected Hurst not only lead to consistent losses, but also in both cases are the worst performing methods among the five possible choices. Instead the Empirical Hurst and the Corrected Empirical Hurst, that for indices are always the worst performing ones, here behaves quite well leading to interesting profits. Finally, again the strategy using the Fractal Dimension places midway, realising a marginal loss on the Euro FX and the higher profit among all the five alternatives on the British Pound. However, the MH indicator does not seem to be a good trading strategy for currencies, as it is possible to see looking at the graphs. While on indices profits have kind of monotone trends (either upwards or downwards), on currencies they have much more volatility and their time series seem to change direction quite frequently. This aspect is much more visible on the Euro FX graph, which highlights the jagged patterns followed by profit time series. Hence, looking at the plots, profits on currencies seem to be much more "unstable" than profits on indices, so applying this strategy in currency markets is not recommended in our opinion.

Finally, looking at the performance of the benchmark strategies, it is clear that they are always outperformed by the MH indicator.
More in detail, the MACD trading strategy always leads to lower or higher losses, or to a breakeven at most, with the only exception that occurs when applied to the Dow Jones Industrial Average.
The same can be said for the strategy considering the two Moving Averages does not perform particularly well when applied on indices. On the other hand, when applied to currencies it behaves a little better, especially with the British Pound, but still does not stand out too much from the others.
Finally, the CCI strategy leads to a profit when applied to all the indices but the S&P 400 MidCap, although the gains are still smaller compared to the ones obtained by other strategies using the MH indicator. Vice versa, when applied to currencies this strategy leads to consistent losses, especially on the Euro FX.
Notice that these strategies used as a benchmark were built using specific values for their parameters, but can lead to very different results when these values are changed. So, it is necessary to specify that the efficiency of these strategies may be improved performing an optimisation process for the parameters considered.

Chapter 8

Conclusions

This dissertations is mainly focused on demonstrating that the classical *Efficient Market Hypothesis*, although being still widely used worldwide, is partly outdated, and other valid and interesting theories could be considered. Among the various alternative models proposed over the years by many scholars, we have chosen to focus on the *Fractal Market Hypothesis* developed by Edgar E. Peters. Then, before presenting in details the FMH, we made a brief introduction to the world of fractals describing their characteristics and properties. Later, we illustrated thoroughly Peters' theory analysing all its features and highlighting the differences from the classical EMH. Finally, considering the characteristics and the constraints of the FMH, we developed a trading strategy based on a technical indicator called Moving Hurst. To compute the MH we followed the procedure proposed by Petr Kroha and Miroslav Škoula calculating the Hurst exponent in five different ways, and we applied the same trading strategy to six different futures, two American indices and two currency couples.

Results were discussed and analysed, and the Moving Hurst indicator superiority compared to other classical indicator has been proved for every security. However, better results were obtained when applied to indices rather than currency couples, that exhibited more jagged patterns of profits. This instability over time suggests that the MH indicator may not be the best indicator for currencies, or at least that could be used along with other techinical indicators. Hence, further analyses are required in order to find a better application of the Moving Hurst to currency couples.

It is also important to precise that all these strategies do not take into accounts transaction fees. These can significantly affect the final profit, especially considering the great number of operations completed for the strategy using the Fractal Dimension. For example considering the Dow Jones Industrial Average, which is best performing index among the instruments considered, between the beginning and the end of the analysed time period there are 3939 days. Considering 250 working days per year and computing a proportion, we found that there are about 2700 working days in this period of time. Hence, having almost 55000 operations in this time interval, it means that the strategy completes more than 20 trades per day. This is an extremely high number for a private investor, who usually suffers prohibitive transactions fees, and it makes this trading strategy unsustainable unless the investor has particular agreements on fees and transactions costs with his broker.

This leads to the possibility of changing the parameters of the strategy, and in particular the number of observations considered by the “fast” and the “slow” Moving Hurst. Considering again the Dow Jones industrial Average, it is sufficient to slightly change the values of these parameters to completely change the output. For example, the following are profit results of the trading strategy that differs from the previously analysed one just for the value of the “fast” Moving Hurst, which now is 18 instead of 16 (MACD strategy remained the same).

	Net profit	# trades	% wins	Avg win	Avg loss	Avg trd
Fd	11632.92	53202	48.61	19.73	-19.53	0.22
Hs	17200.08	8632	50.06	48.7	-46.44	1.99
Hrs	20285	16896	49.38	35.34	-33.57	1.2
He	8481	16324	49.33	37.62	-36.78	0.52
Hal	2854	17914	49	35.73	-35.21	0.16
MACD	5816	9566	33.24	74.11	-36.39	0.61
MA	959	4307	28.37	122.62	-48.51	0.22
CCI	2418.04	7006	29.17	62.76	-25.99	0.35

Table 8.1: Trading results for the Dow Jones Industrial Average

It is possible to see that patterns are slightly different for all the computation methods, but profit results do not change too much considering that the time interval is almost 11 years long.

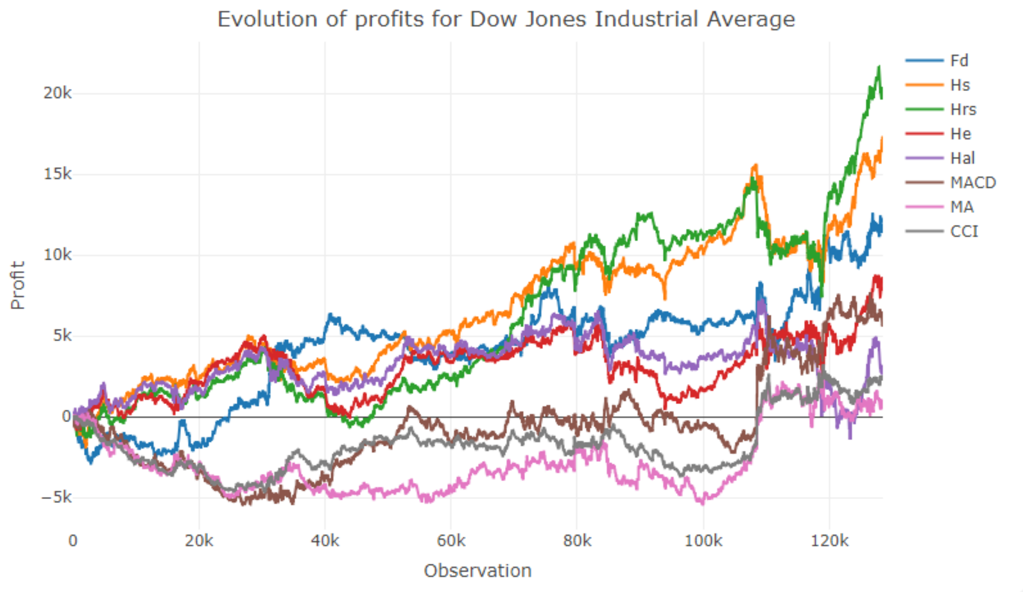

Figure 8.1: Profit on the Dow Jones Industrial Average

However, doing a little change in both the parameters, and setting for example to 20 and 40, results are totally different.

	Net profit	# trades	% wins	Avg win	Avg loss	Avg trd
Fd	4152	50033	48.95	20.18	-20.4	0.08
Hs	18616	6724	49.97	58.12	-53.72	2.77
Hrs	14176	13568	50.22	39.68	-39.54	1.04
He	17672	14049	49.99	41.13	-39.97	1.26
Hal	17860	15447	50.16	38.75	-38.15	1.16
MACD	5816	9566	33.24	74.11	-36.39	0.61
MA	959	4307	28.37	122.62	-48.51	0.22
CCI	2418.04	7006	29.17	62.76	-25.99	0.35

Table 8.2: Trading results for the Dow Jones Industrial Average

Now the strategies using the Empirical Hurst and the Corrected Empirical Hurst are no longer the worst performing ones. Indeed, only the Simplified

Hurst one performs a little better, while the *Hrs* and above all the *Fd* strategies perform worse than before.

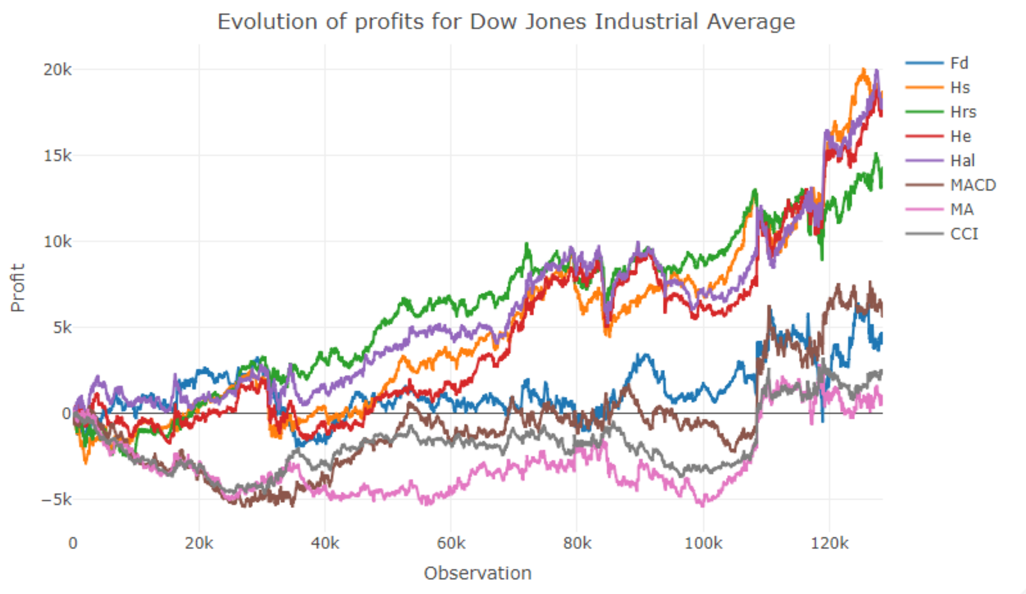

Figure 8.2: Profit on the Dow Jones Industrial Average

Further analyses should be done before actually applying this indicator to a real trading strategy, but the results obtained for the selected indices are certainly very interesting. A complete strategy optimization would be the next step to follow in order to produce a valid and efficient indicator, optimising not only the values of the parameters but also the time interval between two different observations. Eventually, for what concerns currency couples, doing the same optimisation would be useful not only to identify the best parameters combination but also to evaluate whether it is worth or not to apply this indicator on currencies time series.

Chapter 9

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