Ca' Foscari University of Venice

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Final Thesis

Downside Risk in a Tracking Error Minimization Problem

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SUMMARY

Introduction

In the following thesis it will be presented a problem of tracking error minimization with respect to a benchmark, the SPX - S&P 500 Top 50. Given the complexity of the problem, it will be used a meta-heuristic: the Particle Swarm Optimization. The tracking error function will not be the classical one, which uses the overall volatility in the computation but it will focus on the downside risk. This thesis is divided in five parts: the first one, concerning the studies behind the selection of a financial portfolio and the theory behind the tracking error; the second one, concerning two important ratio that will cover a very important role in the definition of the tracking error measure that will be used; the third one, which focuses the attention on the definition and derivation of the tracking error measure; the fourth one, which will cover the literature of the particle swarm optimization, describing the model that will be used in the next chapter and the fifth one, which will cover the application of the model to the problem and then the application of the parameters obtained to the virtual future to assess the validity of the model. In particular, the first chapter will start from describing the Markowitz's mean-variance criterion and its criticalities and it will introduce the concept of Tracking error as a possible solution of Markowitz's model (Roll's and Jorion's models). The second chapter will describe the Sharpe ratio and the Sortino ratio, analyzing the differences and implication of the two ratios in two examples. In particular, a great importance will be given to the Sortino ratio, which will the core of the proposed tracking error measure. In the third chapter, such intuitions will be used to define the tracking error measure that will be the objective function of the minimization problem: this measure will focus on the downside risk rather than the overall risk because it could be easier to investors to understand and evaluate it. Furthermore, the ETF that will be used in the problem will be presented. The fourth chapter will describe the Particle Swarm Optimization (PSO), the biology-inspired meta-heuristic which will be used to solve the problem with the Kennedy-Ebenhart-Shi model. The fifth chapter will cover the application of the PSO to the problem. In the first part, after a brief view on the economic and financial period of the observations, the model on which the PSO will be applied will be presented: the PSO will be performed through a Matlab code, suitably created for the problem. The process of parameter estimation will be divided in two parts: in the first one, several runs will be performed to obtain the in-sample population/iteration parameters which lead to the best fitness value. In the second part, the combination will be used to perform the out-of-sample test to assess the validity of the model. The out-of-sample results will be compared to the performances obtained by the ETF and the index and then, the final comparison and comments will be made to assess whether the model is suitable to the problem or not.

1. Starting the analysis

The starting point of the analysis involves the introduction of Markowitz's Mean-Variance criterion, the foundation of the Modern Portfolio Theory, with the definition of optimal portfolio under Mean-Variance criterion. Moving forwards, the definition of tracking error will be introduced and defined under two interesting point of view on the same problem: the Roll's one and the Jorion's one.

1.1 Markowitz mean-variance criterion

The classical financial theory asserts that the investor's utility in a certain investment is given by the performance and the risk of the return of the considered investment. The return can be measured in many ways, such as the percentage variation of the asset value in a given period, whereas the performance and the risk can be respectively seen as the higher return and the toll to pay in order to obtain that return in the investment. In 1952, Harry Markowitz introduced the ***Modern Portfolio Theory*** (***MPT***)***,*** a framework for assembling a portfolio in which the expected return is maximized for a given level of risk by selecting the proportions of assets in the portfolio. The basic assumptions of the MPT are:

- *Returns are normally distributed random variables*
- *Investors are rational (i.e. they avoid unnecessary risk)*
- *There are no asymmetries in information*
- *Investors attempt to maximize their expected utility*
- *There are no taxes or commissions*
- *There is no investor large enough to influence the price*
- *Investors have unlimited access to borrow money at the risk-free rate*

A **financial portfolio** composed by n assets can be assembled by investing a certain capital Z into n assets, such that $\sum_{i=1}^{n} Z_i = Z$ where Z_i is the proportion of wealth invested in the i^{th} asset (where $= 1, 2, \dots, n$). Each asset has its own weight w (such that $\sum_{i=1}^{n} w_i = 1$) and a certain return R . The (expected) portfolio return can therefore be expressed as the weighted average of the returns of each single asset, meaning that every single asset contributes to the overall portfolio return $R_P$with respect to its weight:

$$E(R_P) = \sum_{i=1}^{n} \frac{R_i * w_i * Z_i}{Z_i} = \sum_{i=1}^{n} E(R_i) * \ w_i.$$

As mentioned before, the portfolio's expected return is only one of the two components to measure the investor's utility. Every asset in the portfolio has a certain dispersion of return with respect to its mean, the ***standard deviation*** σ_{i} *,* calculated as the squared root of the ***variance*** and it contributes to the overall portfolio variance with respect to its weight:

$$\sigma_P^2 = \sum_{\mathrm{i}=1}^{n} \sum_{j=1}^{n} w_i w_j * cov\big(R_i, R_j\big).$$

In order to choose the best portfolio for a certain investor, it's fundamental to indicate both a measure of profitability and a measure of riskiness. The chosen measure of profitability is obtained by considering the expected return of each assets. The reason why it has been considered the expected value of the asset instead of the actual return lies in statistic: since the its returns are volatile and aleatory, an important information about the asset return is given by its mean value. The standard deviation of the portfolio returns is the measure of riskiness for the model. The model also considers the correlation between the assets: investors should choose assets that are less likely to lose value at the same time. The MPT introduces the concept of "***efficient frontier***" which is the set of optimal portfolios gathers with the best risk-return combination (the definition of such frontier will be discussed in the next pages). For a given return, every portfolio in the efficient frontier has the lowest variance possible and, conversely, for a given risk it has the highest return possible. Graphically, it can be shown that the efficient frontier is represented as follows:

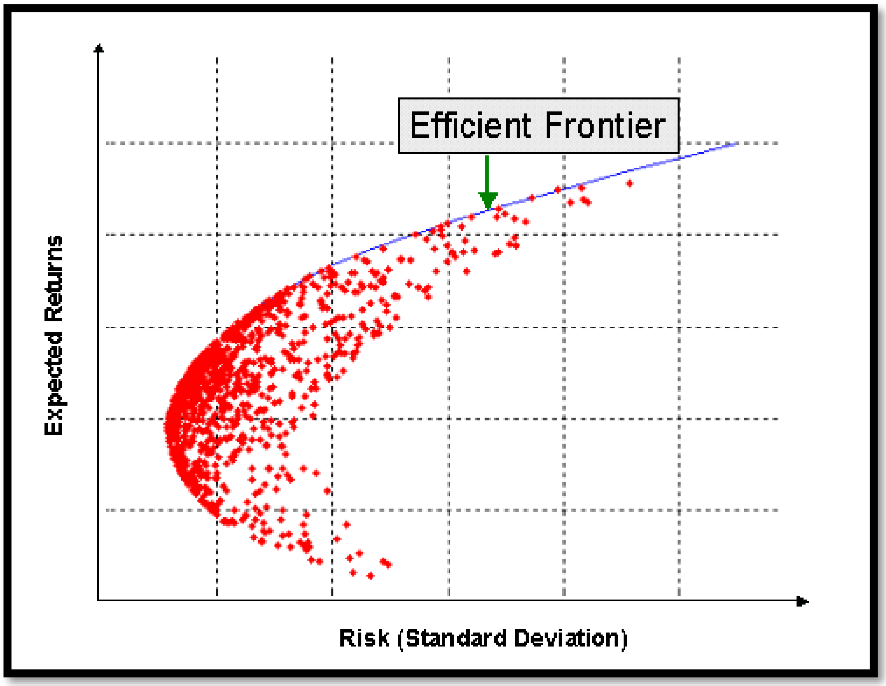

Figure 1: The efficient frontier. The red dots represent possible portfolios with different risk-return combinations. The blue line is the efficient frontier, which gathers portfolios with the best risk-return combination for the same level of risk

Portfolios allocated along the efficient frontiers are called ***efficient portfolios***, because they have the best risk-return combination. Since for a given level of risk there are different portfolios, the one which has the highest return for a given level of risk is said to be the ***dominant*** portfolio and therefore it lies on the efficient frontier. For a given level of risk, a portfolio A *dominates* portfolio B if the expected return of A is greater than the expected return of portfolio B. These portfolios are considered "Pareto optimum", in the sense that it is not possible to improve the expected return without improving the riskiness as well. For a given value of variance, investors will chose the portfolio with the higher expected return and the lowest variance (risk-aversion). The next step is to determinate a mathematical process to trace the efficient frontier. To determine the frontier, it is necessary to solve the following problem:

$$min_x \, x'Vx$$

$$s.t. \begin{cases} x'\bar{r} = \pi \\ x'e = 1 \end{cases}$$

where:

- x is the n-vector of assets' weights
- V is the variance-covariance matrix
- $\bar{r}$ is the vector of the expected returns
- e is the unit vector of order n
- π is the desired return of the investor

The two constraints impose that the minimization problem has to consider the investor's expected return π and that the investor invests all of his available capital in the portfolio. The optimization process traces a parabola in the mean-variance graph. It is possible to notice in the figure below that there is exactly one portfolio which has the minimum variance among all the others: the ***Global Minimum Variance Portfolio*** (***GMVP***). It not only represents the less risky portfolio in the efficient frontier, but it divides the parabola in two sectors: the upper sector gathers all efficient portfolios, whereas in the lower sector lie inefficient portfolios since, for a given level of risk, it is possible to choose an efficient portfolio with a higher return.

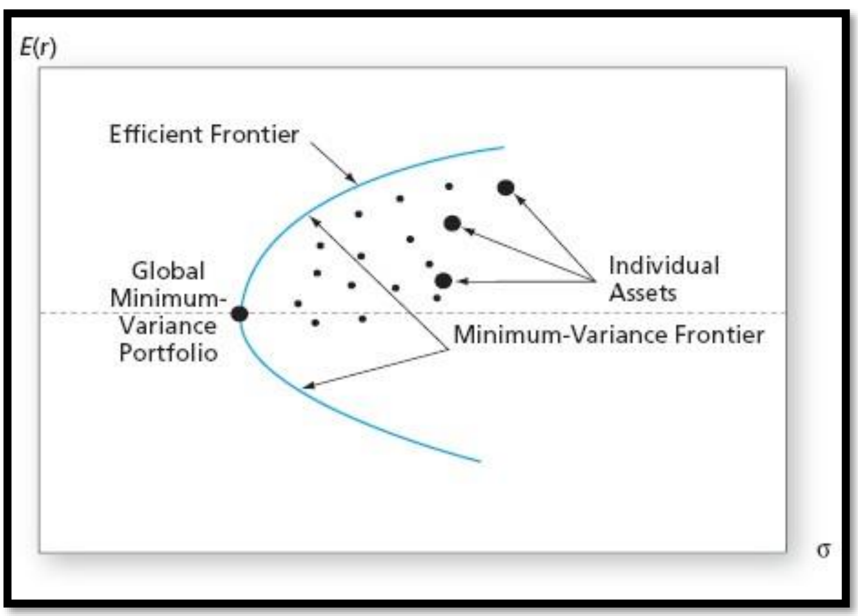

Figure 2: The global minimum variance portfolio: the GMVP represents the threshold between the efficient and inefficient frontier.

A further step is to choose the best portfolio for the investor in the frontier and this process needs the definition of investor's utility. A utility function is a function *U* defined on real numbers which represent levels of wealth and giving a real value. All different random wealth levels are ranked by computing their expected values, so that it is possible to state that:

$$E[U(x)] \geq E[U(y)]$$

There is not a unique utility function suitable to all investors, since it depends on their financial environment and risk aversion. The only restriction is that it must be an ***increasing*** (or ***strictly increasing***) ***function***. The most used utility functions are:

- **Exponential**: $U(x) = -e^{-ax}$
- **Logarithmic**: $U(x) = \ln(x)$
- **Power**: $U(x) = bx^b$
- **Quadratic**: $U(x) = x - bx^2$

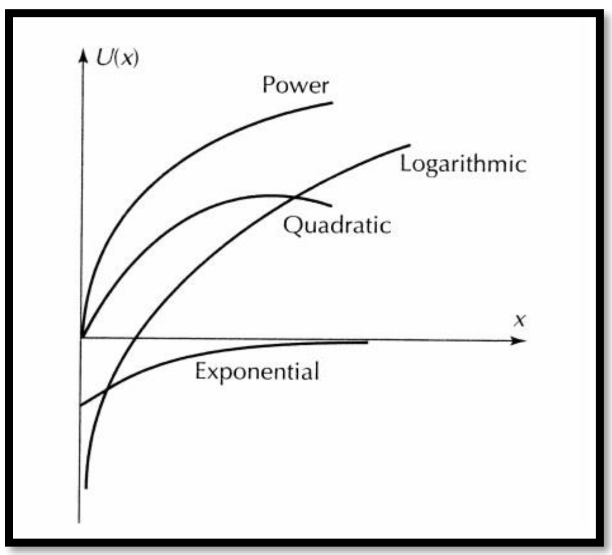

Figure 3: Representation of different utility functions

Markowitz uses the quadratic utility function $U = R_P - AR_P^2$ where R_P is the portfolio return, *A* is the risk-aversion coefficient for the investor and σ_P^2 is the portfolio variance. The objective of the process is to find the portfolio which maximizes the expected utility of the investor, but the chosen portfolio must lie on the efficient frontier.

The ***portfolio diversification*** plays an important role in this process. To diversify a portfolio implies to collect assets with low levels of ***correlation*** to the others. Suppose the case of an investor willing to invest the same amount on N>1 assets (in this case we assume the portfolio to be ***equally-weighted***), the variance of the portfolio is defined as:

$$\sigma_P^2 = \frac{1}{N}\sum_{i=1}^{N}\frac{\sigma_i^2}{N} + \frac{N-1}{N}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\frac{\sigma_{i,j}}{N(N-1)};$$

the covariance is:

$$\overline{Cov} = \frac{1}{N(N-1)}\sum_{j=1}^{N}\sum_{i=1}^{N}Cov\left(R_i, R_j\right).$$

Therefore, the variance of the portfolio can be expressed as:

$$\sigma_P^2 = \frac{1}{N}\bar{\sigma}_i^2 + \frac{N-1}{N} * \overline{Cov}_{i,j}$$

where $\bar{\sigma}_i^2$ is the variance of the equally-weighted portfolio and $\overline{Cov}_{i,j}$ is its covariance.

When the $\overline{Cov}_{i,j} = 0$, the only risk is the so-called ***systematic risk***, i.e. the risk associated with the market. The first addend tends to 0 only when the number of assets in our portfolio N tends to ∞: in such way, it is possible to diversify and then decrease the portfolio's risk. When $\overline{Cov}_{i,j} \leq 0$, the portfolio variance decreases: this phenomenon is called ***risk contraction***. This model, however, has different intrinsic limits, which make it useless in real-market applications, that can be summarized in three main topics:

- Operative choice of the target return;
- Estimation risk related to the model's parameters;
- Instability of the mean-variance solutions.

The mean-variance approach tends to overweight assets with large expected returns, small variances and negative correlations: unfortunately, those assets tend to bear large estimation errors. By increasing the number of assets in a portfolio (N), the number of estimates increases exponentially: for 100 assets, an investor must estimate 100 expected returns, 100 variances and 4950 covariances (i.e. $(n^2-n)/2$). Furthermore, the investor must specify a target return π, but it may be difficult to choose a coherent target return, especially within unstable financial environments, because it may choose a portfolio characterized by too much variance or leave expected returns on the table. Asset's mean, variance and N-1 covariances are unknown to the investor and they must be estimated from historical data. Every estimation implies estimation errors, which can degrade the properties of the efficient frontier. Input estimations are contained in a confidence interval, meaning that there can be a certain degree of inaccuracy in estimating the expected return, the variance and covariance; therefore, the efficient frontier will have an upper and a lower bound as well.

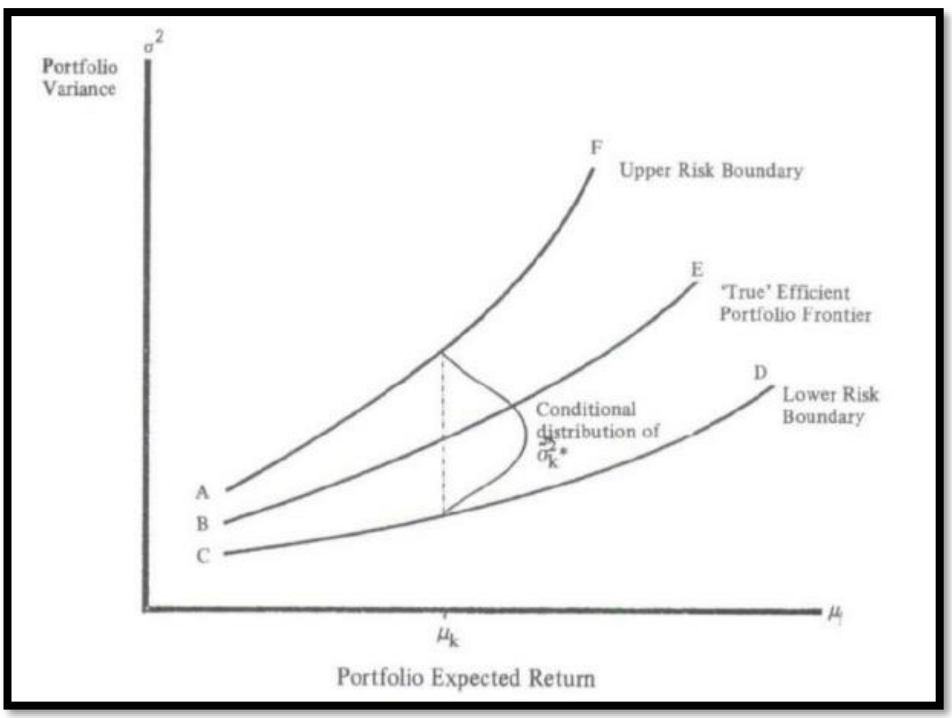

Figure 4: Confidence interval on the efficient frontier

The central line in the graph above represents the computed efficient frontier, bounded by the lower and upper boundary. The risk-aversion of the investor plays an important role in this process. A risk-averse investor, indeed, will care the most on variance-estimation errors rather than on return-estimation ones. A risk-lover investor, on the other hand, will stress more the importance of returns-estimation errors rather than variances-estimations ones, because he is willing to sustain a higher risk in return of a higher reward. Another problem is presented by Best & Grauer (1991), in which they assert the sensibility of the return and variance to new market predictions, which imply a change in input estimations. In their article, they state that "*When only a budget constraint is imposed on the investment problem, (…) an MV-efficient portfolio's weights, mean, and variance can be extremely sensitive to changes in asset means. When nonnegativity constraints are also imposed on the problem, (…) a positively weighted MV-efficient portfolio's weights are extremely sensitive to changes in asset means, but the portfolio's returns are not.*"[1] As Markowitz stated, each portfolio is tailored on the specific investor's risk-aversion, which makes it virtually impossible to compare with the portfolio of another investor with a different utility function. All the above-mentioned critical issues prevented the Markowitz mean-variance model to be consistently applied in real market scenarios. For this reason, different models have been developed to overcome those limits.

1.2 The tracking error

Portfolio managers evaluate their fund's performances by comparing them with a certain benchmark. One of the most used risk measurement of a portfolio with respect to a benchmark is the ***tracking error*** (TE), which is computed starting from the assets weights and the estimated variance-covariance matrix. The tracking error analysis can reveal whether a fund is actively (a management team make decisions about how to invest the fund's money) or passively managed (which simply follows a market index): in the first case there will be important deviation from the benchmark, whereas in the other case there will be just a benchmark's performances replication. The ***tracking error variance*** (TEV) represents the differential riskiness the investor has to bear to invest in the fund instead of the benchmark. Markowitz's mean-variance model presents some

[1] Best, M J, & Grauer, R R (1991) On the sensitivity of mean-variance-efficient portfolios to changes in asset means: some analytical and computational results *Review of Financial Studies,* 4(2), 315-342

conflicts with the real financial markets behavior. For example, one of its basic assumptions is that returns are normally distributed, which is generally not coherent with the modern volatile markets. Furthermore, the quadratic utility function is not very adaptable to the different investors' risk aversion. Therefore, it is necessary to define some models which allow to compensate these limitations.

1.2.1 The Roll's model

One of the most important contributions about the tracking error study was provided by Richard Roll which, in his 1992 "*A Mean/Variance Analysis of Tracking Error"* article, suggested that investors should rely on the deviation of the portfolio from a certain benchmark to get reliable information on their portfolio's performances. The minimization of the variance of the difference between the returns of the managed portfolio and the one of the benchmark can lead the portfolio's performances to be more coherent with the benchmark. This process is typical of an active-managed portfolio. The passive-managed portfolio on the other hand, fulfills the objective of transaction costs minimization and the reduction of the taxation on capital gains. The solution for the minimization problem can be achieved by obtaining the portfolio with the minimum tracking error variance (TEV) for a certain level of extra performance. Starting from Markowitz studies, Richard Roll has defined the minimum TEV frontier by assuming the TE variance as the measure of risk and the expected return as the profit. As Roll suggests (this intuition can be seen in *Figure 5*), the active-managed portfolio and the benchmark are at the same distance from the Markowitz's efficient frontier, implying that if the benchmark would be efficient in mean-variance criterion, the active-managed portfolio would be efficient as well.

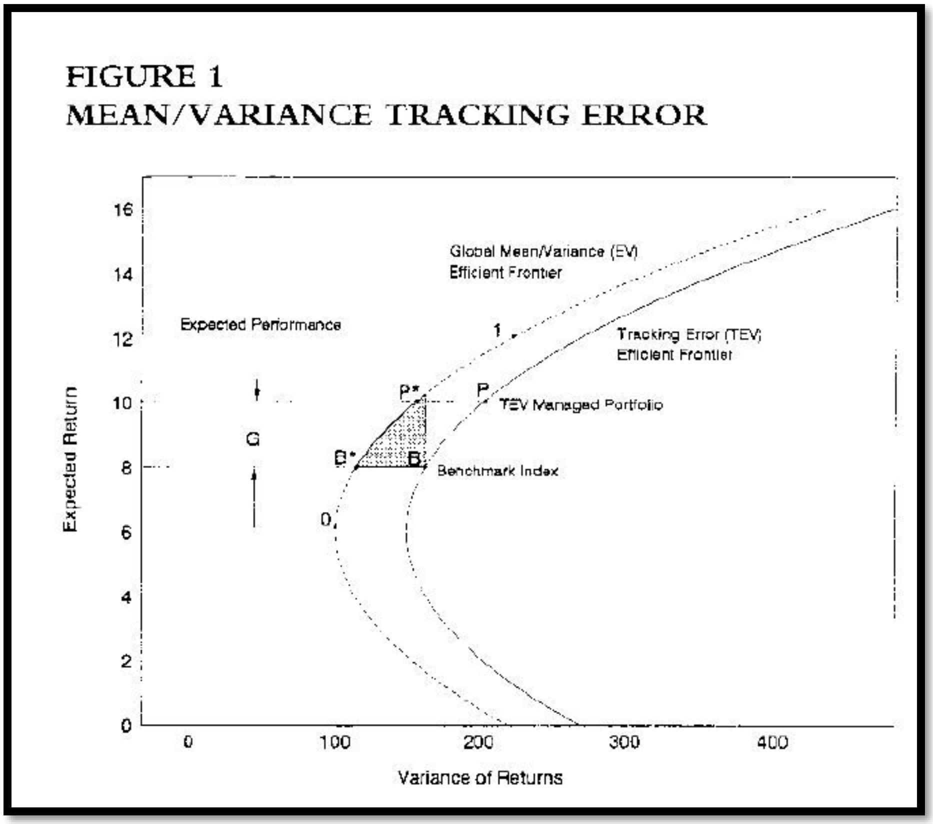

Figure 5: Mean/Variance tracking error (Roll, 1992, "A Mean/Variance Analysis of Tracking Error")

The TE measures the difference between the portfolio return R_P and the benchmark return R_B:

$$TE = R_P - R_B$$

Therefore, the TEV can be expressed as:

$$TEV = Var\ (TE)$$

$$= (w_P - w_B)'\mathrm{V}(w_P - w_B)$$

$$= (w'_P V w_P + w'_B V w_B - 2w'_P V w_B = (w_P - w_B)'V(w_P - w_B) = x'Vx$$

where w_P and w_B are the weight-vectors of the portfolio and the benchmark respectively, V is the assets' variance-covariance matrix and $x = w_P - w_B$ is the difference in returns between a portfolio's asset and a benchmark's one . The expected return of the TE is given by:

$$\mathrm{E(TE)} = Y = x'K$$

where K is the vector of assets' expected returns and Y is the portfolio manager's target return. Therefore, the portfolio's expected return is:

$$E_P = E_B + Y$$

An important condition is that the sum of all weights is equal to 1 (or 100%), which can be expressed in the following way:

$$w'_P 1 = w'_B 1 = 1 \leftrightarrow x'1 = 0$$

The solution of the following optimization problem gives the minimum TEV frontier:

$$min_x Var_{TE} = x'Vx$$

$$s.t. \begin{cases} x'K = Y \\ x'1 = 0 \end{cases}$$

The solution in terms of extra-weight is:

$$x = (w_P - w_B)\frac{cYV^{-1} - bYV^{-1}1}{ac - b^2}$$

where $= K'V^{-1}K$, $b = K'V^{-1}1$, $c = 1'V^{-1}1$ (Roll, 1992). It is possible to express x in terms of the GMV portfolios (denoted with subscript "0") and portfolio "1" which is the portfolio that lies where a ray from the origin through the GMV portfolio intercepts the efficient frontier:

$$x = \frac{Y}{R_1 - R_0}(w_1 - w_0)$$

By denoting with "K" the expected return vector, with "σ^2" the variance and with "w_0" and "w_1" the weight-vectors, the two portfolios are:

$$R_1 = \frac{a}{b} \qquad \sigma_1^2 = \frac{a}{b^2} \qquad w_1 = \frac{V^{-1}Y}{b}$$

$$R_0 = \frac{b}{c} \qquad \sigma_0^2 = \frac{1}{c} \qquad w_0 = \frac{V^{-1}1}{c}$$

Thanks to the two funds theorem, (which states that two efficient portfolios can be established so that any (efficient) portfolio can be seen (in terms of mean/variance) as a combination of these two) the “x” can be determined as a linear combination of portfolio “1” and “0”, and therefore the TEV frontier can be tracked.

The variance of the TE can be re-arranged as:

$$Var(TE) = x'Vx = (w_1 - w_0)'V(w_1 - w_0)$$

By denoting the target performance as $D = \frac{Y}{R_1 - R_0}$ and using the equation

$$x = \frac{Y}{R_1 - R_0}(w_1 - w_0),$$

the TEV equation can be re-written as:

$$D^2(w_1 - w_0)'V(w_1 - w_0) = D^2(w'_1Vw_1 + w'_0Vw_0 - 2w'_1Vw_0$$

But, since $w_0 = V^{-1} * \frac{1}{c}$, the TEV can be expressed as

$$w_1'Vw_0 = w_0'Vw_0 = \frac{1}{c} = \sigma_0^2.$$

Therefore, the TEV frontier can be expressed as:

$$V_P = V_B + 2x'Vw_b + x'Vx$$

$$V_P = V_B + \frac{2(E_P - E_B)(cE_B - b)}{ac - b^2} + \frac{c(E_P - E_B)^2}{ac - b^2}$$

Putting $d = \frac{ac-b^2}{c}$ and $\delta_1 = E_B - \frac{b}{c}$, the former expression can be re-written as:

$$V_P = V_B + \frac{2\delta_1(E_P - E_B)}{d} + \frac{(E_P - E_B)^2}{d};$$

$$V_P = V_B + \frac{2\delta_1 Y + Y}{d}.$$

As shown in Figure 5, the benchmark is in the TEV frontier. The distance between the TEV frontier and the EV one is the difference between the variance of the benchmark and the variance of an efficient portfolio with the same expected return (in Figure 5, $\overline{BP^*} = \overline{BB^*}$):

$$\Delta_{B,MV} = V_B - V_{MV} = V_B - \frac{cE_B^2 - 2bE_B + a}{ac - b^2}.$$

The benchmark in the TEV frontier is usually inefficient and so is the managed portfolio. However, in the case in which $R_B > R_0$ and the target return Y > 0, the managed portfolio has both a higher return and variance than the benchmark. In the opposite case, if $R_B < R_0$, the managed portfolio could dominate the benchmark, with a higher expected return and a lower variance if Y is a small value. Roll improved this method with a further constraint, to overcome the lack of reference to the portfolio absolute risk (i.e. the risk independent from the benchmark). Fund sponsors are generally concerned that their managers do not display systematic biases relative to the benchmark in up versus down markets. Therefore, for the sake of a greater transparency, it should be optimal for managers to maintain the ***portfolio's beta*** (i.e. the tendency of a security's returns to respond to swings in the market[2]) within certain bounds. The beta of a portfolio represents the non-diversifiable risk and it determines whether the managed portfolio is riskier or safer than the benchmark and it is defined as:

$$\beta_{PB} = \frac{Cov(R_P, R_B)}{\sigma_B^2}$$

If $\beta > 1$, the security's price is generally more volatile than the benchmark; if $\beta < 1$, the security is less volatile than the benchmark; if $\beta = 1$ the price moves with the benchmark. As Roll stresses in his article, dominating portfolios must have betas strictly lower than 1; if the expected return of the benchmark is greater than the GMV portfolio, the managed portfolio could not dominate it. Furthermore, there would exist portfolios with the same volatility of the managed one, but with higher returns. To consider the systematic risk in his essay, Roll added a third constraint:

[2] Investopedia. (2018, September 25th). Investopedia. *Retrieved from Investopedia Web Site: https://www.investopedia.com/terms/b/beta.asp*

$$x'Vw_B = \sigma_B^2(\beta - 1)$$

or

$$\beta = \frac{x'Vw_B}{V_B} + 1.$$

The portfolio β will be different from 1 if the extra-return will be different from 0 or in the extremely improbable situation in which the expected return of the portfolio is equal to the GMV one. The improved minimization problem is:

$$min_x Var_{TE} = x'Vx$$

$$s.t. \begin{cases} \dfrac{x'Vw_B}{V_B} + 1 = \beta \\ x'K = Y \\ x'1 = 0 \end{cases}$$

It is possible to express the solution of the problem using the Lagrange multiplier technique:

$$x = \ w_{\mathrm{P}} - w_{\mathrm{B}} = \gamma_1 \frac{V^{-1}Y}{b} + \gamma_0 \frac{V^{-1}1}{c} + \gamma_B w_B$$

where $\gamma_1, \gamma_0, \gamma_B$ are:

$$\gamma_1 = b \frac{\delta_2(E_P - E_B) - \delta_1 V_B(\beta - 1)}{d\delta_2 - \delta_1^2},$$

$$\gamma_0 = \frac{(E_P - E_B)(E_P - bV_B) + (bE_B - a)V_B(\beta - 1)}{d\delta_2 - \delta_1^2},$$

$$\gamma_B = \frac{-\delta_1(E_P - E_B) + dV_B(\beta - 1)}{d\delta_2 - \delta_1^2}.$$

where $a = K'V^{-1}K$, $b = K'V^{-1}1$, $c = 1'V^{-1}1$, $\delta_1 = E_B - \frac{b}{c}$, $\delta_2 = V_B - \frac{1}{c}$. Also in this case, it is possible to define the efficient frontier by applying the beta-constraint to the solution formula of the tracking error minimization problem (with the Lagrange multiplier):

$$x = \gamma_1 \omega_1 + \gamma_0 \omega_0 + \gamma_B w_B$$

$$s.t.\begin{cases}\gamma_1+\gamma_0+\gamma_B=0\\ \gamma_1R_1+\gamma_0R_0+\gamma_BR_B=Y\end{cases}$$

This frontier is defined as the linear combination of two portfolios belonging to the Markowitz frontier and the benchmark. The equation of the frontier is therefore:

$$V_P=V(R_P)=w'_PVw_P,$$

that is

$$V_P=V_B+2x'Vw_B+x'V,$$

that is

$$V_P=V_B+2V(\beta-1)+\frac{\delta_2(E_P-E_B)^2-2\delta_1(E_P-E_B)V_B(\beta-1)+dV_B^2(\beta-1)^2}{d\delta_2-\delta_1^2}.$$

1.2.2 The Jorion's model

Another important essay on portfolio optimization with tracking error constraint was provided by Philippe Jorion in 2002. In this paper, Jorion considers active managers who are given to beat a benchmark. The problem he stresses is that with this setup the overall portfolio risk is usually put aside, resulting in inefficient portfolios, unless some constraints are imposed. He shows that TEV-constrained portfolios are described by an ellipse in the MV plane and this particular shape can improve the performance of the active portfolio when there are low values of TEV or the benchmark is relatively inefficient[3]. Usually, when an outperformance is observed, the primary concern is whether the benefit is in line with the risk undertaken. Jorion tried to find a solution for portfolios with a higher variance with respect to the benchmark, imposing the portfolio's volatility to be equal to the benchmark one:

$$\sigma_P^2=\sigma_B^2$$

[3]Jorion, P. (2003). Portfolio optimization with constraints on tracking error. *Financial Analysts Journal*, 70-82;

The optimization problem is defined as:

$$\max x'E$$

$$s.t.\begin{cases} x'1 = 0 \\ x'Vx = T \\ (q+x)'V(q+x) = \sigma_P^2 \end{cases}$$

Where x represents the deviations from the benchmark, T is a scalar and represents the desired TEV, $(q+x)$ is the vector of the weights and V is the variance-covariance matrix. The last restriction imposes the portfolio variance to be equal to the given value of σ_P^2. As previously said, the ***constant TEV frontier*** is ellipse-shaped as in Figure 6:

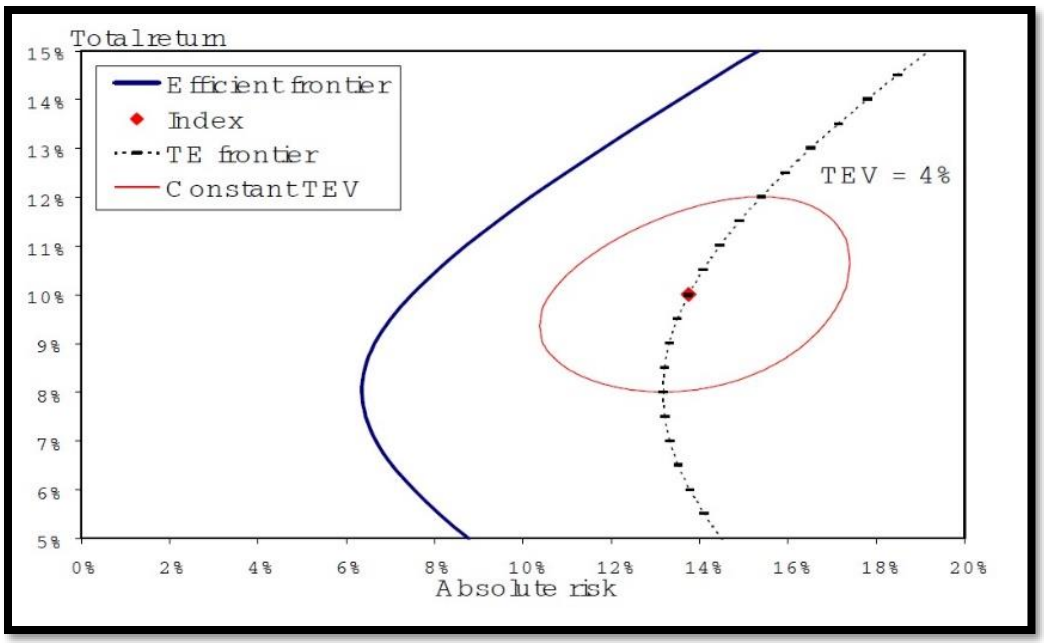

Figure 6: Constant TEV frontier (Jorion, 2002, "Portfolio optimization with constraints on Tracking Error")

The main axis of the ellipse has different inclination with respect to different values of the difference between the expected benchmark return μ_B and the expected MV return μ_{MV}: if the difference is positive, the inclination is positive; if the difference is negative, the inclination is negative; if the difference is 0, the ellipse is horizontal. The return of the expected minimum variance portfolio is:

$$\mu_{MV} = -\frac{E'^{V^{-1}}1}{1'^{V^{-1}}1},$$

the portfolio variance is defined as:

$$\sigma_P^2 = \sigma_B^2 + T + 2\Delta_1\sqrt{\frac{T}{d}},$$

where $d = \frac{a-b^2}{c}$ and $\Delta = \mu_B - \mu_{MV}$.

The last equation shows that the portfolio riskiness is not only related to the TEV but to the difference in expected return as well. Then it is possible to enunciate the constant TEV frontier:

$$dy^2 + 4\Delta_2 z^2 - 4\Delta_1 zy - 4T(d\Delta_2 - \Delta_1^2) = 0$$

where:

$y = \sigma_P^2 - \sigma_B^2 - T$,

$z = \mu_P - \mu_B$,

$\Delta_2 = \sigma_B^2 - \sigma_{MV}^2$.

Once defined the TEV frontier, Jorion adds the restriction:

$$\sigma_P^2 = \sigma_B^2$$

which can improve the performance of the portfolio in the case of a lover TEV or an inefficient benchmark.

2. One step further: the downside risk

This chapter will focus on how the conception of volatility as a measure of risk can be misleading for investors, since the important difference between upside volatility and downside volatility may have been not stressed enough. Therefore the concept of downside semivariance will be introduced. Then, the Sharpe ratio will be recalled, for its undeniable importance in relating variance to portfolio performances. The chapter will move to the Sortino ratio, a modified Sharpe ratio, which incorporates the considerations on the semivariance.

The models exposed so far look at the variance of a portfolio as a measure of volatility (i.e. risk), but they do not discern between a "positive" fluctuation and a "negative" one with respect to the mean. By putting into investors' shoes, it is evident that positive fluctuation of an asset represent a profit opportunity rather than a threat and therefore investors should not feel particularly menaced if their asset (or a portfolio) performs better than expected. On the other hand, if their assets perform worse than expected, they would probably feel this as the real risk in their portfolio's management. The aim of this essay is to analyze how the introduction of semivariance as a measure of volatility instead of variance modifies the choices in asset proportions in a fund/portfolio *ceteris paribus* and if the new mix performs better or worse than the previous. Semivariance is defined as "*a representation in statistics of the analysis of data that fall below the mean value of a set of data. Semivariance refers to an average of the squared deviation of figures that fall below the mean return (…)*"[4], but it can be referred to a chosen target value as well:

$$\sigma_{1/2}^2 = \frac{1}{N}\sum_{i=1}^{N}(T - r_i)^2$$

where T can be either the mean value or the target value. If the variance is the overall portfolio's riskiness, the negative semivariance is the riskiness associated with below-the-mean-only fluctuations and therefore represents the real risk of losing money for the investor. Following this

[4] Business Dictionary. (2018, July 16th). Semivariance. Retrieved from Business Dictionary Web Site: http://www.businessdictionary.com/definition/semivariance.html

reasoning, several theories has been developed. The Sortino ratio (1994) is a measure of risk-adjusted return of an investment/portfolio and it is a modified Sharpe ratio, in the sense that it penalizes only the returns falling below a critical value (usually chosen by the investor as its minimum hurdle rate).

2.1 The Sharpe ratio

The Sharpe ratio (from now on S-ratio) has been devised in 1966 by William F. Sharpe, which called it "*reward-to-volatility*" at first. It is the average return earned in excess of the risk-free rate divided by the volatility (i.e. the excess return per unit of volatility):

$$S = \frac{R_A - R_F}{\sigma_A}.$$

Therefore, a portfolio with a return equal to the risk-free one will have a S-ratio of zero: the greater the S-ratio, the more attractive the portfolio. However, the interpretation of the the ratio alone can be misleading: it is possible, indeed, that high outlier in the portfolio will increase the volatility (i.e. the denominator) more than excess return (i.e. the numerator). This lead to a decrease in the ratio, even if the excess return is notable. The (paradoxical) conclusion should be to remove those assets which provided the higher excess return to decrease the overall volatility of the portfolio. However, this could not be the only problem with the S-ratio. It is widely known that assets' returns do not follow a normal distribution and there might be different degrees of skewness. The skewness, or the asymmetry from the normal distribution, can be either positive or negative: in the first case, mean and median are greater than the mode, in the second case the opposite. Skewness is crucial in estimating how future returns will fall around the mean. As *Figure 7* shows, S-ratio is not able to tell the difference between a positive skewed distribution (very attractive for an investor) and a negative skewness (very repulsive for an investor) and this is a serious problem when someone is looking for an unbiased index. Nonetheless, there are some scenario in which computing a S-ratio can be useful. Consider, for example, the case in which an investor is willing to add a new asset in his equally-weighted portfolio (which consists initially of just two assets) and he wishes to know how the risk-return performance will change. Suppose that the two-assets portfolio has a S-ratio of 0.67. In the new portfolio which has a proportion of 40-40-20, for example, the S-ratio is 0.87. In this case, since the new asset would produce an increase in the S-ratio, there is a diversification benefit and therefore the asset should be added to the portfolio. The S-ratio can be useful to tell whether an excess return is due to an increase in risk or to smart decisions: an investment is a good investment only if it does not come from an increase in portfolio riskiness. In other words, if an investor decides to include in his portfolio to get a higher return, by looking at the S-ratio he can assess whether the higher return is generated by a higher riskiness provided by the asset or not. In the second case, if the overall riskiness does not increase, it should be a good idea to include the new asset.

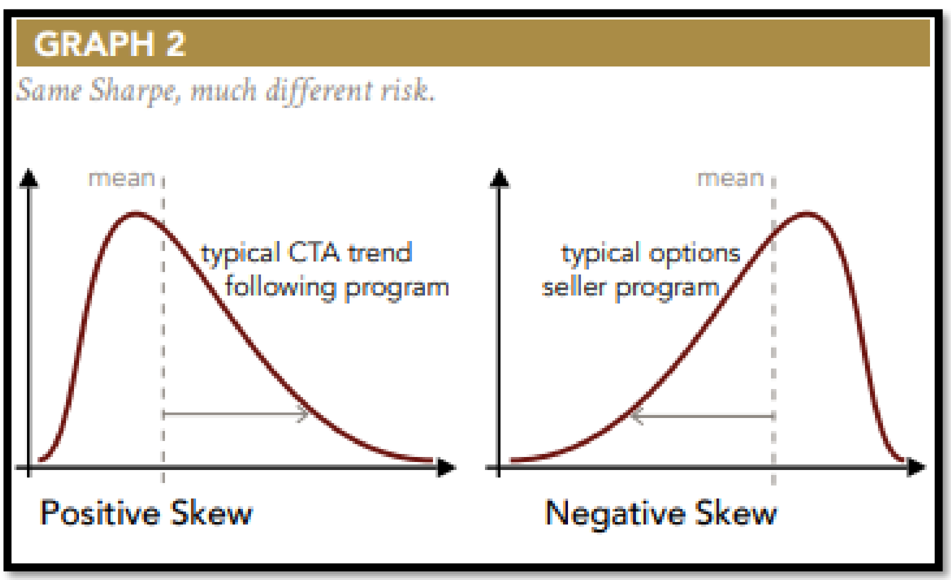

Figure 7: Same Sharpe but different skewness[5]

[5] Red Rock Capital. (2018, October 11th). Red Rock Capital. Retrieved from Red Rock Capital Web Site: http://www.redrockcapital.com/Sortino__A__Sharper__Ratio_Red_Rock_Capital.pdf

2.1 The Sortino ratio

If the aim of the analyst is to compare and measure performances of portfolios showing skewness in their returns distribution, the Sortino Ratio (from now on Sortino) is a better choice. It takes its name from Frank Sortino, it can be seen as an evolution of the S-ratio, in the sense that is uses downside deviation rather than the overall standard deviation. Furthermore, the downside deviation is computed with respect to a desired target return rather than the mean:

$$SO = \frac{R_A - R_F}{\sigma_{DA}}$$

where σ_{DA} is the downside deviation of the asset or the portfolio with respect to the desired return. As previously stated, the downside deviation can be more useful for investors rather than the classic standard deviation, because it isolates the real risky component in investment. Therefore, the Sortino can be more useful than S-ratio for investors. For example, consider the following set of returns:

$$7\% \text{ }, 5\%, 3\%, -5\%, 12\%, 6\%, -2\%$$

the total target return for the investor is 3% per annum.

The computation of the Sortino is straightforward:

1. Compute the mean of the set to get the numerator of the ratio:
   $\frac{7\%+5\%+3\%-5\%+12\%+6\%-2\%}{7} = 3{,}7\%$;
2. Compute the downside deviation with respect to the target return of 3% and then square it;
3. Compute the mean of the calculated square deltas:
   $\frac{0{,}16\%+0{,}04\%+0\%+0{,}64\%+0{,}81\%+0{,}09\%+0{,}25\%}{7} = 0{,}28\%$;
4. Compute the square root of the mean to get the denominator of the S-ratio:
   $$Square\ root = 5{,}291\%;$$
5. The S-ratio is:

$$Sortino\ Ratio = \frac{3{,}7\%}{5{,}291\%} = 6{,}993.$$

The following table presents the calculations performed for the example above:

Average return	*Target return*	*Delta*	*Delta^2*
7.00%	*3.00%*	*4.00%*	*0.16%*
5.00%	*3.00%*	*2.00%*	*0.04%*
3.00%	*3.00%*	*0.00%*	*0.00%*
-5.00%	*3.00%*	*-8.00%*	*0.64%*
12.00%	*3.00%*	*9.00%*	*0.81%*
6.00%	*3.00%*	*3.00%*	*0.09%*
-2.00%	*3.00%*	*-5.00%*	*0.25%*

Figure 8: Average return and semi-variance of set A

Consider now another set of returns:

$$13\%\ , 11\%, 9\%, -11\%, 18\%, 12\%, -8\%$$

In addition, the investor's target is still 3%. This new set has an expected return of 6.29% and it has a downside variance (= 1.18%), higher than in the previous case. The next table shows the calculations performed for the second example:

Average return	*Target return*	*Delta*	*Delta^2*
13.00%	*3.00%*	*10.00%*	*1.00%*
11.00%	*3.00%*	*8.00%*	*0.64%*
9.00%	*3.00%*	*6.00%*	*0.36%*
-11.00%	*3.00%*	*-14.00%*	*1.96%*
18.00%	*3.00%*	*15.00%*	*2.25%*
12.00%	*3.00%*	*9.00%*	*0.81%*
-8.00%	*3.00%*	*-11.00%*	*1.21%*

Figure 9: Average return and semi-variance of set B

The computed Sortino ratio is 0.58, which is slightly lower than the previous one, despite the fact that average returns are much higher than before. Even though the second set is actually earning more than the first one, it is not doing it as efficiently as the first one, since the downside variance is higher. By looking at the table below, it is possible to see why the Sortino is "a better" Sharpe ratio: the second set is clearly dominating the first one in Sharpe's sense, because the excess return per unit of volatility is higher. However, by distinguishing between "good" and "bad" volatility with the S-ratio, set B is performing worse than set A, meaning that the trade-off between the additional return and additional risk is better in set A. By an investor's point of view, investing in the set with the higher trade-off (performance per unit of risk) is safer and makes the investment more efficient. The next table presents a comparison between the Sortino ratio and the Sharpe ratio for the two sets used in the example:

	Sortino	*Sharpe*
Set A	*0.6966*	*2.1930*
Set B	*0.5797*	*2.6376*

Figure 10: Sortino ratio and Sharpe ratio of set A and set B

The table above shows that set B has a better Sharpe ratio, meaning that these assets have a higher excess return per unit of volatility with respect to set A. On the other hand, it has a lower Sortino ratio with respect to set A, implying that the assets in portfolio A provide a better risk-return trade-off. Therefore, an investor is more attracted by portfolio A rather than B, since the extra return per additional unit of risk is higher.

As the former examples have shown, considering the negative semi-variance as the criterion for investment decision can reverse the situation: it is true indeed that higher returns could provide higher benefit for the investor than lower ones, but at the same time, it is true that lower returns can provide a better risk-reward trade-off, because of their lower risk. Using the negative variance when measuring performances provides clearer information to investors, resulting in a more understandable model, because a high downside variance is immediately associated with greater losses, whereas a high variance is not immediately identifiable with great possible losses, since it can also imply greater returns: therefore, the definition of a tracking error measure based on negative covariance could be more useful. Both Roll and Jorion have considered the volatility as the core of their debates: for the former, the volatility is the core of the minimization problem, since it aims to minimize the volatility regardless the index's one; for the latter, the variance is the restriction to be imposed to maximize the expected return of the portfolio since its objective is to force the portfolio's volatility to be equal to the index's one. It is possible, however, that the application of such considerations can waste opportunities given the restriction of the overall variance without discerning between a positive and a negative volatility. On the other hand, returns could be higher if the restriction is made on the negative volatility, since there is no upper bound on positive variance. In the real world, asset returns are generally non-normally distributed: they can be either positive or negative skewed. Skewness is a coefficient that can be positive, null or negative and it measures the degree of symmetry in a distribution. It is particularly important for investors to know whether the majority of their returns is falling in the positive area or in the negative one. The mean of the distribution is also important to define if the skewness is good or bad: a positive mean in a positive skew is good, since it shifts returns to the positive area even more (higher positive returns), whereas a negative mean in a positive skew is bad, since it shifts negative returns to the negative area even more (higher negative returns). Skewness is probably one of the best reasons why the application of ratios based on a normal distribution-assumptions can be risky. Hedge funds, for example, tend to produce small positive returns and occasional large losses. Illiquidity is also a big issue for the Sharpe ratio, which cannot measure it, because illiquid assets tend to be less volatile (i.e. the lower the volatility, the higher the ratio) and difficult to be priced. Volatility clustering, serial correlation and sudden changes in fund management and size can overstate the Sharpe ratio as well. Several financial crises have occurred during the last 30 years, starting from the Latin American debt crisis in the early '80s to the last financial crisis in 2008. Even if these events are relatively rare, conventional derived portfolios are not able to deal efficiently with them, implying a higher level of downside risk that is harder to identify. Perhaps the primary reason for this could be the fact that asset allocation decision making is based on Markowitz's theory. The basic assumption that future returns will be independent from period to period is overly simplistic, but its application is widely used because it only needs one assumption for the mean and one for the standard deviation, and one for each pair (i.e. covariance). Assuming that asset returns are normally distributed implies that the model "ignores" sporadic events with huge backfires. There are three primary causes for non-normality:

- ***Serial Correlation***: the assumption that returns from period to period are *iid* (Independent and identically distributed) is one of the main critical issue of traditional theory. It is almost impossible to state that the consequences of management behavior (or external event) in a certain day or week are confined to that day/week without any further aftereffect.
- ***Negative skewness and Leptokurtosis:*** the graph of asset returns distribution usually shows fat left tails, meaning that there is a higher probability to observe negative return than implied by a normal distribution. Figure 11 compares a sample distribution with the normal one, highlighting the presence of fat left tails. However, it is not the only evidence: the blue line is more peaked and has lower density on the right tail, meaning that there is a higher probability of extreme negative events.

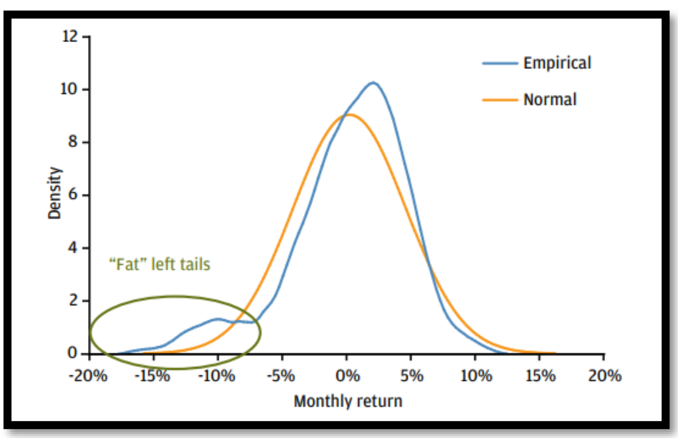

Figure 11: International equities - Historical returns[6]

[6] Abdullah Z Sheikh and Hongtao Qiao, (2010) Non-Normality of Market Returns: A Framework for Asset Allocation Decision Making, The Journal of Alternative Investments;

- ***Correlation breakdown***: traditional asset allocation models often assume that different asset classes are bounded by a linear relationship, meaning that there is the same relationship both in the extreme values and in the less extreme ones. This assumption is similar to assuming that the joint distribution of asset returns is multivariate normal. The consequences of this assumption is that the model tends to overestimate the benefits of diversification during high volatility periods.

In conclusion, the need to shift from a risk model based on the normal distribution is justified by the fact that non-normality is empirically observed with much greater frequency than normality, and by ignoring this, investors understate downside portfolio risk dangerously. Furthermore, incorporating non-normality in asset allocation has proven that there is less need for external constraints, since there is no need to adapt normal frameworks to provide non-normal solutions.

3. A downside risk-based tracking error

This chapter will introduce a proposal of a tracking error which will be used in a minimization problem, to be applied to an ETF, with the intent of recalibrate the weights of the assets to get the lowest tracking error possible.

The aim of this thesis is to show that it is possible to replicate an index's performances when considering negative variance rather than general variance in the minimizing the tracking error of the portfolio. The next chapters will analyze a new proposal of tracking error, which starts from Sortino's intuitions and will optimize an existing fund's composition by minimizing the negative variance. The key assumption of the project is that, assuming the negative semi deviation is the real concern for investors, it is possible to re-calibrate the weights of a portfolio by minimizing the T.E. associated and still tracking a benchmark efficiently.

As seen in previous chapters, the process to find the weights of a portfolio under certain constraints starts with the definition of such limitations. The proposed T.E. takes into considerations three assets:

- An ***Exchange-Traded fund***: it represents the portfolio to be optimized. ETFs are particular investment funds which holdings are usually stocks, bonds or commodities and they are assembled to replicate the performances of a relative index (which cannot be bought or sold in financial markets). The ownership of the ETF is divided in shares held by shareholders, which can be bought and sold in the market. Due to its particular composition and dissemination, it can be seen as a "n-asset portfolio".
- An ***Index***: which is the index to whom the ETF is referred. It is a measurement of a section of the stock market, used to describe the performances of a market division. In a *price-weighted index* (*Dow Jones Industrial Average, NYSE Arca Major Market Index* for example) the value of the index is determined by the price of each component. In an *equal-weighted index* (the *Barron's 400 Index* for example) each component has the same weight.

- A ***Treasury Bond*:** it is a bond issued by national government and it is assumed to be risk-free.

The classical tracking error was defined in chapter 2 as the difference between the expected return of the asset (the ETF in this case) and the index but the T.E. proposed in this chapter will include the risk-free rate into the analysis. It is possible to start from this point to delineate the relationship between the three assets described before. Given the fact that a T.E. measure is defined as a difference between a component relative to the portfolio performance and a component relative to an index, the starting structure can be written as follows:

$$TE = x_{ETF} - x_{index}$$

It is now necessary to express the x to take into account both the risk-free and the unique enhancement of the downside risk. By recalling the Sharpe ratio presented in chapter 2.1:

$$S_A = \frac{R_A - R_F}{\sigma_A}$$

is possible to see that it is structured to accommodate both assumptions since R_F is the risk-free rate and σ_A is the risk associated with asset A. The only substitution to be performed is to switch the overall risk with the downside risk: $\sigma_A \rightarrow \sigma_A^-$ and replicate the relationship for the index as well:

$$TE = \frac{R_{ETF} - R_F}{\sigma_{ETF}^-} - \frac{R_{index} - R_F}{\sigma_{index}^-}$$

The last adaptation to be applied is to make the measure easier to be read, removing the need to interpret negative results and to stress more the attention on big values rather than smaller ones. Therefore, all of the subtraction will be squared:

$$TE = \left(\frac{R_{ETF} - R_F}{\sigma_{ETF}^-} - \frac{R_{index} - R_F}{\sigma_{index}^-}\right)^2$$

By taking a closer look the formula, it is possible to re-write it as:

$$TE = (MSS_{ETF} - MSS_{index})^2$$

Where MSS is the acronym of "*Modified Sharpe-Sortino Ratio*". Indeed, the ratio can be seen as an enhancement of the Sharpe ratio with the innovating consideration of Frank Sortino: the standard deviation is replaced with the downside risk (following the considerations of chapter 2); the difference between the two MSS is a difference between two risk-return trade-offs (one of the ETF and one of the index) implying that if it is negative, the index is performing better than the ETF, and vice versa).

3.1 The case study: the XLG - INVESCO S&P 500 top 50 ETF

To perform the test about the proposed TE minimization problem, the chosen financial instrument is an ETF. An ETF, or *Exchange-traded Fund*, is a particular investment fund which aims to replicate a specific index through a passive management and it can be easily bought and sold in the market. By investing in an ETF, the investor is able to operate in the whole market index (S&P 500 for example). This particular fund merges the qualities of both an investment fund and a stock:

- It allows to diversify (and therefore decrease) the overall risk of the investment;
- It can be exchanged easily and transparently in the market.

Given the fact that an ETF allows to invest in a market index, it allows to differentiate the risk more efficiently than other financial tools, and the low fees increase the attractiveness for investors. An ETF tends to replicate the performances of a "*benchmark*" and the composition of the fund changes along with the components of the benchmark: therefore, as one of the benchmark factor changes, the ETF will conform to it. One of the main difference with investment funds is that whereas traditional mutual funds sell and redeem their shares at NAV, an ETF share can be bought and sold by financial institutions directly from the ETF in *creation units* (batch of shares). Since the shares are traded all the day and they are created and redeemed in such creation units at the NAV calculated the next day, they provide a better tax-protection than conventional mutual funds: due to the fact that ETF shares are traded throughout the day there should be adverse effects that arise from cash creation and redemptions. Federal income taxation can be avoided by distributing gains coming from the rebalance of securities to shareholders. Such instrument can either pay dividends or reinvest them. Generally, they are declared and paid annually, as well as capital gains. Such distributions can be reinvested in additional shares if the option is available. These funds aim to qualify as ***RIC*** *(Regulated Investment Companies)* not to be subject to entity-level taxation on income and capital gains. The taxation is shifted to investors, if taxable. The exchange-traded fund chosen for the case study is the *XLG - Invesco S&P 500 Top 50 ETF.* This fund is based on the S&P 500 Top 50 Index, which gather the 50 largest companies from the S&P 500. The main component is the IT sector (Apple Inc. , Microsoft Corp. , Facebook Inc. A, Alphabet Inc. , ….) which represents the 36,8% of the total assets. The 13,8% is given by the Consumer Discretionary sector (Amazon.com Inc. , Walt Disney Co/The, Netflix Inc, McDonald's Corp. , …) ; the Financials sector is the 12,6% of the ETF (JPMorgan Chase & Co, Bank of America Corp, …) and the Health Care weights the 11,8% (Johnson & Johnson, Pfizer Inc). This fund is passive-managed and therefore its aim is to replicate the underlying S&P 500 Top 50 Index. The "replication" implies to invest in all of the assets of the underlying index in the same proportions as in the index. However, the fund does not invest in the underlying entirely: it is investing approximately the 90% of its total assets in the index, whereas the remaining 10% is invested in other securities, futures and options contracts[7] . Furthermore, the fund holds repo-agreements and other collaterals on a day-by-day basis. The correlation between the fund and the index is declared to be over 95% before fees and taxes.

[7] Invesco S&P 500® Top 50 ETF prospectus (2018, October 22nd). Retrieved from Invesco Web Site: https://www.invesco.com/portal/site/us/investors/etfs/product-detail?productId=XLG&ticker=XLG&title=invesco-s-p-500-top-50-etf

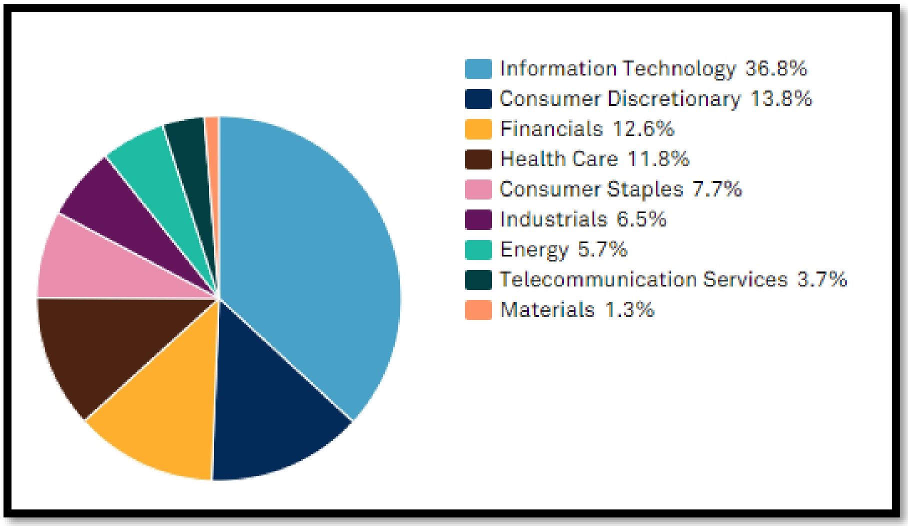

Figure 12: S&P 500 Top 50 sector breakdown

The time interval chosen for the observation is from 2015-01-01 to 2018-01-01 and the following plot shows the trend over the time interval:

Figure 13: XLG closing prices from 2015-01-01 to 2018-01-01. (Candle bar obtained in R Studio)

As the graph shows, there is an increasing pattern which culminates with the $ 190.1995/share in 2017-12-18, whereas the minimum closing price was $ 121.4822/share in 2015-08-25. As the graph notes, the last adjusted closing price was $ 190.0200/share.

The principal risks the ETF is exposed to are:

- **Correlation and Tracking Error Risk**: there is no guarantee that the XLG will achieve a perfect correlation with the underlying index, due to rounding of share prices, fund expenses, regulatory policy, changes in the index composition and the use of leverage. These risk prevents a high correlation between the fund and the index either in short and long period;
- **Equity Risk**: market and economic conditions associated with negative perception of industries can decrease the value of equity securities and equity-based derivatives in the fund**;**
- **Liquidity and Valuation Risk**: there is no guarantee that the fund may be able to purchase/sell assets at a fair price. Furthermore it may be difficult or almost impossible to liquidate assets of the fund under certain conditions. The Adviser may be in difficulties when evaluating the accurate daily value to investments;
- **Authorized Participant Concentration Risk**: given the fact that only APs can create and redeem transactions with the fund and that only few institutions can act as Authorized Participants can represent an issue for the Fund. Furthermore, Aps have no obligation to submit creation or redemption orders, therefore there is no guarantee that there will be a continue trading market for the shares;
- **ETF Trading Risks**: market disruptions as well as unanticipated closing of exchange markets, where either the trade of shares or the portfolio holdings trade may result in the shareholder's inability to buy or sell the fund's securities. Market disruptions can also make it difficult for the Adviser to accurately evaluate its investments and it may be impossible to rebalance the portfolio, involving possible losses**;**
- **Non-Diversification Risk**: the fund's ability to invest a higher portion of its assets in securities than a diversified fund exposes it to amplified effects of the fluctuations in share's value;

- **Passive Investment Risk**: in declining markets, there are no attempts to take defensive position from the Adviser, involving a higher risk of losses;

Weight (%)	XLG - Invesco S&P 500 Top 50 ETF Companies		Weight (%)
8,821	Apple Inc	Coca-Cola Co/The	1,470
6,818	Microsoft Corp	Citigroup Inc	1,468
6,253	Amazon.com Inc	Walt Disney Co/The	1,378
3,439	Facebook Inc	Comcast Corp	1,347
3,305	Berkshire Hathaway Inc	PepsiCo Inc	1,343
3,224	JPMorgan Chase & Co	DowDuPont Inc	1,296
3,008	Alphabet Inc	AbbVie Inc	1,234
2,997	Alphabet Inc	NVIDIA Corp	1,224
2,977	Johnson & Johnson	Oracle Corp	1,187
2,730	Exxon Mobil Corp	Walmart Inc	1,166
2,393	Bank of America Corp	Netflix Inc	1,134
2,127	Wells Fargo & Co	Philip Morris International Inc	1,089
2,081	Visa Inc	Amgen Inc	1,083
2,078	UnitedHealth Group Inc	McDonald's Corp	1,042
2,030	Pfizer Inc	International Business Machines Corp	1,027
1,980	AT&T Inc	Medtronic PLC	1,011
1,868	Verizon Communications Inc	3M Co	1,001
1,860	Home Depot Inc/The	Adobe Systems Inc	0,997
1,855	Chevron Corp	Union Pacific Corp	0,955
1,809	Intel Corp	Honeywell International Inc	0,955
1,779	Cisco Systems Inc	Altria Group Inc	0,951
1,735	Procter & Gamble Co/The	General Electric Co	0,880
1,566	Boeing Co/The	United Technologies Corp	0,824
1,536	Mastercard Inc	Booking Holdings Inc	0,730
1,531	Merck & Co Inc	Schlumberger Ltd	0,717
		Broadcom Inc	0,691

Figure 14: XLG - Invesco S&P 500 Top 50 ETF company composition

- **Large-Capitalization Securities Risk**: the high capitalization of the companies in the ETF may prevent them to react quickly to inedited competitive challenges, while smaller companies can be more flexible and obtain a higher growth rate;

Besides these general risks, there is also a specific risk connected with the sectors in which the fund is investing:

- **Consumer Staples Sector Risk**: it includes the manufacturers and distributors of tobacco, food and beverages. The risk is associated to the fact that the securities can underperform under legislative changes, increased competition, consumer confidence, adverse market conditions, as well as supply lowering;
- **Financials Sector Risk**: it includes companies involved in finance/specialized finance, asset management, banking. Companies operating in the financial sector are particularly subject to massive government regulation and this can set limitation of their financial commitments, involving a decrease in profitability. Profitability depends on the availability and cost of capital funds and fluctuates due to interest rate changes;
- **Health Care Sector Risk**: it includes companies operating in health care providing and health care technologies. The approval of health products and services by government can affect the price and availability of such products;
- **IT Sector Risk**: it includes software houses, cellular phones producers, hardware and equipment distributors. The primary source of risk is the sensitivity to short product cycles and aggressive pricing, as well as an extremely aggressive market competition.

4. Application of the PSO to the tracking error minimization problem

This chapter will present the Particle Swarm Optimization algorithm, a biologic-inspired meta-heuristic and its fundamental. Then, the PSO will be introduced to constrained optimization problems, which will be used in this work's analysis.

The minimization problem exposed in chapter 4 will be solved through the metaheuristic PSO, *Particle Swarm Optimization*, created by Kennedy, Ebenhart and Shi in 1995 to represent the movements of fish schools and bird flocks. A metaheuristic is a particular algorithm that aims to solve a problem by focusing its activity in the more promising areas of the solutions space. The inspiration for such an algorithm derives from the natural environment, in particular from the behavior of birds flocks or fish schools which, thanks to a relatively basic set of rules coordinates the movements of each component of the group. This metaheuristic makes close to no assumption about the problem and, while this may eventually lead to no solutions at all, it allows to search larger spaces of solution with respect to traditional solving processes. The *Swarm Intelligence* is the "*collective behavior of decentralized, self-organized systems, natural of artificial"* and the actors are particles which interact in the environment and within the environment with other particles in the same way of living, social beings. They are called "*particles"* because of each object in the SI can "move" towards the best solution with a certain speed and acceleration. Therefore each particle is a candidate solution of the problem and has its own *fitness value* , which represents its attribute. Moving through space involves that each particle has a starting position that is used both by the particle and the other ones to guide. In the animal world, a member of a bird flock uses information about its speed and position to coordinate with the others, along with the speed and position of the rest of the group: in this way, it can pursue its objectives efficiently. When the particle (i.e. the bird) finds a source of food, for example, it can carry out two opposite behavior to reach it: it can pursue individualism, moving away from the flock and going on its own or it can move towards it while remaining a part of the group. There are then two different behavior to be implemented: one I based on the individualism and exploration, while the other is connected to the exploitation of others and based on sociability. At the start of the simulation, particles are randomly placed in the solution space: progressively, each particle analyze its current position and plans the next movement by analyzing its position, its best past position (called *personal best fitness*) and the flock's best past position (called *global best fitness*). During the process it exchanges these information with the other members. Thanks to this behavior, the flock will tend to move towards the global best position. The group is characterized by three main qualities:

- Robustness: when a member of the flock moves away, the group will follow it;
- Flexibility: it is possible to solve different problem with the same set of rules;
- Auto-organization: the flock's behavior is ruled by basic laws.

The PSO is characterized by M particles representing possible solutions, N variables of the optimization problem and the fitness function $f : S \in R^n \rightarrow R$ which is used to evaluate the goodness of the particle position. The PSO is an evolution in space and time k:

- x_j^k is the position in the space of solutions of the j-th particle at time k;
- v_j^k is the speed of the j-th particle at time k;
- p_j is the best past position of the j-th particle;
- $f(x_j^k)$ is the goodness of the actual position;
- $pbest_j = f(p_j)$ is the goodness of the fit in the best past position of the j-th particle

The last assumption allows the particles to beat the former best solution. The original algorithm develops in 8 phases:

1. A particles population is randomly set in the solutions space with random positions and speeds;
2. Iteration starts;
3. The fitness function $f(x_j^k)$ in the current position x_j^k is evaluated;
4. The fitness function is compared with the $pbest_j$, which is updated if the current fitness function is better than it;
5. Identify the nearest particle with the best global fitness and assign the g-index to it (global index);

6. The position and speed of particles is updated through the equation:

$$(1)\begin{cases} v_j^{k+1} = v_j^k + U(0,\theta_1) \otimes \left(p_j - x_j^k\right) + U(0,\theta_2) \otimes \left(p_g - x_j^k\right) \\ x_j^{k+1} = x_j^k + v_j^{k+1} \end{cases}$$

   Where

- $U(0,\theta_1)$ and $U(0,\theta_2)$ are two vectors of uniformly distributed random numbers in $[0,\theta_1]$ and $[0,\theta_2]$ and θ_1, θ_2 are acceleration coefficients;
- $\otimes$ is the tensorial product;

7. If a sufficient number of iterations or an acceptable fitness value is reached, it is possible to move to phase 8, otherwise, restart from phase 3;
8. End of iterations.

The formula (1) considers the following elements:

- Current speed v_j^k;
- Distance between the actual position and the past visited positions;
- Past experience of each particle $U(0,\theta_1) \otimes \left(p_j - x_j^k\right)$;
- Contribution of each particle to the group $U(0,\theta_2) \otimes \left(p_g - x_j^k\right)$

The algorithm needs some parameters to be fixed by the operator. The population is usually fixed between 20 and 150 particles. The parameters θ_1 and θ_2 determine the intensity of the forces that attract the particle towards the rest of the group and the best solution. Given the fact that a particle is attracted by both the best solution it found $\left(p_j - x_j^k\right)$ and the best global solution $\left(p_g - x_j^k\right)$ (Corazza, Fasano, Gusso, 2013), it is important to choose coherent parameters. Initially the chosen value was $\theta_1 = \theta_2 = 2$ which, in practice, prevent the algorithm to be instable due to sudden changes in the speed of particles. There was the idea that by choosing a value $[-V_{max}, +V_{max}]$ it could have been possible to balance exploration and exploitation. However, it was difficult to trade-off the need for exploration and the need for exploitation, since a great V_{max} implies a higher degree of exploration, whereas a small V_{max} encourages exploitation. To overcome the need of V_{max}, Shi and Eberhart introduced the *inertia weight* parameter w^k in 1998, modifying the formula (1) such that:

$$v_j^{k+1} = w^k v_j^k + U(0,\theta_1) \otimes \left(p_j - x_j^k\right) + U(0,\theta_2) \otimes \left(p_g - x_j^k\right)$$

In this way, a correct choice of the w^k value allows to find a correct trade-off between local and global exploration/exploitation: great values of the inertia weight transform the PSO in a global research algorithm, while small ones transform the PSO in a local research algorithm. The implications of the inertia weight are extremely important: by progressively reducing the magnitude of the inertia weight it is possible to improve the research enormously. The *linearly decreasing strategy* reduces the value of the w^k proportionally to the number of iterations: in this way, the process starts with a high value of w^k implying a great freedom of movement for particles, exploring a big space of solution; progressively the w^k decreases, focusing the solution' research in smaller spaces and making the operation more efficient. However, the values of θ_1 and θ_2 have to be set accordingly to the inertia weight parameter. The mathematical way to set the inertia weight is:

$$w^k = w_{max} + \frac{w_{min} - w_{max}}{K} * K$$

Where K is the number of iterations and w_{max} and w_{min} are the highest and lowest value of inertia weight. Another way to eliminate the need for V_{max} was proposed by Clerc in 1999 and it involves the application of a *restriction coefficient* χ to the speed formula:

$$v_j^{k+1} = \chi(v_j^k + U(0,\theta_1) \otimes \left(p_j - x_j^k\right) + U(0,\theta_2) \otimes \left(p_g - x_j^k\right))$$

Where

$$\theta = \theta_1 + \theta_2 > 2$$

$$\chi = \frac{2}{\theta - 2 + \sqrt{\theta^2 - 4\theta}}$$

and θ is generally fixed at 4.1 such that $\chi = 0.729$. It is evident that the PSO with restriction is equivalent to the PSO with inertia weight and they can be converted each in the other. With respect to the flock, it is possible to tell two different ways the particles interact each other:

- Statically
- Dynamically

Initially, the particles of the PSO were considered to be static meaning that those in the around another particle do not change along with the increment of iterations. Subsequently, algorithms moved towards a *local best* (*lbest)* topology, allowing for parallel researches. The opposite of the lbest is the *global best (gbest)* in which each particle's behavior is influenced by the others. The lbest is probably the best method for the exploration of the solution spaces, whereas the gbest tends to converge more rapidly. Therefore, it should be optimal to start the process with a lbest topology and move progressively towards a gbest-oriented topology.

4.1 Constrained non-linear minimization problems

The PSO has shown a great utility and efficiency for the resolution of non-constrained optimization problems; however, sometimes it is necessary to optimize the results under some constraints. These constraints specify the spaces in which variables have to operate. A Constrained Non-Linear Optimization problem is defined as follows:

$$find\ \vec{x}\ that\ optimizes\ f(\vec{x})$$

$$s.t.\begin{cases} g_i(\vec{x})\ \leq 0 & i = 1, \dots, m \\ h_i(\vec{x}) = 0 & i = 1, \dots, p \end{cases}$$

where:

- $\vec{x} = [x_1, x_2, \dots, x_n]$ and represents the vector of solutions;
- m is the number of constraints expressed as inequality
- p is the number of constraints expressed as equality.

It is possible to identify three main components of a constrained non-linear optimization problem: a set of variables, the fitness function to be optimized and a set of constraints. The main issue of the whole optimization problem is how to deal with such constraints. Theorists have proposed different approaches over years. Probably, the simplest solution to the problem is the one proposed by Hu and Eberhart in 2002 paper "Solving Constrained Nonlinear Optimization

Problems with Particle Swarm Optimization" in which the idea is to generate a set of acceptable solutions and then verify whether they are in accordance with constraints. In this process, particles explore all of the solution space, but they keep only feasible solutions in their memory:

1. The cycle starts;
2. Particles are initialized;
3. If every particle is located in the admittable space, the process keep going. Otherwise, it goes back to step 2;
4. The cycle ends;
5. The cycle starts;
6. The process computes the fitness function $f\left(x_j^k\right)$ for every particle;
7. If the current fitness function value is better than the pBest and the particle is located in the space of admissible solutions, this fitness function value is set as the new pBest;
8. The process detects the particle with the highest pBest and marks it as the gBest;
9. The process computes and then updates the speed and position of every other particle with the formula:

$$\begin{cases} v_j^{k+1} = v_j^k + U(0,\theta_1) \otimes \left(p_j - x_j^k\right) + U(0,\theta_2) \otimes \left(p_g - x_j^k\right) \\ x_j^{k+1} = x_j^k + v_j^{k+1} \end{cases}$$

10. When the maximum number of iterations is reached (or the fitness function value is reached), the process ends the cycle; otherwise it goes back to step 6;
11. The cycle ends.

There are two main differences with the original PSO algorithm: first, all particles are initialized such that they respect the constraints; second, every particle keep only feasible solutions in its memory when computing both lbest and gbest. However, this approach may lead to high-computational costs in the case, for instance, that the initial random generation of particles is not able to produce any admissible solution (step 2).

Another approach, proposed by Zhang et al in 2004, introduces a "move operator" with the intent to increase the explorative capacity of particles in CNPO PSOs. The interesting aspect of such approach is that it tries to manage the boundaries of the admissible region efficiently. Given the fact that the gbest continuously changes its position inside the admittable space, there is the possibility that it lies in the neighborhood of the boundary. This implies that some particles could be located outside the admittable space. Therefore, there are three main control procedures for boundaries control: boundary mode, random mode and periodic mode. In the boundary mode, a particle $\overrightarrow{x_M} \in S_m$ that lies outside the boundaries are re-placed ON the boarder $(\overrightarrow{x_s})$ through the rule:

$$\widetilde{M_B}\ (x_n)\!:\!\begin{cases} x_n = 1_n \ if \ x_n < I_n \\ x_n = \ u_n \ if \ x_n > u_n \end{cases}$$

where I_n and u_n represents the lower and upper bound associated with the n-th dimension respectively (therefore the particle is located inside the admissible region).

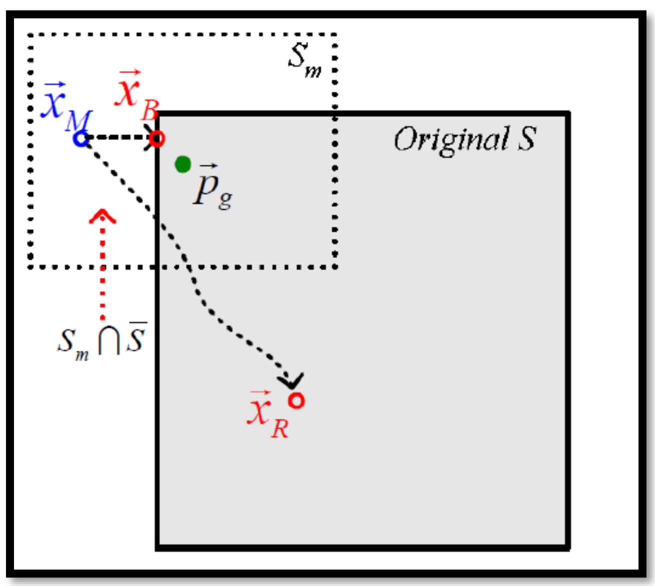

Figure 15: Traditional approach in the management of boundaries (Zhang, Xie, & Bi, 2004)

Furthermore, the solution is modelled to reduce as much as possible the shock, meaning that the particle is re-directed to the nearest boundary. The random method the shift $\overrightarrow{x_M} \rightarrow \overrightarrow{x_R}$ follows the rule:

$$\widetilde{M_B}\ (x_n)\colon x_n = U_R(l_n, u_n)$$

where $U_R(l_n, u_n)$ is a random value inside the interval (l_n, u_n). This method repositions the particle inside the admissible space of solutions regardless the shocks it produces. However, in both cases, the excessive use of move operators can disrupt the capability of the flock to "re-organize" itself and therefore their exploration. The periodic mode replicates the admissible space of solutions infinitely, creating an infinite space made by periodic replicas of the original space, with the same trend of the fitness function (Figure 16). The original space is the grey one $S^{(0)} = S$ while the near spaces are its periodic copies $S^{(C)}$. The main difference with former approaches is that there is not a move operator: each particle $\vec{x} \in S^{(E)}$ (where $S^{(E)}$ represents the copy of S) is related to each other through the relationship:

$$\widetilde{M_B}\,(x_n \to z_n): \begin{cases} z_n = u_n - (l_n - x_n)\%s_n, & if\ x_n < l_n \\ z_n = l_n - (x_n - u_n)\%s_n & if\ x_n > l_n \\ z_n = x_n & if\ \ x_n \in [l_n, u_n] \end{cases}$$

where $s_n = |u_n - l_n|$ is the variability interval associated to the n-th dimension.

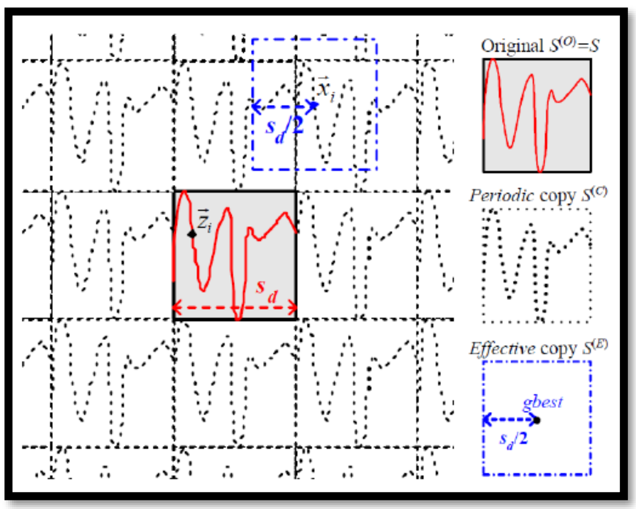

Figure 16: Periodic mode. Infinite replication of the space of solutions. (Zhang, Xie, & Bi, 2004)

The main implication of this model is that the upper and lower boundaries are $x_n + \frac{s_n}{2}$ and $x_n - \frac{s_n}{2}$ respectively: furthermore, the periodic model reduces the maximum distance between the current gbest and the absolute gbest from S_n to $\frac{s_n}{2}$. Another possible approach is based on considering a random local subset of particles instead of the global one (Cabrera & Coello, 2007). This approach is also called "micro PSO", because the population is very small (5 particles): the neighborhood is randomly generated, and the high degree of exploration of the PSO and the heterogeneity of the set is obtained through a mutation operation and the re-initialization process. The algorithm identifies a leader particle in each subset based both in the space of solutions and the fitness value of the particle: whenever two or more admittable solutions are compared, in the case in which only one of the two solutions is admittable, the admittable one wins or, in the case in which both solutions are admittable, the one with the lowest fitness values wins. The aim of the process is to choose those leader particles that, where not admittable, lie near the admittable ones. The fitness value of solutions is assigned through the equation:

$$fit(\vec{x}) = \begin{cases} f_i(\vec{x}) \; if \; \vec{x} \; is \; admittable \\ \sum_{j=1}^{n} g_j(\vec{x}) \sum_{k=1}^{p} |h_k(\vec{x})| \; if \; \vec{x} \; is \; not \; admittable \end{cases}$$

Even if the use of such a small population is not widely used (since it concentrates the population rapidly at each iteration), the re-initialization process allows its use because it replaces particles with high fitness values with new randomly generated ones, keeping in memory the pbest of the substituted ones.

5. The application of the PSO to the XLG – INVESCO S&P 500 top 50 ETF

This chapter will present the application of the PSO to XLG – INVESCO S&P 500 top 50 ETF assets, according to different setups. Then, the best computed portfolio will be analyzed to check if it is effective in emulating the benchmark performances in the out-of-sample analysis.

Before moving towards the application of the PSO to the case study, it is mandatory to resume the major economic and political events that happened during the observation period 01.01.2015 – 01.01.2018. It is important to keep in mind that financial markets are extremely susceptible to political and economic announcements and events to whom they respond more or less reactively, transforming the management concerns in relocations of money and assets: when there are fears of a possible nefarious scenario for a certain company, investors usually tends to sell their company's assets to prevent losses; vice versa, when a positive outcast is setting up, they tends to increase their shares. One of the most notable economic events of 2015 was the Greek-government debt crisis in July 2015 as Greece became the first advanced economy to miss a payment to the IMF (International Monetary Fund) in over 70 years. The impact of this event was huge on Eurozone markets, because of the fear of a potential Greek exit from the Eurozone ("Grexit") and the initial rejection of the bailout terms. The turmoil unavoidably affected the markets outside the Eurozone. The uncertainty increased even more in 2016 due to a slowdown of emerging economies and a general skepticism in the capability of the EU to be effective in the economy and inflation management. Furthermore, there was a downside revision of the profitability forecasts of European and American banks. The most notable politic event of 2016 was the British referendum of June 23rd 2016, where the majority of British voters supported the withdrawal of the United Kingdom from the European Union: this fact increased the sense of uncertainty about the future of one of the most advanced economy in the world, which resulted in financial markets to be more cautious. 2017 started with the election of Donald Trump as the 45th president of the United States, which started an expansive and protectionist policy to boost the economy (which resulted in S&P500 assets to increase their performances).

The following analysis is focused on the historical data of the assets that compose the benchmark *S&P 500 Top 50 Index* and the *XLG – INVESCO S&P 500 top 50 ETF* during the time span 01.01.2015 – 01.01.2018. The price considered for both the ETF and the benchmark are the closing prices of each market day. The assets considered in the simulation are *3MCo, AbbVie Inc, Adobe Inc, Alphabet Inc, Alphabet Inc, Altria Group Inc, Amazon.com Inc, Amgen Inc ,Apple Inc, AT&T Inc, Bank of America Corp, Berkshire Hathaway Inc, Boeing Co/The, Booking Holdings Inc, Broadcom Inc, Chevron Corp, Cisco Systems Inc, Citigroup Inc, Coca-Cola Co/The, Comcast Corp, DowDuPont Inc, Exxon Mobil Corp, Facebook Inc, General Electric Co, Home Depot Inc/The, Honeywell International Inc, Intel Corp, International Business Machines Corp, Johnson & Johnson, JPMorgan Chase & Co, Mastercard Inc, McDonald's Corp, Medtronic PLC, Merck & Co Inc, Microsoft Corp, Netflix Inc, NVIDIA Corp, Oracle Corp, PepsiCo Inc, Pfizer Inc, Philip Morris International Inc, Procter & Gamble Co/The, Schlumberger Ltd, Union Pacific Corp, United Technologies Corp, UnitedHealth Group Inc, Verizon Communications Inc, Visa Inc, Walmart Inc, Walt Disney Co/The, Wells Fargo & Co.* As anticipated in the former chapters, the optimization problem will be solved through the Kennedy-Ebenhart-Shi PSO which develops in eights phases:

1. Particles' initialization;
2. Iteration starts;
3. The fitness function $f(x_j^k)$ in the current position x_j^k is evaluated;
4. The fitness function is compared with the $pbest_j$, and eventually updated;
5. Identification of the nearest particle with the best global fitness;
6. Update of particle's speed and position through the formula:

$$(1)\begin{cases} v_j^{k+1} = w^k v_j^k + U(0,\theta_1) \otimes \left(p_j - x_j^k\right) + U(0,\theta_2) \otimes \left(p_g - x_j^k\right) \\ x_j^{k+1} = x_j^k + v_j^{k+1} \end{cases}$$

7. If a sufficient number of iterations or an acceptable fitness value is reached, it is possible to move to phase 8, otherwise restart from phase 3;
8. End of iterations.

The optimization problem to solve is:

$$\min TE\ (h)$$

$$s.t \begin{cases} \sum_{i=1}^{n} h_i = 1 \\ h_i \geq 0 \end{cases}$$

where:

- $TE(h) = \left(\frac{R_{ETF} - R_F}{\sigma_{ETF}^-} - \frac{R_{index} - R_F}{\sigma_{index}^-}\right)^2$

  R_{ETF} is the expected return of the ETF;

  R_{index} is the expected return of the benchmark;

  R_F is the risk-free rate of return (in this case it is a U.S. 1 Years Treasury Bill);

  σ_{ETF}^- is the negative semi-variance of the ETF;

  σ_{index}^- is the negative semi-variance of the benchmark;

- $\sum_{i=1}^{n} h_i = 1$ means that the sum of weights has to be equal to 1;
- $h_i \geq 0$ means that the short selling is not allowed.

It has been necessary to compute several parameters before starting the process, such as both the ETF and benchmark daily returns, mean and the (semi)variance, which have been used to compute the portfolio's tracking error and therefore to solve the problem. Another important parameter for the objective function is the desired annual return, which represents the minimum hurdle rate the investor is willing to accept before investing. In this case, the hurdle rate is 2.00% per annum or equivalently 0.00736% daily return, which is a realistic return for a U.S. 1 Years Treasury Bill.

The procedure pursued for the analysis has been the following:

1. A reasonable set of population and number of iterations for the PSO have been chosen;
2. The algorithm has been run in Matlab for each combination of population and number of iterations for the period 01.01.2015 – 30.10.2017 (the ***in-sample*** period) to make a selection of the portfolio that will be used in the period 31.10.2017 – 29.12.2017 (the ***out-of-sample*** period) to assess the validity of the model;

3. The combination of a) number of particles and b) number of iterations resulting in the best fitness value (among all the previous simulations) has been chosen;
4. 15 more runs have been made with the best combination and the one resulting in the best fitness value has been chosen for the out-of-sample analysis;
5. The performances in terms of returns of the *XLG – INVESCO S&P 500 top 50 ETF* , the benchmark *S&P 500 Top 50 Index* and the estimated portfolio have been compared to check whether the model has proven to be effective in emulating the performances of the benchmark or not and its performances with respect to the original ETF.

Returns for the out-of-sample for the XLG - Invesco S&P 500 Top 50 ETF and SPX - S&P 500 Top 50 have been computed by downloading the historical data on closing prices and computing the logarithmic returns for both series through the formula:

$$R_{ln,t} = ln\left(\frac{P_{i,t}}{P_{i,t-1}}\right).$$

The initial PSO parameters have been set as follows:

- Cognitive acceleration coefficient (i.e. the deviation of the particles in the search space): c1 = 1,49618;
- Social acceleration coefficient (i.e. the convergence to the present global best): c2 = 1,49618;
- Inertia weight parameter: w = 0,7298.

The simulation has been run through the following sets of population (P) and iterations (I):

Population	**50**	**90**	**120**	**150**
Number of iterations	**100**	**1000**	**5000**	**10000**

The Matlab code output consists of the best portfolio generated by the PSO under the starting conditions, the tracking error graph and the fitness value graph for the best portfolio. The next paragraph will focus on the in-sample analysis, displaying a sample of the best portfolios computed for each particle parameter and the overall results. There is the possibility that some combinations produce results that are in contrast with the initial conditions (no short-selling and sum of weights equal to 100%), due to the low number of iterations or the insufficient number of particles. In this cases, results have been refined by imposing a weight of 0% to those assets with negative weights and then normalizing the sum of all weights to 100%. The subsequent one will concentrate on the out-of-sample analysis, studying the best portfolio in relationship with the benchmark and the original ETF.

5.1 The in-sample analysis

This paragraph contains the best results (in terms of fitness value) obtained for each population parameters and it collects the portfolio's composition, the tracking error's graph and the fitness value's graph. It has been chosen to gather the results by population rather than number of iterations (and then choosing the best one among them) because it is easier to see the improvement in the fitness value in each class of population.

Best fitness-value portfolio for P = 50:

Population	50	**Non neg. constraints**	0.000706
Iterations	1000	**Violation to sum (x) = 1**	1.00021
Fitness	9.164429	**Assets considered**	49
Tracking error value	69.4647		
Asset weights (Matlab results)		***Asset weights (normalized)***	
AAPL	6.215%	6.210%	
ABBV	0.433%	0.433%	
ADBE	0.400%	0.400%	
AMGN	0.318%	0.318%	
AMZN	0.210%	0.210%	

AVGO	0.156%	0.156%
BA	2.315%	2.313%
BAC	0.151%	0.151%
BKNG	0.296%	0.296%
BRK/B	2.426%	2.423%
C	0.359%	0.359%
CMCSA	0.410%	0.410%
CSCO	3.161%	3.158%
CVX	0.011%	0.011%
DIS	2.574%	2.572%
DWDP	0.309%	0.309%
FB	0.100%	0.100%
GE	-0.010%	0.000%
GOOG	1.103%	1.102%
GOOGL	6.992%	6.986%
HD	1.638%	1.637%
HON	0.017%	0.017%
IBM	2.944%	2.942%
INTC	0.644%	0.644%
JNJ	1.397%	1.395%
JPM	0.262%	0.261%
KO	0.176%	0.176%
MA	1.370%	1.368%
MCD	0.229%	0.229%
MDT	18.765%	18.748%
MMM	0.364%	0.364%
MO	1.667%	1.665%
MRK	0.629%	0.628%
MSFT	0.442%	0.441%
NFLX	0.038%	0.038%
NVDA	2.815%	2.813%
ORCL	0.642%	0.642%
PEP	-0.061%	0.000%
PFE	0.146%	0.146%
PG	1.224%	1.223%
PM	0.081%	0.081%
SLB	3.291%	3.288%
T	0.321%	0.320%
UNH	1.404%	1.403%
UNP	0.639%	0.638%
UTX	21.061%	21.041%
V	0.051%	0.051%
VZ	4.588%	4.584%

WFC	1.359%	1.358%
WMT	1.717%	1.715%
XOM	2.229%	2.227%

The improvement of the fitness value is progressive and smooth and it reaches its minimum approximately at iteration n.450. The tracking error value collapses soon, way before iteration n.450:

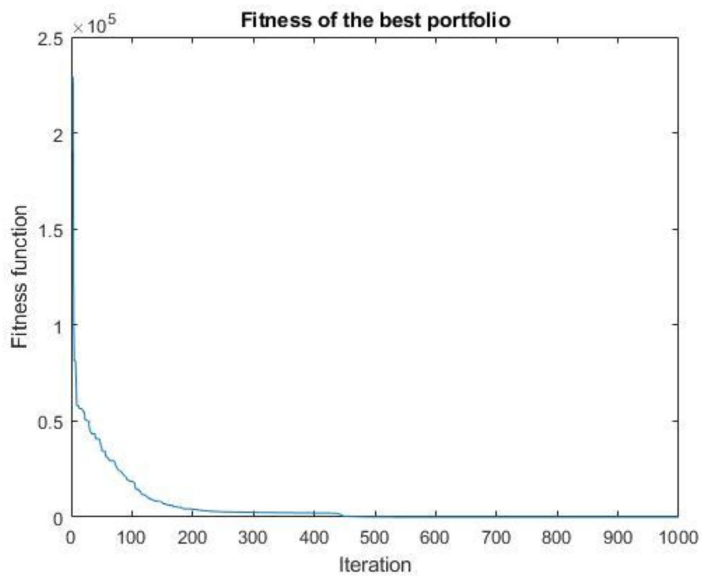

Figure 17: Fitness value for P = 50, I = 1000

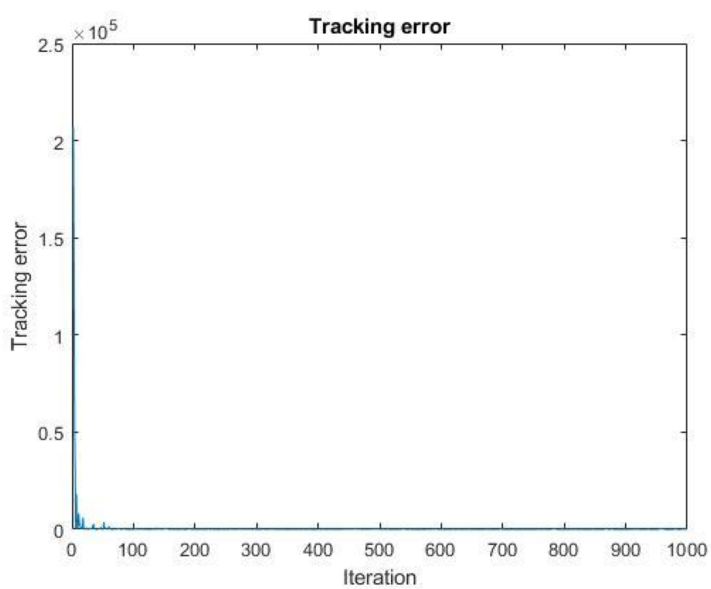

Figure 18: Tracking error value for P = 50, I = 1000

Best fitness-value portfolio for P = 90:

Population	90	**Non neg. constraints**	0
Iterations	1000	**Violation to sum (x) = 1**	0.999995
Fitness	0.047576	**Assets considered**	51
Tracking error value	7.054619		
Asset weights (Matlab results)		***Asset weights (normalized)***	
AAPL	0.513%	0.513%	
ABBV	1.921%	1.921%	
ADBE	0.330%	0.330%	
AMGN	0.724%	0.724%	
AMZN	0.970%	0.970%	
AVGO	0.015%	0.015%	
BA	0.101%	0.101%	
BAC	1.096%	1.096%	
BKNG	0.218%	0.218%	
BRK/B	0.265%	0.265%	
C	1.657%	1.657%	
CMCSA	0.021%	0.021%	
CSCO	0.240%	0.240%	
CVX	1.597%	1.597%	
DIS	0.613%	0.613%	
DWDP	0.447%	0.447%	
FB	0.156%	0.156%	
GE	0.034%	0.034%	
GOOG	2.054%	2.054%	
GOOGL	1.513%	1.513%	
HD	0.023%	0.023%	
HON	6.090%	6.090%	
IBM	2.725%	2.725%	
INTC	0.104%	0.104%	
JNJ	0.461%	0.461%	
JPM	3.489%	3.489%	
KO	0.223%	0.223%	
MA	0.252%	0.252%	
MCD	0.737%	0.737%	
MDT	0.197%	0.197%	
MMM	1.920%	1.920%	
MO	0.342%	0.342%	
MRK	3.434%	3.434%	
MSFT	2.222%	2.222%	

NFLX	0.418%	0.418%
NVDA	0.161%	0.161%
ORCL	0.297%	0.297%
PEP	0.272%	0.272%
PFE	1.046%	1.046%
PG	0.256%	0.256%
PM	0.438%	0.438%
SLB	1.142%	1.142%
T	54.502%	54.502%
UNH	0.009%	0.009%
UNP	1.777%	1.777%
UTX	0.106%	0.106%
V	1.740%	1.740%
VZ	0.727%	0.727%
WFC	0.290%	0.290%
WMT	0.021%	0.021%
XOM	0.096%	0.096%

The improvement of the fitness value is less progressive than the previous simulation, but it reaches its minimum first, approximately at iteration n.350. The tracking error value, however, collapses later.

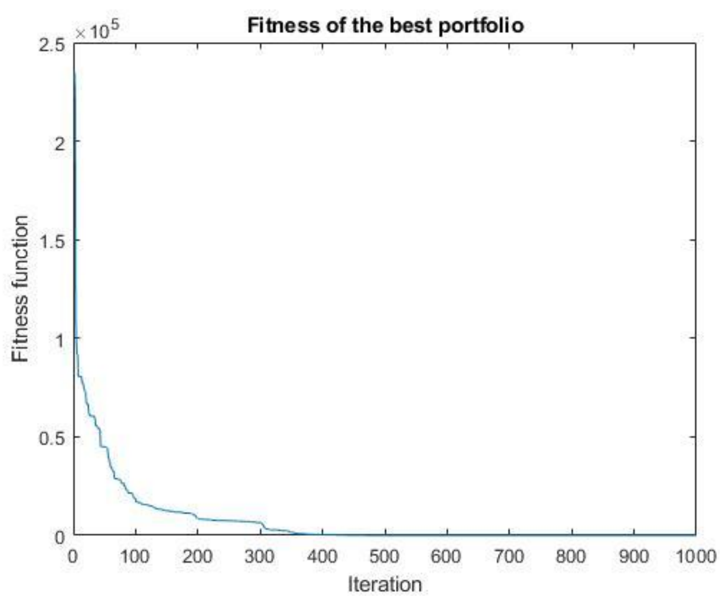

Figure 19: Fitness value for P = 90, I = 1000

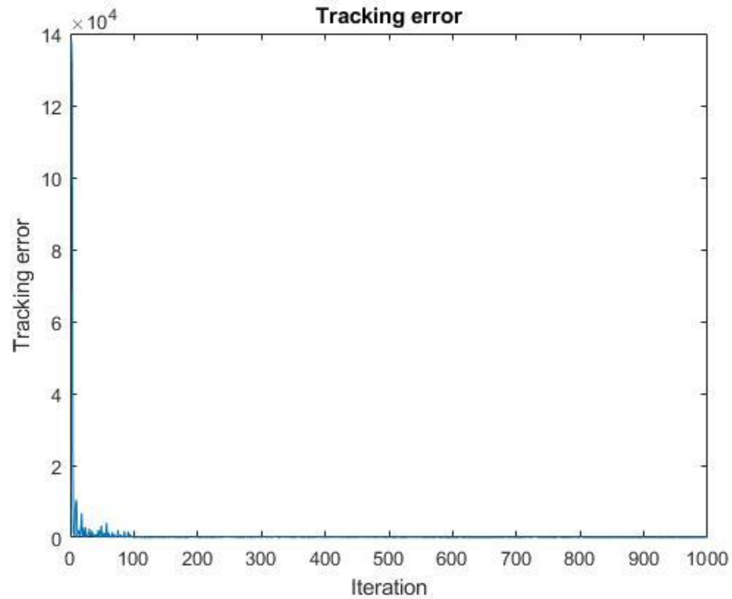

Figure 20: Tracking error value for P = 90, I = 1000

Best fitness-value portfolio for P = 120:

Population	120	**Non neg. constraints**	0
Iterations	10000	**Violation to sum (x) = 1**	1.000001
Fitness	0.011137	**Assets considered**	51
Tracking error value	0.100281		
Asset weights (Matlab results)		***Asset weights (normalized)***	
AAPL	3.441%	3.441%	
ABBV	1.126%	1.125%	
ADBE	1.865%	1.865%	
AMGN	0.534%	0.534%	
AMZN	0.006%	0.006%	
AVGO	1.701%	1.701%	
BA	0.282%	0.282%	
BAC	0.036%	0.036%	
BKNG	1.355%	1.355%	
BRK/B	0.053%	0.053%	
C	5.283%	5.283%	
CMCSA	0.407%	0.407%	
CSCO	4.276%	4.276%	
CVX	4.763%	4.763%	
DIS	0.371%	0.371%	

DWDP	1.163%	1.163%
FB	4.038%	4.038%
GE	0.582%	0.582%
GOOG	1.854%	1.854%
GOOGL	0.861%	0.861%
HD	1.557%	1.557%
HON	0.416%	0.416%
IBM	17.763%	17.763%
INTC	0.758%	0.758%
JNJ	1.305%	1.305%
JPM	1.325%	1.325%
KO	3.721%	3.721%
MA	2.854%	2.854%
MCD	0.433%	0.433%
MDT	2.082%	2.082%
MMM	2.139%	2.139%
MO	0.461%	0.461%
MRK	2.229%	2.229%
MSFT	1.392%	1.392%
NFLX	0.790%	0.790%
NVDA	2.782%	2.782%
ORCL	1.039%	1.039%
PEP	2.881%	2.881%
PFE	2.594%	2.594%
PG	0.474%	0.474%
PM	1.924%	1.924%
SLB	0.080%	0.080%
T	0.963%	0.963%
UNH	0.188%	0.188%
UNP	3.787%	3.787%
UTX	1.942%	1.941%
V	1.341%	1.341%
VZ	0.422%	0.422%
WFC	0.746%	0.746%
WMT	5.244%	5.244%
XOM	0.374%	0.374%

The improvement of the fitness value is sudden, and it reaches its minimum, approximately at iteration n.200/250. The greater number of particles, as well as a higher number of iterations, allows to get to the point first:

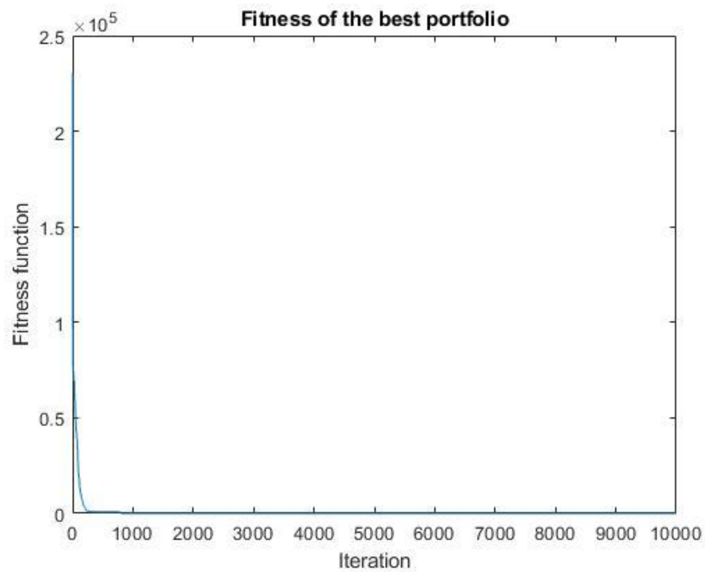

Figure 21: Fitness value for P = 120, I = 10000

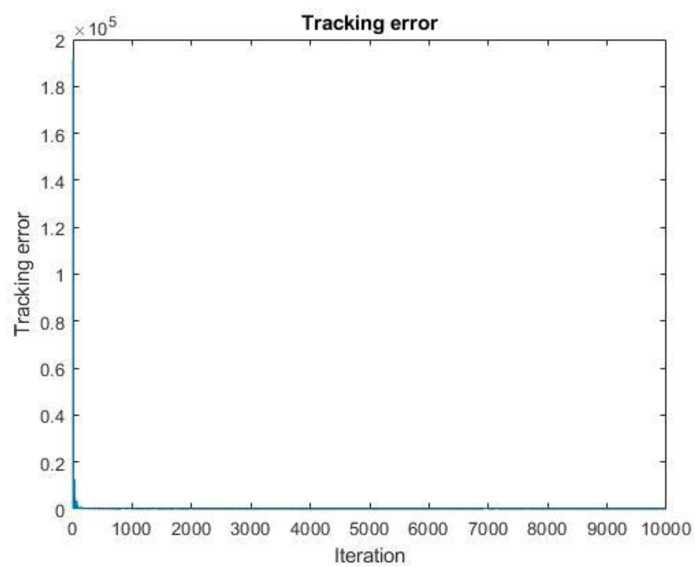

Figure 22: Tracking error value for P = 120, I = 10000

Best fitness-value portfolio for P = 150:

Population	150	**Non neg. constraints**	0
Iterations	5000	**Violation to sum (x) = 1**	1

Fitness	0.002971	**Assets considered**	51
Tracking error value	0.02135		
Asset weights (Matlab results)		***Asset weights (normalized)***	
AAPL	45.795%	45.795%	
ABBV	9.168%	9.168%	
ADBE	2.052%	2.052%	
AMGN	0.150%	0.150%	
AMZN	0.458%	0.458%	
AVGO	0.863%	0.863%	
BA	0.002%	0.002%	
BAC	0.079%	0.079%	
BKNG	0.189%	0.189%	
BRK/B	0.117%	0.117%	
C	0.026%	0.026%	
CMCSA	0.615%	0.615%	
CSCO	1.531%	1.531%	
CVX	0.256%	0.256%	
DIS	0.929%	0.929%	
DWDP	2.116%	2.116%	
FB	0.001%	0.001%	
GE	0.981%	0.981%	
GOOG	0.384%	0.384%	
GOOGL	0.190%	0.190%	
HD	0.208%	0.208%	
HON	0.006%	0.006%	
IBM	0.332%	0.332%	
INTC	0.217%	0.217%	
JNJ	0.140%	0.139%	
JPM	0.526%	0.526%	
KO	0.107%	0.107%	
MA	1.793%	1.793%	
MCD	0.642%	0.642%	
MDT	2.200%	2.200%	
MMM	0.261%	0.261%	
MO	0.448%	0.448%	
MRK	0.825%	0.825%	
MSFT	16.290%	16.290%	
NFLX	0.984%	0.984%	
NVDA	0.243%	0.243%	
ORCL	0.724%	0.724%	
PEP	0.786%	0.786%	
PFE	0.023%	0.023%	
PG	1.578%	1.578%	

PM	2.070%	2.070%
SLB	0.509%	0.508%
T	0.009%	0.009%
UNH	0.051%	0.051%
UNP	0.451%	0.450%
UTX	0.218%	0.217%
V	0.090%	0.090%
VZ	0.857%	0.857%
WFC	0.493%	0.493%
WMT	0.245%	0.245%
XOM	0.774%	0.774%

Also in this case, the improvement of the fitness value is sudden and it reaches its minimum, approximately at iteration n. 250.

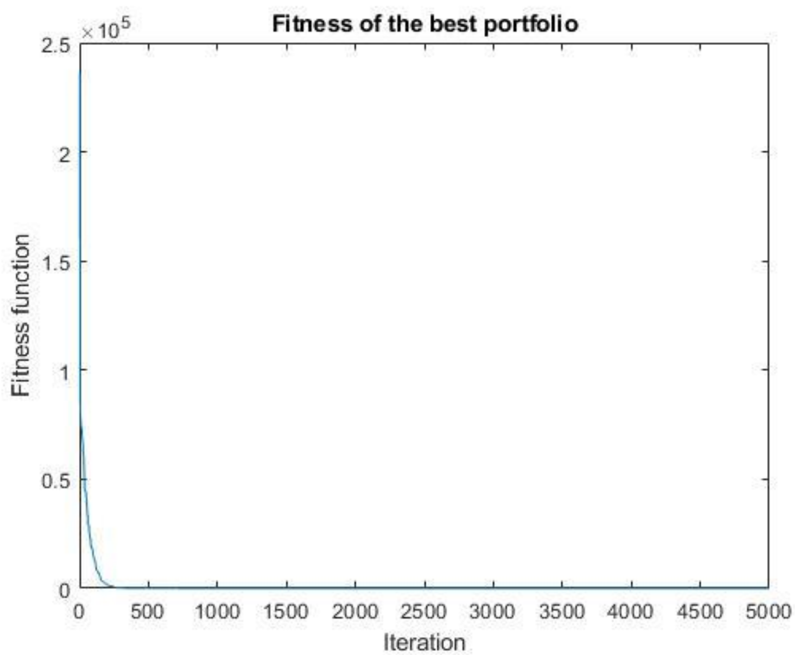

Figure 23: Fitness value for P = 150, I = 5000

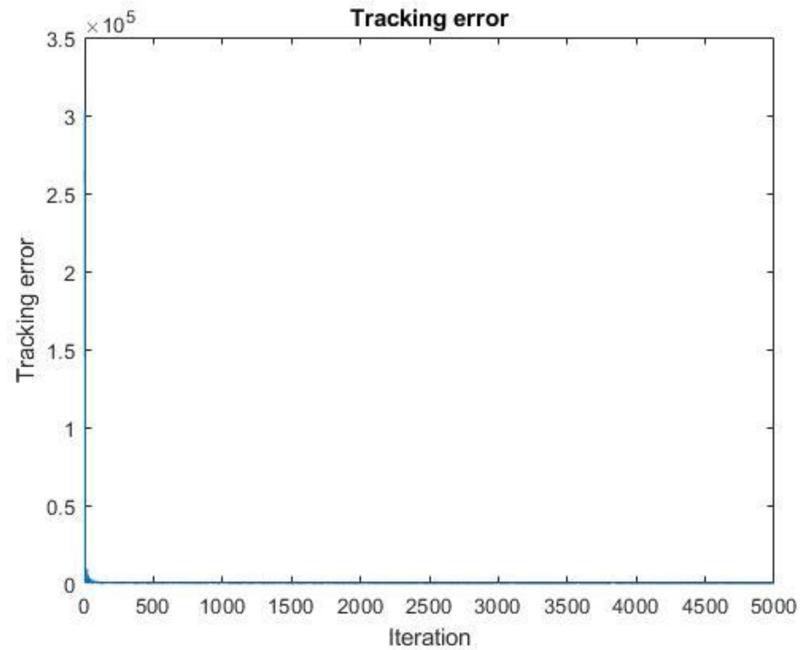

Figure 24: Tracking error value for P = 150, I = 5000

The overall results of all runs is the following:

Fitness function		**Population**			
		50	**90**	**120**	**150**
Iterations	**100**	25656.786076	16602.596610	21051.093852	9923.555389
	1000	9.164429	0.047576	529.711222	1.647869
	5000	3216.243905	0.070223	0.119493	0.002971
	10000	53.918174	32.483554	0.011137	0.014280

It is clear that, for a low number of iterations, the algorithm is not particularly efficient in finding the optimal portfolio without violating the constraints of the problem: as much as iterations increase, the algorithm performs better and better. By looking at the results, combination P = 50 - I = 1000 provides the best result for the class of population P = 50, but it gets worse for P = 5000. A possible explanation could be that, during the initialization process (which is step n.1 of Kennedy-Ebenhart-Shi process), particles were in the near of the best solution and they needed less iterations to converge. In the case P = 50 - I = 5000, they have been initialized farer and the number of iterations has not been sufficient. In the case P = 50 - I = 10000, the high number of iteration allowed to obtain a better result. Eventually, the best population-iteration combination is 150- 5000. Therefore, this will be the parametrization used in the next paragraph to test the out-of-sample data.

5.2 The out-of-sample analysis

In this paragraph, the algorithm is run 15 more times with the best parametrization obtained in the previous paragraph, in order to get the best portfolio to be compared with the benchmark and the original ETF. In particular, the out-of-sample analysis will be pursued as following:

1. run the algorithm for 15 times, to get a significant number of optimal portofolios;
2. choose the portfolio with the best fitness value among the set to perform a virtual future analysis from 31.10.2017 to 29.12.2017;
3. the returns of the portfolio will be computed by considering the real returns of each asset in the considered period and the assets' weights;
4. a graphic representation will be used to assess whether the portfolio can be considered a good model in emulating the performances of the benchmark;
5. a similar representation will be used to compare the portfolio with the original ETF, to check if some improvements have been obtained in managing the negative semivariance.

The following figure show the results obtained in the 15 runs. It presents, for each run, the composition of the optimal portfolio obtained with the PSO, as well as information about its fitness value, the violation of the sum (x) = 1 and the non-negative constraint. Twelve runs over fifteen presents a total sum of weights equal to 1 (100%) and all respect the non-negative constraint. The best fitness value has been obtained in the 7^{TH} launch, with Fitness = 0.002372, whereas the worst value has been obtained in the 14^{th} launch, with Fitness = 0.087473

Out-of-sample	RUN							
	1	2	3	4	5	6	7	8
Fitness value	**0.02507**	**0.008838**	**0.00272**	**0.004117**	**0.015535**	**0.020113**	**0.002372**	**0.06903**
Sum = 1	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00001**
Non-neg. Cond.	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**
Tracking error	**0.01376**	**0.19843**	**0.25932**	**0.12118**	**0.02132**	**1.00866**	**0.02642**	**0.68291**
AAPL	1.9620%	2.1620%	3.4238%	1.6554%	3.2565%	2.9107%	3.8413%	1.6318%
ABBV	4.7161%	0.4741%	0.3435%	0.9452%	3.4114%	0.0352%	5.1493%	1.6295%
ADBE	3.5034%	0.7301%	0.6495%	0.8942%	0.4666%	0.6060%	0.4091%	1.3942%
AMGN	0.0510%	0.6632%	7.6716%	0.2831%	0.8953%	14.3550%	1.2869%	1.8710%
AMZN	0.0676%	2.8757%	0.4611%	3.9639%	0.6954%	0.9152%	1.8772%	1.3380%
AVGO	3.3350%	0.0341%	2.6033%	0.5436%	0.0060%	3.0808%	0.8241%	0.6590%
BA	8.9224%	0.3337%	0.2205%	6.3065%	1.1403%	0.2261%	1.2978%	0.4250%
BAC	0.4802%	0.6848%	4.7898%	2.8138%	0.8885%	4.0332%	3.0654%	1.0447%
BKNG	2.0294%	2.1219%	3.0408%	2.3125%	1.5739%	3.3694%	0.0921%	0.4496%
BRK/B	2.6622%	0.2092%	0.2182%	0.5339%	0.5136%	0.0489%	1.5596%	2.4452%
C	0.1878%	0.0321%	2.6498%	0.9288%	2.6030%	1.4957%	2.4490%	0.4461%
CMCSA	0.6064%	0.7322%	0.0916%	0.5851%	2.8929%	4.1486%	0.2252%	1.3225%
CSCO	0.4378%	2.2499%	0.3487%	1.0776%	0.4634%	1.5001%	0.3017%	0.3081%
CVX	1.5356%	0.5403%	1.3594%	0.3817%	3.8930%	1.9638%	0.7843%	0.9541%
DIS	3.9432%	0.5516%	1.5049%	2.4088%	0.4794%	0.9821%	0.7030%	0.0386%
DWDP	0.5614%	0.7637%	5.5138%	1.8010%	2.2972%	0.1611%	8.4361%	2.5614%
FB	0.5194%	10.5388%	1.7292%	0.0325%	0.4681%	0.9535%	0.5951%	1.7198%
GE	1.1219%	0.2824%	4.5505%	0.8582%	5.0212%	2.7284%	1.3509%	2.1124%
GOOG	0.5078%	0.1154%	0.4751%	0.3064%	0.1134%	0.1579%	0.8398%	5.4718%
GOOGL	1.9478%	4.1924%	1.4707%	0.3936%	0.7615%	5.1856%	1.3135%	1.1743%
HD	0.6241%	1.7115%	2.4057%	0.1921%	2.6412%	0.2463%	0.7873%	8.7855%
HON	0.9288%	1.1474%	0.5195%	2.1652%	0.7413%	0.4506%	4.1999%	0.0106%
IBM	0.1675%	0.0852%	2.7447%	0.1976%	0.4448%	0.4494%	7.2561%	0.4550%
INTC	7.7722%	2.1098%	0.5387%	4.5329%	0.5247%	0.0471%	0.4744%	1.7474%
JNJ	0.8933%	0.7660%	0.4935%	0.8015%	0.7678%	1.7206%	0.1051%	2.3202%
JPM	5.3081%	1.0105%	1.0455%	4.5078%	2.3132%	3.1917%	4.2463%	0.5518%
KO	3.1237%	1.3222%	2.3579%	3.1277%	0.0167%	0.2497%	3.4567%	0.7421%
MA	0.3546%	3.3486%	2.8025%	4.5697%	1.6136%	0.7918%	0.5139%	3.0381%
MCD	1.8229%	0.4077%	0.4754%	0.0403%	0.4233%	0.4682%	7.6972%	1.5410%
MDT	0.0397%	0.8247%	2.7516%	1.4005%	0.0580%	1.4330%	0.3257%	6.3130%
MMM	1.1946%	0.0393%	0.0111%	4.0833%	2.5949%	0.2953%	2.1412%	0.5466%
MO	1.7759%	0.3699%	0.0661%	0.5497%	10.0647%	0.6575%	4.6935%	0.0028%
MRK	0.2701%	6.5707%	4.2607%	12.0066%	0.1202%	3.4665%	0.4054%	0.0871%
MSFT	8.0339%	2.3981%	0.2049%	1.4388%	0.1960%	0.2112%	0.3020%	1.6730%

NFLX	3.0570%	0.9695%	0.9068%	4.1728%	0.9618%	4.0436%	1.4814%	3.4964%
NVDA	3.3045%	1.2128%	0.1240%	1.0313%	2.2146%	0.0856%	1.7987%	3.2376%
ORCL	2.7020%	0.2959%	0.3258%	3.3822%	1.5322%	2.3463%	2.7174%	0.1017%
PEP	2.1251%	0.2157%	0.7690%	0.1509%	1.3361%	1.6015%	0.1671%	0.9140%
PFE	5.3682%	5.3139%	1.5997%	0.4190%	8.8128%	3.1571%	2.4068%	0.8014%
PG	0.0398%	0.1321%	8.5734%	0.9459%	1.7627%	0.2138%	0.0974%	7.4085%
PM	1.9945%	2.2989%	2.4740%	0.0549%	0.1230%	0.3014%	1.0577%	10.6508%
SLB	0.0925%	1.9676%	0.0900%	3.2483%	3.0901%	1.9347%	5.2679%	4.9896%
T	0.0245%	0.5406%	4.0598%	0.7805%	0.0603%	0.0006%	2.8437%	0.1611%
UNH	0.9832%	3.6170%	1.5749%	3.3345%	1.0279%	14.4919%	0.6060%	2.8927%
UNP	0.2435%	1.4167%	2.1171%	2.2981%	1.3658%	3.5147%	0.4059%	2.1356%
UTX	1.5664%	0.2882%	0.5263%	1.5736%	6.9425%	0.8331%	2.4157%	0.0806%
V	4.5855%	0.2340%	2.7450%	4.3659%	0.0418%	1.7999%	1.4459%	0.2928%
VZ	0.3761%	3.1768%	4.8329%	0.5258%	0.7500%	0.2542%	1.6802%	0.2054%
WFC	1.3416%	18.6904%	3.1606%	3.1487%	6.0159%	1.4691%	0.1790%	0.7369%
WMT	0.0388%	6.6262%	1.4122%	1.1593%	3.6863%	0.6422%	1.2591%	3.3552%
XOM	0.7494%	0.6047%	0.9151%	0.7689%	5.9152%	0.7741%	1.1641%	1.7289%

Out-of-sample	**RUN**						
	9	**10**	**11**	**12**	**13**	**14**	**15**
Fitness value	**0.011225**	**0.010861**	**0.007698**	**0.078819**	**0.017485**	**0.087473**	**0.007545**
Sum = 1	**1.00000**	**1.00000**	**1.00000**	**1.00001**	**1.00000**	**1.00001**	**1.00000**
Non-neg. Cond.	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**	**0.00000**
Tracking error	**0.01261**	**0.97201**	**1.22398**	**2.10089**	**0.02353**	**2.90981**	**1.50918**
AAPL	0.2409%	0.0931%	0.1733%	1.9743%	0.6259%	0.4776%	0.0061%
ABBV	0.2560%	3.1788%	0.6205%	0.0335%	0.2232%	1.0892%	3.2624%
ADBE	4.5912%	0.4945%	3.9404%	3.6809%	0.0575%	2.4706%	0.3403%
AMGN	0.3879%	2.6397%	6.2219%	1.2732%	0.0520%	0.1569%	11.4203%
AMZN	2.7724%	0.9045%	3.5200%	0.3091%	1.6331%	0.6480%	0.0717%
AVGO	0.3564%	0.1395%	3.8784%	0.6893%	0.0024%	0.3808%	0.0145%
BA	0.5879%	2.9819%	1.1073%	0.8262%	0.8559%	1.5051%	1.0077%
BAC	0.3854%	3.6406%	0.0775%	0.2489%	2.1117%	1.1103%	0.2004%
BKNG	0.5849%	0.7061%	0.3620%	0.7831%	0.0612%	0.8551%	0.8211%
BRK/B	2.0676%	0.0879%	4.4258%	3.9262%	0.0184%	1.6242%	0.7700%
C	0.3069%	0.6851%	0.5613%	0.2130%	0.5274%	0.2570%	8.0938%
CMCSA	4.7407%	3.0784%	10.2004%	0.6331%	61.4733%	5.4428%	0.6317%
CSCO	0.1320%	1.6269%	2.9068%	1.1996%	0.9416%	0.3334%	0.5107%

CVX	10.0097%	0.5513%	3.4337%	1.4515%	0.0363%	22.1587%	0.0747%
DIS	0.3400%	1.0195%	1.6653%	0.8647%	0.0643%	0.2498%	1.5022%
DWDP	0.7254%	1.7657%	0.6863%	3.7437%	0.0560%	0.1527%	2.6201%
FB	0.1290%	0.6464%	1.2602%	0.9766%	0.4855%	1.1540%	2.0920%
GE	0.6436%	17.9602%	1.3689%	3.5610%	0.2720%	0.9439%	12.1036%
GOOG	0.0707%	2.5760%	0.9341%	0.1575%	0.7487%	2.6302%	0.4505%
GOOGL	4.6992%	3.7086%	3.5790%	10.3862%	0.5972%	0.0337%	4.0534%
HD	0.0975%	0.0276%	2.1360%	0.8699%	0.0771%	0.9175%	0.6634%
HON	1.2172%	0.4935%	3.5110%	4.5004%	0.5237%	0.3231%	6.5917%
IBM	4.0329%	1.2621%	1.9563%	0.5273%	0.0858%	1.9549%	0.1997%
INTC	6.0631%	3.3650%	0.1837%	0.4637%	0.0108%	0.2226%	1.2634%
JNJ	0.3867%	4.1325%	0.6547%	0.2774%	0.0426%	0.3786%	0.5284%
JPM	0.7928%	0.1722%	0.1202%	0.0043%	0.0191%	1.3003%	0.2859%
KO	1.0465%	1.5266%	0.6882%	3.1237%	0.6428%	0.4112%	4.9951%
MA	0.2474%	0.3861%	0.0749%	2.7066%	0.4186%	0.0046%	0.3006%
MCD	0.4937%	3.2190%	0.3663%	2.3326%	0.1319%	0.2272%	6.0906%
MDT	4.8813%	4.1055%	0.8751%	0.1973%	0.9286%	1.0828%	0.5737%
MMM	9.7279%	0.4089%	0.4902%	3.9750%	0.6445%	4.9092%	0.3452%
MO	1.7845%	0.9550%	5.4730%	0.3047%	0.2137%	3.4531%	3.8110%
MRK	1.4606%	1.7883%	0.1902%	0.1111%	0.0397%	0.5399%	2.8656%
MSFT	1.5566%	1.4262%	1.0094%	0.6923%	0.0195%	11.5473%	6.2923%
NFLX	0.1759%	0.5720%	0.4346%	0.9013%	0.3160%	1.8313%	0.7097%
NVDA	0.0028%	0.6364%	0.0177%	3.7453%	0.6775%	0.6244%	1.3018%
ORCL	2.7357%	0.7526%	0.4083%	3.9237%	0.2992%	0.1132%	0.7651%
PEP	6.4068%	4.1962%	5.9143%	0.7253%	1.0559%	0.2042%	0.8835%
PFE	0.8914%	3.1503%	0.6412%	0.3599%	6.4687%	0.3304%	0.3327%
PG	0.5535%	2.2552%	0.1950%	4.3784%	0.0045%	0.0976%	0.5867%
PM	0.5246%	3.4937%	0.0103%	1.3743%	0.9982%	0.0961%	0.0604%
SLB	2.3246%	1.6671%	0.4972%	0.3992%	0.0446%	1.9511%	5.5538%
T	1.9410%	0.0980%	0.0571%	1.3642%	5.7770%	5.7862%	1.6245%
UNH	1.4945%	0.3520%	1.1974%	2.4478%	1.4397%	3.5324%	0.5561%
UNP	3.7768%	1.9303%	3.1166%	6.6415%	1.3436%	0.7042%	0.1112%
UTX	4.6600%	2.9260%	1.2759%	1.6106%	0.1567%	0.8784%	0.1487%
V	1.4026%	0.2145%	1.3636%	3.8585%	3.4289%	1.1982%	0.0310%
VZ	0.6312%	4.7292%	9.6582%	7.9339%	0.7302%	0.1108%	0.0002%
WFC	1.7011%	0.8907%	0.3493%	1.7404%	0.1065%	1.8220%	0.7063%
WMT	2.7965%	0.2237%	1.9328%	1.5724%	0.8753%	9.5330%	1.7313%
XOM	0.1643%	0.1591%	4.2783%	0.0062%	1.6355%	0.2410%	0.0430%

The composition of the portfolio, compared to the XLG - Invesco S&P 500 Top 50 ETF is:

Population	150	
Iterations	5000	
Fitness	0.020113	
Tracking error value	0.024623	
Asset weights	*PSO - portfolio*	*XLG - Invesco S&P 500 Top 50 ETF*
AAPL	3.841%	8.821%
ABBV	5.149%	1.234%
ADBE	0.409%	0.997%
AMGN	1.287%	1.083%
AMZN	1.877%	6.253%
AVGO	0.824%	0.691%
BA	1.298%	1.566%
BAC	3.065%	2.393%
BKNG	0.092%	0.730%
BRK/B	1.560%	3.305%
C	2.449%	1.468%
CMCSA	0.225%	1.347%
CSCO	0.302%	1.779%
CVX	0.784%	1.855%
DIS	0.703%	1.378%
DWDP	8.436%	1.296%
FB	0.595%	3.439%
GE	1.351%	0.880%
GOOG	0.840%	3.008%
GOOGL	1.314%	2.997%
HD	0.787%	1.860%
HON	4.200%	0.955%
IBM	7.256%	1.027%
INTC	0.474%	1.809%
JNJ	0.105%	2.977%
JPM	4.246%	3.224%
KO	3.457%	1.470%
MA	0.514%	1.536%
MCD	7.697%	1.042%
MDT	0.326%	1.011%
MMM	2.141%	1.001%
MO	4.694%	0.951%
MRK	0.405%	1.531%
MSFT	0.302%	6.818%
NFLX	1.481%	1.134%

NVDA	1.799%	1.224%
ORCL	2.717%	1.187%
PEP	0.167%	1.343%
PFE	2.407%	2.030%
PG	0.097%	1.735%
PM	1.058%	1.089%
SLB	5.268%	0.717%
T	2.844%	1.980%
UNH	0.606%	2.078%
UNP	0.406%	0.955%
UTX	2.416%	0.824%
V	1.446%	2.081%
VZ	1.680%	1.868%
WFC	0.179%	2.127%
WMT	1.259%	1.166%
XOM	1.164%	2.730%

The tracking error obtained by the PSO, with the proposed tracking error function is 0.024623 or 2.4623%. By looking at the former table, it can be seen that the PSO has redistributed the weights of certain assets quite heavily, in particular:

- AAPL's weight has shrink from 8.821% to 3.841%;
- AMZN's weight has fallen from 6.253% to 1.877%;
- FB's weight has reduced by almost 7 times, from 3.439% to 0.595%;
- The same has happened to GOOG, from 3.008% to 0.840% and MSFT, from 6.818% to 0.302%.

Their weights have been redistributed mainly to:

- ABBV, from 1.234% to 5.149%;
- DWDP, from 1.296% to 8.436%;
- HON, from 0.955% to 4.200%;
- MO, from 0.951% to 4.694%;
- SLB, from 0.717% to 5.268%.

One of the possible reason for this redistribution is that the algorithm aims to minimize the downside risk and AAPL, AMZN, FB, GOOG appear to be more susceptible to downside risk with respect to others. They are some of the biggest companies in the world as well as the most publicly-known and, for example, the notification of infringement to workers' rights or working conditions (Apple and Amazon are infamous for such behavior) or the unauthorized disclosure of private information (Facebook and Google are exposed to informatic hackers and/or Governments disclosure requests) are real issue for them and can incur in enormous losses. Vice-versa, those companies which increased their shares in the portfolio, represents the pharmaceutical and industrial world, sectors which have not seen any dramatic event or scandal in last years. On the contrary, the presence of patents (AbbVie Inc) and the discovery of new oil field (for Schlumberger, the world's greatest oil company) have probably increased the upper bound of positive returns, reducing the risks of negative ones.

The daily returns for the optimized portfolio have been obtained by multiplying the computed returns and the weights the algorithm has produced. The results are the following:

Date	Logarithmic returns		
	SPX - S&P 500 Top 50	PSO-portfolio	XLG - Invesco S&P 500 Top 50 ETF
31/10/2017	0.10%	0.05%	0.05%
01/11/2017	0.16%	0.35%	0.24%
02/11/2017	0.03%	-0.23%	-0.03%
03/11/2017	0.32%	0.17%	0.49%
06/11/2017	0.14%	0.08%	0.05%
07/11/2017	-0.02%	0.22%	0.08%
08/11/2017	0.15%	0.08%	0.16%
09/11/2017	-0.35%	-0.47%	-0.31%
10/11/2017	-0.05%	-0.14%	-0.12%
13/11/2017	0.10%	0.02%	0.02%
14/11/2017	-0.22%	-0.49%	-0.28%
15/11/2017	-0.53%	-0.51%	-0.53%
16/11/2017	0.85%	0.97%	0.89%
17/11/2017	-0.26%	-0.35%	-0.57%

20/11/2017	0.13%	0.22%	0.13%
21/11/2017	0.66%	0.61%	0.91%
22/11/2017	-0.07%	0.01%	0.02%
24/11/2017	0.21%	0.23%	0.38%
27/11/2017	-0.03%	-0.15%	0.13%
28/11/2017	0.99%	1.10%	0.89%
29/11/2017	-0.02%	0.13%	-0.50%
30/11/2017	0.86%	0.99%	0.98%
01/12/2017	-0.20%	-0.03%	-0.21%
04/12/2017	-0.10%	0.15%	-0.37%
05/12/2017	-0.37%	-0.31%	-0.15%
06/12/2017	-0.01%	-0.16%	0.19%
07/12/2017	0.31%	0.06%	0.18%
08/12/2017	0.56%	0.53%	0.54%
11/12/2017	0.32%	0.41%	0.53%
12/12/2017	0.16%	0.25%	0.31%
13/12/2017	-0.04%	-0.06%	0.00%
14/12/2017	-0.39%	-0.44%	-0.22%
15/12/2017	0.90%	0.46%	0.94%
18/12/2017	0.54%	0.75%	0.57%
19/12/2017	-0.32%	-0.14%	-0.38%
20/12/2017	-0.07%	-0.11%	-0.15%
21/12/2017	0.20%	0.21%	0.22%
22/12/2017	-0.05%	0.04%	-0.08%
26/12/2017	-0.10%	-0.18%	-0.26%
27/12/2017	0.09%	0.19%	0.13%
28/12/2017	0.20%	0.10%	0.18%
29/12/2017	-0.51%	-0.51%	-0.59%

The following graphs shows the behavior of the PSO with respect to 1) the benchmark, 2) the existent ETF and 3) to both:

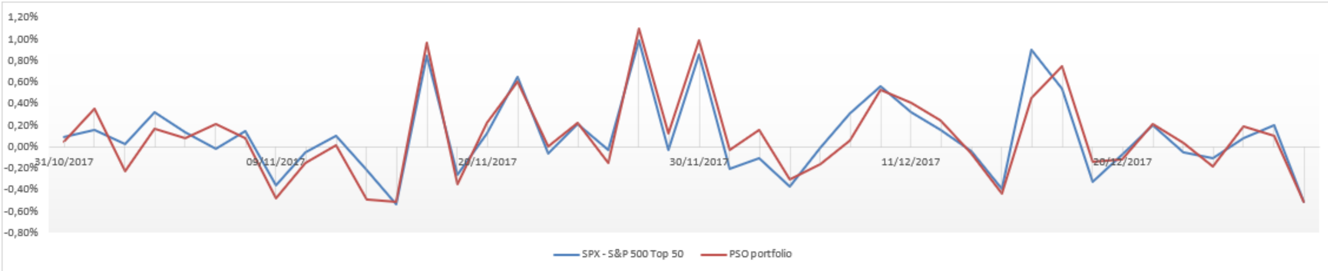

Figure 25: Returns of SPX - S&P 500 Top 50 index and the PSO - portfolio

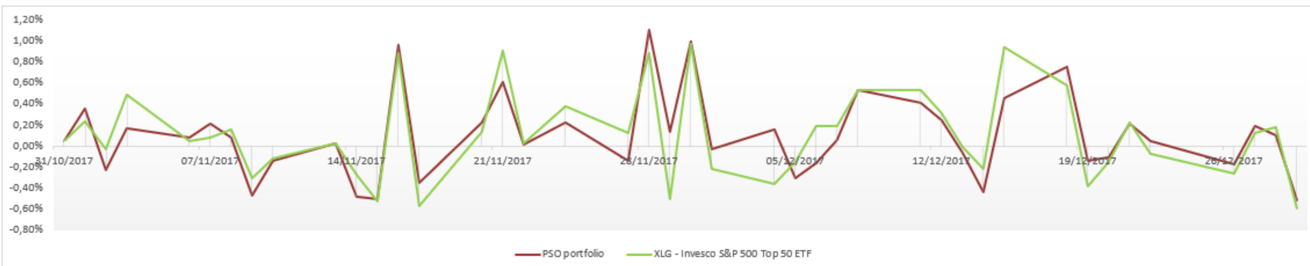

Figure 26: Returns of XLG - Invesco S&P 500 Top 50 index and the PSO - portfolio

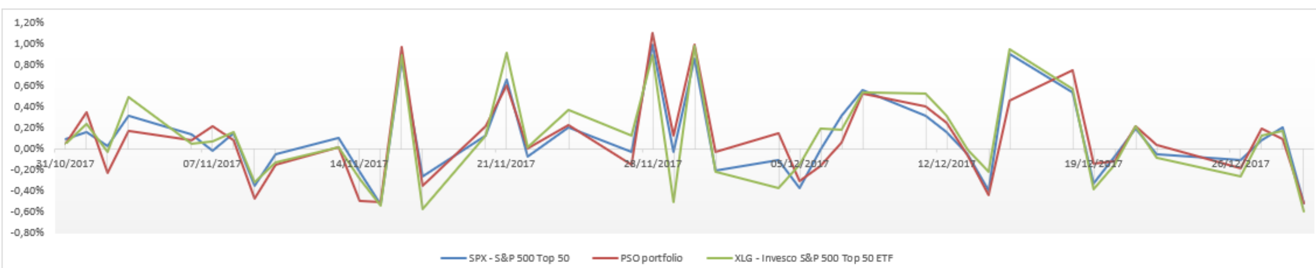

Figure 27: Returns of XLG - Invesco S&P 500 Top 50 index and the PSO - portfolio and the SPX - S&P 500 Top 50 index

By looking at the first graph, it is possible to see that the portfolio's setup shows a satisfying capability to replicate the benchmark trend: in the first 10-15 days of the out-of-sample analysis, the portfolio performs worse than the benchmark, even if there are some days of negotiations in which it performed very well (01.11.2017 and 07.11.2017). From 15.11.2017 to 04.11.2107 the portfolio started to replicate the index almost perfectly, even if portfolio's returns were slightly higher. In the following days, the bias has increased, but there always have been a synchronous movement between the two.

The same observations can be made by looking at the second graph, the comparison between XLG - Invesco S&P 500 Top 50 index and the PSO - portfolio, mainly because the ETF appears to follow the index pattern closely (*Figure 28*). However, one interesting observation is that the portfolio in most cases, when a sudden dive happens, is losing less with respect to the ETF: this fact is probably a consequence of both the imposition of the objective function which minimizes the downside risk and the fact that the ETF has generally performed worse than the benchmark in managing falls.

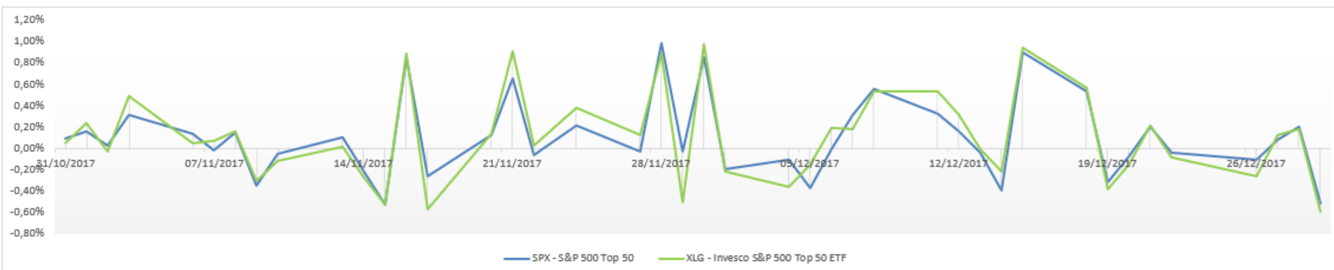

Figure 28: Returns of XLG - Invesco S&P 500 Top 50 index and the SPX - S&P 500 Top 50 index

Figure 27 pictures a more complete representation: the portfolio has shown to be effective in replicating the trend of the benchmark, therefore the model is consistent in this sense. Moreover, by focusing the attention on the downside risk, it can be shown that, when there is a sudden dive in returns, the portfolio is more efficient in reducing the losses than the ETF, reducing the spread with the index (Figure 29, points 1,2,3,4,5) and even performing better than the benchmark. On the other hand, when the contraction of returns is slower, the portfolio appears to be less reactive and it tends to perform worse (Figure 29, points. A possible explanation of this behavior could be that the model intercepts sudden dives efficiently, since the downside risk increases at a higher pace, but when it comes to reducing the downside risk at a slower pace, it has more difficulties at recognize it.

Figure 29: The model shows dynamicity when it comes to react to sudden dives

This behavior could not be totally a bad news: if the contraction of returns is slow, managers could have the opportunity to intervene and modify the portfolio composition to respond efficiently to the event. In the same sense, when the dive is sudden, there could not be enough time to intervene and, thus, this model could prove to be more efficient than others in limiting the drawdown. It can be interesting to make a comparison between the original ETF and the obtained portfolio in terms of tracking error computed with the classical formula, to check for substantial differences in the capability of tracking the benchmark efficiently:

$$TE = \sqrt{\frac{\sum_{i=1}^{n}(R_P - R_B)^2}{N-1}}$$

where:

R_P is the portfolio return;

R_B is the benchmark return;

N is the number of observations.

The tracking error obtained for the ETF and the portfolio with respect to the benchmark are:

Tracking error (XLG - Invesco S&P 500 Top 50 ETF)	**0.145646%**
Tracking error (PSO - portfolio)	**0.149807%**

There are no substantial differences between the two portfolios in replicating the benchmark, since the spread is only 0.004162%: this result is a positive news, because it means that the PSO portfolio is able to track the SPX - S&P 500 Top 50 as efficiently as the XLG - Invesco S&P 500 Top 50 ETF. The tracking error function used in the PSO has produced fairly the same results as the classical function presented above in terms of efficiency in replicating the benchmark performances, even if the assets' weights are (in some cases) quite different from the ETF. This means that the meta-heuristic can be applied to this tracking error minimization problem efficiently and the results it produces are quite as good as the ones of a traded ETF.

6. Conclusion

This thesis has analyzed a tracking error minimization problem of a portfolio with respect to the index SPX - S&P 500 Top 50, through the application of the PSO. In particular, the objective was to check if it is possible to minimize a tracking error function based on the downside risk rather and still keep tracking the index efficiently, starting from an existent ETF, the XLG - Invesco S&P 500 Top 50 ETF. The function has been developed starting from the Sortino ratio which, in fact, uses the downside risk rather than the overall volatility to assess the portfolio's performance: by considering the downside component of the volatility as risk, it is possible to provide investors a better information about the risk of losses because the upper side volatility is not seen as a real risk but rather, an opportunity for higher returns. The TE measure has been assembled by computing the Sortino ratio for both the ETF and the benchmark against a U.S. 1 Years Treasury Bill and then squaring the difference:

$$TE = \left(\frac{R_{ETF} - R_F}{\sigma_{ETF}^{-}} - \frac{R_{index} - R_F}{\sigma_{index}^{-}}\right)^2$$

The method that has been used is the Kennedy-Ebenhart-Shi PSO which, by making close to no assumption about the problem, allows to search larger spaces of solution with respect to traditional solving processes. Almost every in-sample portfolio generated by Matlab respected the constraints of 1) non negative weights (i.e no short-selling) and 2) total sum of weight equal to 100%, with the exclusion of some runs made with a low number of particles and/or iterations. By increasing both the population and the iterations, the fitness value reduced greatly and it reached the value of 0.020113. The resulting portfolio has then been compared with the original ETF by computing the out-of-sample returns for the XLG, the SPX and the portfolio. It has been checked the capability of the portfolio to track the benchmark as the XLG. The results have been satisfactory. The portfolio has shown to be able to track the SPX almost as efficiently as the XLG but there is more: as the graphs pictures, by using the downside volatility instead of the overall volatility, the portfolio has shown (in most cases) to be less susceptible to sudden drops in returns. By taking an overall comparison between the portfolio, the SPX and the XLG, it can be seen that its returns are lower than the XLG but (generally) in line with the SPX, whereas its losses are lower than the ETF but slightly higher than the benchmark. The portfolio has been compared with the XLG in terms of one of the most widely used tracking error measure:

$$TE = \sqrt{\frac{\sum_{i=1}^{n}(R_P - R_B)^2}{N-1}}.$$

The results are satisfactory, since the TE for the ETF is 0.145646% whereas the TE for the portfolio is 0.149807%: the difference is very low and it means that the portfolio tracks the benchmark almost as efficiently as the ETF. In light of the above, it can be said that the PSO is a meta-heuristic that can be applied to this tracking error minimization problem efficiently.

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Appendix

The Matlab code for the PSO that has been applied to the tracking error minimization problem.

```
clc
clear all
close all

format long


annual= 2/100; % desired annual return
pi = (1+annual)^(1/252) - 1; % daily annual return


after = 42; % number of days for the out-of-sample analysis


%% loading historical data

prices = load ('assets.csv');
rendetf = (prices(2:end,:) - prices(1:end-1,:))./prices(1:end-1,:);

benchmark = load ('index.csv');
rendbenchmark = (benchmark(2:end,:) - benchmark(1:end-1,:))./benchmark(1:end-1,:);


rendetf1 = rendetf(1:end-after,:);
rendetf2 = rendetf(end-after+1:end,:);
[n,numvaretf] = size(rendetf1);

rendbenchmark1 = rendbenchmark(1:end-after,:);
rendbenchamrk2 = rendbenchmark(end-after+1:end,:);
[m,numvarbenchmark] = size(rendbenchmark1);


% etf statistics

meanetf = mean(rendetf1);
semivaretf = zeros(numvaretf,numvaretf);
for i = 1:numvaretf
   for j = 1:numvaretf
      for k = 1:n
         semivaretf(i,j) = semivaretf(i,j) + min(0,rendetf1(k,i)-meanetf(i))*min(0,rendetf1(k,j)-meanetf(j));
      end
   end
end
```

```
semivaretf = semivaretf/(n-1);

% benchmark statistics

meanbenchmark = mean(rendbenchmark1);
semivarbenchmark = zeros(numvarbenchmark,numvarbenchmark);
for i = 1:numvarbenchmark
   for j = 1:numvarbenchmark
      for k = 1:m
         semivarbenchmark(i,j) = semivarbenchmark(i,j) + min(0,rendbenchmark1(k,i)-
meanbenchmark(i))*min(0,rendbenchmark1(k,j)-meanbenchmark(j));
      end
   end
end
semivarbenchmark = semivarbenchmark/(n-1);



%% Initializing PSO parameters

P = 5; % particles
niter = 1000; % iterations
c1 = 1.49618;
c2 = 1.49618;
w = 0.7298;
vmaxx = zeros(1,numvaretf); % service vector
TE(niter,1) = 00;
% other parameters
epsilon1 = 1.0e-004; % penalization parameter


%% Service vectors for PSO optimization (Tracking error, variance, mean, constraint)

T_Error = zeros(P,1);
var_port = zeros(P,1);
med_port = zeros(P,1);
vinc_1 = zeros(P,1); % budgetary constraint
vinc_2 = zeros(P,1); % profitability constraint
app_1 = zeros(P,numvaretf);
vinc_3 = zeros(P,1); % x >= 0

% random initialization of particles' positions, speed and objective function

x = rand(P,numvaretf);
y = rand(P,numvarbenchmark);
vx = rand(P,numvaretf);
vy = rand(P,numvarbenchmark);
f = ones(P,1)*1.0e+015;
rrr = ones(P,1)*1.0e+015;
```

```
% pb=pbest: the vector of particles' best position;

pbx = [x f];
pby = [y f];

% gbest and its relative objective function value

gx = zeros(1,numvaretf+1);
gy = zeros(1,numvarbenchmark+1);


tic;


for k = 1:niter

  % 1) computing the range for maximum speed

  for i = 1:numvaretf
     vmaxx(i) = abs(max(x(:,i))-min(x(:,i)));
  end
  for i = 1:numvarbenchmark
     vmaxy(i) = abs(max(y(:,i))-min(y(:,i)));
  end

  % 2) computing the objective function

  for p = 1:P
     for i = 1:numvaretf
        app_1(p,i) = max(0,-x(p,i));
     end

   % Computing the portfolio's tracking error

   T_Error(p) = ((((((x(p,:)*meanetf')- pi))/semivaretf(p))-(((y(p,:)*meanbenchmark')- pi)/semivarbenchmark)))^2;


   % Under the boundaries

     vinc_1(p) = abs(sum(x(p,:))-1); % total sum of weights = 1
     vinc_2(p) = max(0,-(x(p,:)*meanetf' - pi)); % profitability = pi
     vinc_3(p) = sum(app_1(p,:)); % x_i >= 0


  end

  % Computing the fitness function

 f = T_Error+(1/epsilon1)*(vinc_1 + vinc_2 + vinc_3);
```

```
% 3) Comparing the value of the objective function and the pbest

for p = 1:P
   if f(p) < pbx(p,numvaretf+1)
      pbx(p,numvaretf+1) = f(p);
      T_Err = T_Error(p);
      for i = 1:numvaretf
         pbx(p,i) = x(p,i);
      end
   end
end

% 3b) Memorize the value of the tracking error function

TE(k) = T_Err ;

% 4) Identification of the particle with the best position among the
% population

[minimo,posizione] = min(pbx(:,numvaretf+1));
gx(numvaretf+1) = minimo;

for i = 1:numvaretf
   gx(i) = pbx(posizione,i);
end

% 5) Update of speed and position

for p = 1:P
   for i = 1:numvaretf
      vx(p,i) = w*vx(p,i)+c1*rand*(pbx(p,i)- x(p,i))+c2*rand*(gx(i)- x(p,i));
      if vx(p,i) > vmaxx(i)
        vx(p,i) = vmaxx(i);
      end
      x(p,i) = x(p,i)+vx(p,i);
   end
end
converg(k,:) = gy(:,end);

  for p = 1:P
   for i = 1:numvarbenchmark
      vy(p,i) = w*vy(p,i)+c1*rand*(pby(p,i)- y(p,i))+c2*rand*(gy(i)- y(p,i));
      if vy(p,i) > vmaxy(i)
        vy(p,i) = vmaxy(i);
      end
      y(p,i) = y(p,i)+vy(p,i);
   end
end
converg(k,:) = gx(:,end);
```

```
% 6) Back to step 2

end

toc

figure
plot(converg);title('Fitness of the best portfolio'),xlabel('Iteration'),ylabel('Fitness function');
fitness = gx(end);
figure
plot(TE(:,1));title('Tracking error'),xlabel('Iteration'),ylabel('Tracking error');


%% virtual future

r_vf = rendetf2*gx(1:end-1)'; % mean in the virtual future
sv_vf = 0; % semivariance in the virtual future
for i = 1:after
   sv_vf = sv_vf + min(0,r_vf(i)-mean(r_vf)).^2;
end
sv_vf = sv_vf/(after-1);


%% OUTPUT

frt = '%12.6f\r';
fid1 = fopen('out.txt','w');
fprintf(fid1, '%s\r', 'Population:');
fprintf(fid1, frt, P);
fprintf(fid1, '%s\r', 'Number of iterations:');
fprintf(fid1, frt, niter);
fprintf(fid1, '%s\r', '* * * * * * * *   OUTPUT   * * * * * * * *');
fprintf(fid1, '%s\r', 'Portfolio:');
frt = '%12.6f %12.6f\r';
for l = 1:numvaretf
   fprintf(fid1, frt, [l gx(l)]);
end
frt = '%12.6f\r';
fprintf(fid1, '%s\r', 'Fitness function:');
fprintf(fid1, frt, fitness);
fprintf(fid1, '%s\r', 'Tracking error (portfolio):');
fprintf(fid1, frt, T_Err);
fprintf(fid1, '%s\r', 'Sum(x) = 1:');
fprintf(fid1, frt, sum(gx(1:end-1)));
fprintf(fid1, '%s\r', 'Performance (Wished/Obtained):');
fprintf(fid1, frt, [pi gx(1:end-1)*meanetf']);
fprintf(fid1, '%s\r', 'Non negativity constraints:');
fprintf(fid1, frt, sum(max(0,-gx(1:end-1)')));
fprintf(fid1, '%s\r', '* * * * * * *   VIRTUAL FUTURE   * * * * * *');
fprintf(fid1, '%s\r', 'Risk (Obtained/Virtual future):');
```

```
fprintf(fid1, frt, sv_vf);
fprintf(fid1, '%s\r', 'Performance (Wished/Obtained in sample/Virtual future):');
fprintf(fid1, frt, [pi gx(1:end-1)*meanetf' mean(r_vf)]);
fclose(fid1);
type('out.txt');
```

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