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Risk Prediction in Automobile Insurance

Steady State Premium Calculations

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Introduction

The new XXI century is a historical epoch of high technologies, global changes in the world, integration of the public gross product, as well as redistribution of the world's energy resources and monetary funds. Insurance is one of the essential financial segments in the front lines of the world economic system.

Such a significant attention is paid to insurance by law, since its role in public life is the accumulation and further redistribution of public funds, which form part of the world's gross product. Insurance is one of the largest financial sectors. It is some kind of guarantee for people who can be involved in accident, be injured or damage something.

But there is the 2-nd party of insurance relations. That is the insurance company itself. While people are trying to guarantee their future financial needs, insurance companies are trying to protect themselves from losses and also earn some profit. In order to play safe, insurance companies have to have a really good risk management system and, in particular, the realistic estimation of claim frequencies can help the company to avoid huge losses in the future.

With this thesis I tried to address the problem of claim frequencies prediction and premiums determination based on that. The thesis examines four very important aspects of insurance such as the general insurance theory, vehicle insurance, risk prediction and premium calculations in insurance.

The content of the thesis begins with general insurance theory, its types, definitions of life and non-life insurance, their classification and examples. The contract details of some types of insurance are also discussed followed by the statistics of life and non-life insurance and their development over the recent years in OECD countries in general and in Italy in particular.

The legislative regulation in financial sector is very important. In case of the absence of regulation, insured people and insurance companies can face horrific effects and risks. That is why the legislative framework such as Solvency 2 in European Union is discussed at the end of the chapter.

But the main purpose of the thesis is vehicle insurance and how we can predict claims made by policyholders. As a result we can determine the premiums according to these claims or update these premium systems. Each year more and more cars are consumed increasing the number of insured cars. That is why it is important to know how to predict risks arising because of these policyholders. Unlike life insurance, the prediction of risks in automobile insurance is very complicated. In life insurance insured person is being checked up and the premium is determined based on these characteristics that are easier to expose. But in case of vehicle insurance, after this *a priori* characteristic determination, some heterogeneity will be still left. That is why Poisson regression function of claim frequencies is being replaced with negative binomial regression function. The parameters of this regression are being estimated with maximum likelihood method.

It is not a secret that bonus-malus system is used for the premium calculations of vehicle insurance in a lot of countries. Based on this fact, two methods of premium calculations are being introduced in chapter 2 along with the calculations of relativities.

In the third chapter the main focus is Italian bonus-malus system and how it works with transition probabilities. Because of the private data policies adopted by financial institutions, the main difficulty is to find a data for the numerical illustration of the theory. That is why we use estimates by Denuit et al. (2007) for the calculation of relativities. We take with *a priori* and without *a priori* ratemaking estimates of mean claim frequency and heterogeneity factor, generate gamma distribution based on this heterogeneity factor and try to tackle it with *a posteriori* corrections.

Chapter 1

Classification of Insurance: Insurance Numbers in OECD Countries

Insurance as a socio-economic institution has an interesting history of development, originating in ancient times, at the stage of the emergence of civilization and the formation of the first signs of statehood. The emergence of insurance preceded the objective prerequisites, which prompted people to enter into certain socio-economic relations, later called insurance.

Such preconditions were the dangerous events that occurred in people's lives, and caused irreparable property and physical losses. In other words, these phenomena and events caused harm by destroying the material goods created by people, influenced their personal non-property benefits. It is possible to formulate the objective and subjective factors that became the cause of insurance. Those factors include the presence of hazards that can cause property or other material damage which can not be prevented (objective factor), people's fears for the preservation and prevention of material and personal non-material goods (subjective factor), the necessity, as well as the desire and aspiration of people to fight with the dangers that cause a certain fear and threat of harmful consequences (the combination of objective and subjective factors). Together, these factors have become prerequisites for the establishment of an insurance institution as a socio-economic way of combating the consequences of hazards. Moreover, the aforementioned set of factors is the historically developed model, which, in essence, has not changed and currently contributes to the insurance procedure. And while there is a subjective human factor, there will be an insurance. In other words, insurance will accompany humanity as much as there will be the problem of surviving in complex and dangerous situations.

1.1 Types of Insurance

Insurance as a financial industry includes a significant number of subjects of civil legal relations. This trend is objective, since insurance is the only financial and legal mechanism that fully protects the economic interests of citizens practically from all the negative circumstances that occur in their life path.

The reimbursement principle is the economic basis of insurance relations, due to the availability of certain funds that provide the implementation of the insurance functions.

The reasons to study insurance are different: from macro perspective such as the high cost of medical care, the tort system, costs of automobile insurance to studying it as just consumers of insurance. At the present stage of the development of society, insurance is an indicator of the economic and social well-being of a country.

The main principle of classification of insurance types is that we get groups and subgroups of insurance that have similarities in insurance application conditions, objects and liabilities. Insurance is classified according to different characteristics. According to the insurance application and the contract characteristics, there are *voluntary* and *compulsory* insurance schemes. The combination of these two helps countries to build a protective environment of economy, social relations, private and property interests of citizens.

Compulsory insurance is organized by government. It takes into account issues concerning the protection of society interests and defines the types of insurances for the most vulnerable events. The types of compulsory insurance and its application conditions are defined in laws of compulsory insurance. Insurance objects and people who must insure their risks and liabilities are also listed in laws. The law regulates relationships between parties and defines the size of the liability.

Voluntary insurance contracts are based solely on insureds will.

Insurance can be classified according to the insurance object difference characteristics. This is based on the differences of insurance objects and liabilities. The classification based on differences of insurance objects is used for the whole insurance system. The classification based on differences of liabilities is used only in property insurance.

Insurance can be *private* and *government*. Private insurance includes:

1. life and health insurance,
2. property and liability insurance.

Government insurance includes social insurance and other government insurance.

We will focus mainly on private insurance.

The insured event in case of *life insurance* includes the death of the insured. Upon occurrence of an insured event, the insurance company has to pay to the insured or third beneficiary party insurance benefits. The beneficiary can be any person. For example, when obtaining a mortgage, banks often insure the life of the borrower on their behalf.

Life insurance can be whole life insurance, whole life insurance with time-limited premiums, term life insurance, insurance with variable sum insured and variable premium, group life insurance. Whole life insurance assumes the payment of sum insured to the policyholder in case of his death. The policyholder pays constant premium to the insurance company per annum. In case of whole life insurance with time-limited premiums, the insured person pays premium for a limited time period or he stops paying if he dies before that age. Term life insurance assumes the payment of the sum insured if the person dies before the defined age. In case of insurance with variable sum insured and variable premium, the sum insured in case of the insureds death changes over time. The terms of group life insurance include the joint insurance of the group of people.

The main goal of *health insurance* is to allow the insured to pay high medical expenses in case of illness. Modern medicine can be extremely expensive even for a wealthy person, so this insurance is a very popular type of voluntary insurance.

Property insurance is aimed at protecting the property interests of the individual, therefore incidents include property damage or loss. This type of insurance also provides with liability coverage in case a person different from the owner is injured because of the accident.

Liability insurance is applied in many areas where mistakes of individuals or businesses can cause significant damage - in medicine, among carriers, etc. Liability insurance has a huge demand in the fields where the actions of insureds can lead to the injuries of other people.

Classification can also be done based on the duration of insurance contracts and the size of insurance risks. In European Union this classification includes *life insurance* and *non-life or general insurance*.

Life insurance is cumulative by its nature which means that the premiums are becoming higher with the age of insured person. The contracts are for a long term that is why they belong to the group of long-term private insurance.

Non-life insurance includes short-term private insurance groups such as casualty insurance, illness insurance, as well as property and liability insurance groups.

The latter classification provides more efficient environment for legislative regulation of long-term and short-term insurance contracts, and better tax policy that considers the specifications of different types of insurance.

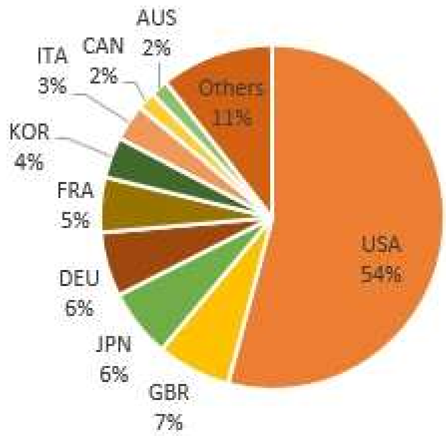


Figure 1.1.1: Gross insurance premium market shares in OECD countries, 2015.

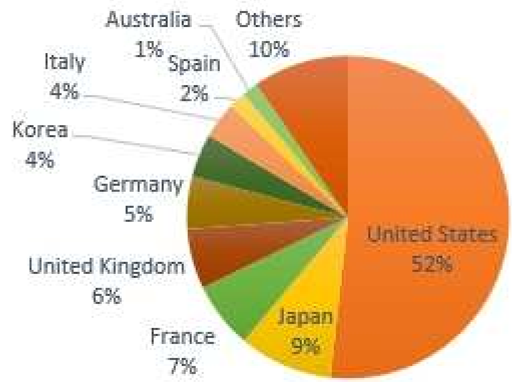


Figure 1.1.2: Gross insurance premium market shares in OECD countries, 2016.

Insurance market is continuing to develop and increases every year. According to the official webpage of data and metadata for OECD, the gross insurance premiums in OECD countries were almost 4840.21 bn USD in 2015. The highest market share belongs to USA with 54% followed by UK with 7% and Japan with 6% (illustrated in fig. 1.1.1).[1]

The same data is absent for some countries in 2016 (Canada, Netherlands, Slovak Republic, for both life and non-life insurance). Excluding those countries, the estimated gross insurance premium was almost 4061.9 bn USD. Again, the biggest share among OECD countries belongs to US (52‘%) followed by Japan (9%). France came up to the 3-rd position with 7% market share (shown in fig. 1.1.2).

Italy has always been a major contributor to the international insurance market. Over the last 10 years its gross insurance premiums were growing. They reach their peak in 2014 with 189.7 bn USD (fig. 1.1.3). It is quite interesting that in 2008-2010 this indicator grows, despite the economic crisis during those years. During 2015 and 2016

[1]Note: All the data on isurance premiums are taken from data.oecd.org.

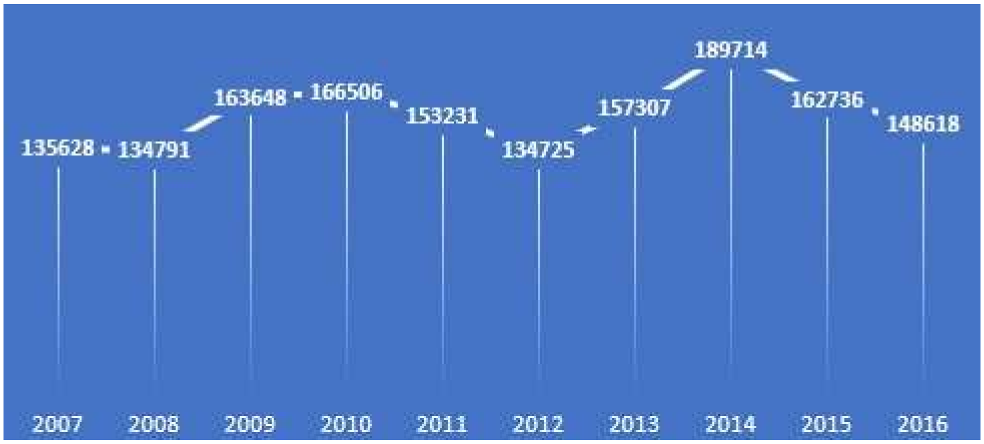


Figure 1.1.3: Gross insurance premiums in Italy, 2007-2016.

gross premiums decreased compared to 2014, but still Italy is one of the major insurance markets in the world.

1.2 Life Insurance

The simplest forms of personal mutual insurance existed in the frames of medieval craftsmen associations. The first mentions of the mutual insurance of craftsmen are discovered in England at X, in Germany at XI, and in Denmark at XII centuries. The statute of associations defined the organization of mutual relations between members, including the payment of membership fees and the procedure for the expenditure of money from the union. Money was spent on burials or on paying benefits to families of the deceased, as well as to sick and disabled people.

Life insurance, as a separate type of entrepreneurship, appeared in Europe in the XVII-XVIII centuries as a complement to marine insurance. Life insurance contracts were signed with ships and cargo insurance for 135 ships' commanders.

In case of life insurance, the insured risk is not the death itself, but its time. The covered risk has three aspects. Those aspects are the likelihood of premature death or death before the average life expectancy, the likelihood of death or disabling in a particular time framework, and the likelihood of having a long life, in which case regular incomes are required with the condition of unemployment. Based on the criteria for risk determination, the types of life insurance are distinguished.

Life insurance has different types of contracts such as pure endowment, life annuity, deferred annuity, term annuity, term deferred annuity, endowment assurance contracts.

In case of pure endowment contracts a person of age h will receive amount M if he lives until some age. If the person does not live up to that age, he will not receive any money and his premium will not be returned. In case of life annuity, a person receives some amount of money periodically until he reaches to some age. This is very similar to deferred annuity, only in case of life annuity the payments are made immediately after buying an annuity. Deferred annuity assumes that there is a delay of some time before the payments will be started. Term annuity is in the contrast with life annuity as the payments are made up to some age. In case of term deferred annuity payments are periodic not starting immediately after buying an annuity. They last up to some age. It is a mixture of term annuity and deferred annuity. The terms of endowment assurance contracts include the payment that is to transferred to the beneficiary of an insured person if the latter dies before the defined age and the sum insured is paid to the insured person if he lives until that age.[2]

Offering a wide range of insurance guarantees and investment services, life insurance allows people to solve a whole system of social and economic problems. These problems can be combined into social and financial groups. The solution of social problems allows to overcome the imperfection of the state, social insurance and security system. The solution of financial issues, on one hand, contributes to the increase of personal income, on the other hand provides necessary guarantees for implementation of a number of financial and credit operations.

Social objectives include family protection, the loss of income of deceased family member, accumulation of funds for the purpose of obtaining financial assistance when children become adults etc. The set of financial objectives includes cumulative objectives concerning investment incomes, protection of businesses, heritage protection, etc.

Thus, a citizen signs a life insurance contract either to support his/her family in the case of his/her premature death, or for investment purposes, to satisfy his/her financial needs in the future. That is why long-term life insurance allows to solve very important socio-economic problems and gets support from the state in this frame. In a market economy, it is one of the most important mechanisms for ensuring economic and social stability.

Nowadays, life insurance is one of the most advanced branches in the international insurance market. In 2015, the gross insurance premium of life insurance over OECD countries was 2001.59 bn USD which includes 260310.81 mln USD pension contracts. Fig. 1.2.1 illustrates the market shares of OECD countries in life insurance. Again, US holds the largest part of the pie, almost 41%.

In 2016, the gross insurance premiums increased up to 2240.22 bn USD. The market

[2]Khmaladze, E. V. (2013), pp. 141-151, 161-165.

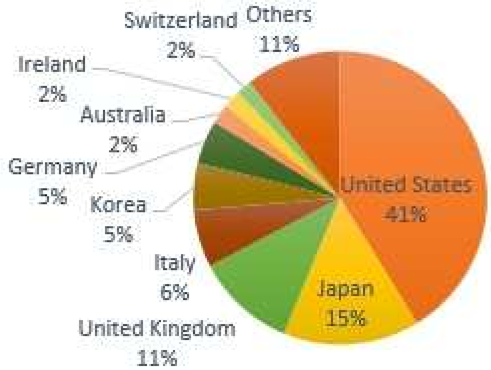


Figure 1.2.1: Life insurance market shares in OECD countries, gross insurance premium, 2015.

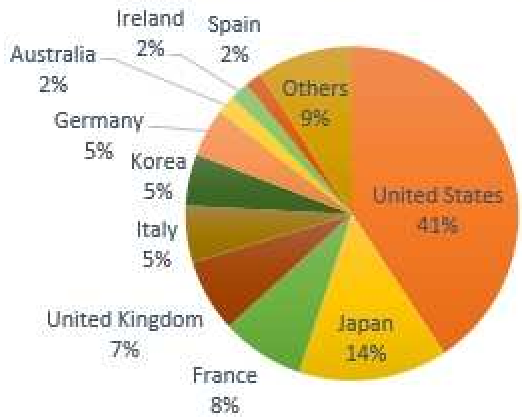


Figure 1.2.2: Life insurance market shares in OECD countries, gross insurance premium, 2016.

shares of this year are shown in fig. 1.2.2. US has 42% market shares in 2016, followed by Japan with 14% and France (8%).

According to International Monetary Fund report, life insurance market in Italy offers very traditional types of insurance. Those products include with-profit endowments, whole life and term life, and linked products (both unit and index linked products) whose risk is generally borne by policyholders. The products cannot guarantee a higher interest rate than the one published by IVASS (Istituto per la Vigilanza sulle Assicurazioni, 60% of the 10 years Italian bonds' return).[3]

The development of life insurance gross premiums are shown in fig. 1.2.3. The highest peak over these 10 years was in 2014 with 146.504 bn USD which was almost 77% of total gross insurance premiums for that year.

[3]IMF (2013), p.13.

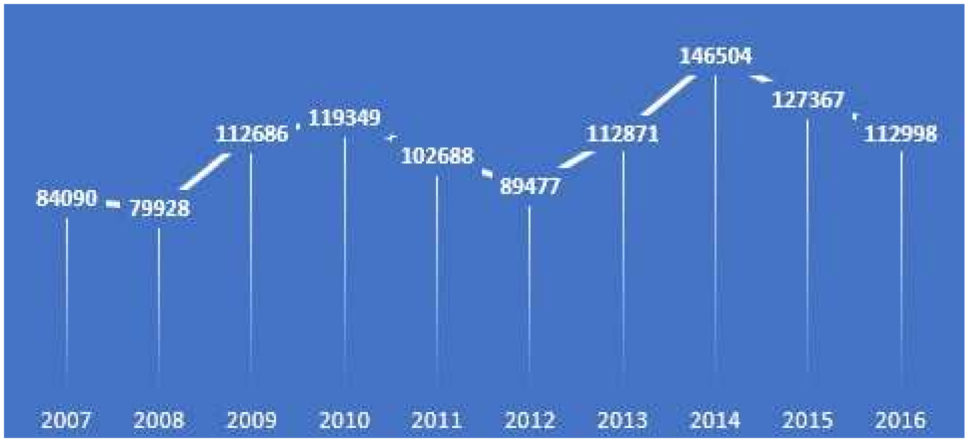


Figure 1.2.3: Life insurance gross premiums in Italy, 2007-2016.

1.3 Non-life Insurance

Non-life insurance includes the elements of private insurance that are not included in life insurance. It also includes property and liability insurances. Unlike life insurance, the classes of non-life insurance are not cumulative.

There is no universal classification of property insurance. However, it is common to consider the following main types of property insurance: property insurance against fire and other risks, cargo insurance, vehicle insurance, technical risk insurance, aviation risk insurance, agricultural risk insurance.

Liability insurance can be classified into liability insurance for the use of vehicles, employer's liability insurance toward employees, professional liability insurance, liability insurance for environmental pollution, liability insurance for manufacturers and sellers, Insurance of other types of civil liability.

1.3.1 Property Insurance

The purpose of property insurance is the compensation of damages. The assessment of insured property and the definition of the *sum insured* are highly important for this type of insurance. In case of voluntary insurance, sum insured is determined by the insurer and insured agreement, and in the case of compulsory insurance, it is the amount defined by law within which insurance object is insured. The *insurance value* is the real (market) value of the property at the moment of signing the contract of property insurance. It should be noted that the sum insured can not be higher than the property insurance value. Otherwise, we would have a situation that pushes insureds to take illegal actions to get insurance compensation which exceeds the fair value of the property. Property insurance contracts usually contain the extent of the insureds involvement in the loss, or the non-reimbursable amount which is known as "franchise". The non-reimbursable amount is set as a certain percentage of the sum insured or it can be in absolute numbers. On one hand, the non-reimbursable amount releases the insurer from the obligation to compensate for minor damages, since in most cases such expenses exceed the amount of damages. On the other hand, it forces the insured to be more careful and attentive to the insurance object. Additionally, the non-reimbursable amount significantly reduces insurance premiums.

Property insurance is the most common type of insurance. It may include buildings and constructions, unfinished construction facilities, equipment, raw materials, fuel, household property insurance, etc. Properties are not usually insured if they are in areas where earthquakes, floods, hurricanes and other natural disasters are common. Insurance risks include fire, lightning, explosions, natural disasters, including earthquakes, hurricanes, floods, water penetration from neighboring areas, etc.

Cargo Insurance is one of the oldest, most common and important types of insurance. In the past, merchants were trying to find ways to protect their goods from various dangers. One of the most effective ways in that period was insurance. At present, cargo insurance is one of the key elements of international trade and the insurance certificate is an inseparable part of any transaction. In international trade practice, there is a large number of agreements for the trades of goods, the most common of which are CIF, C&F, FOB, FAC.

In case of trading goods with conditions of CIF (cost, insurance and freight), the seller is obliged to deliver the cargo to the station, load the ship, pay for the carriage of the goods, insure the goods throughout the transportation until it reaches to the buyer, as well as send all the necessary transportation documents to the buyer. After the carriage of the goods and the formulation of transportation documents in the name of the buyer, the product becomes the buyer's property, so the risks associated with sudden loss of goods fall on the buyer.

In case of C&F (cost and freight) transaction, the seller is obliged to make a contract for the carriage of goods on his/her behalf, as well as to load the vehicle. In this case the buyer is responsible for the cargo insurance.

In case of FOB (free on board) the buyer is responsible for the carriage and the insurance of goods. The seller is obliged to load the board, notify the buyer and send the transportation documents to him/her.

The only difference of FAC (free alongside ship) from FOB is that the seller should load the ship and only after that it becomes the property of the buyer.

It should be noted that the aforementioned transactions are made for sea freight. However, freight transportation is often carried out by rail or automobile. Insurance is carried out in accordance with one of the three types of insurance coverage which include "liability for all risks", "limited liability" and "liability only for the loss of the whole load".

In case of "liability for all risks" (Institute Cargo Clauses "A"), the insurer compensates all losses and damages of the whole cargo or only a part of it, except for some non-indemnified cases. In "limited liability" (Institute Cargo Clauses" "B") the scope of compensated cases is narrower. "Only for the loss of the whole load" (Institute Cargo Clauses "C") covers the whole loss or the loss of a full part of the freight (container, carriage, etc.).

Vehicle insurance is the next type of property insurance. The classes of vehicle insurance include casco insurance, motor third party liability insurance and driver and passengers insurance from accidents. Casco insurance is vehicle insurance from the loss and damage that include physical damage, fire and theft. *Casco* is an international term, and in Spanish it means a car or ship skeleton. In insurance it means a vehicle skeleton insurance. Only the damage caused to the vehicle is insured. In case of insurance from accidents all seats as well as separate seats can be insured.

Technical risks insurance is the 3-rd type of property insurance. In the 1950s, with regard to scientific advancement, increase of production capacities, increase in the share of technological equipment in foreign trade circulation, as well as buildings' construction and installation works in foreign countries, there was a need for technical risks insurance. Today, construction insurance contracts are the largest in terms of sums insured. As a result of buildings' construction and installation works, the requirements for environmental protection have grown, leading to the need of liability insurance. The number of insurance companies involved in technical risk insurance has also grown significantly, especially in developed countries such as USA, Japan, England and so on.

Aviation risk insurance has emerged at the beginning of the 20-th century, during World War I, but started to develop in a rapid pace after the Second World War, along with the development of civil aviation, parallel to the increase in air transport and air traffic. The distinguishing feature of aviation insurance is its disastrous nature and the huge amount of damage. This means that only large insurance companies can engage in aviation insurance and reinsure most of the risks. Aviation risks insurance involves casco insurance, civil liability insurance, airline personnel insurance, aircraft maintenance and exploitation service insurance.

Agricultural risk insurance includes agricultural crops, crop and perennial seed insurance, livestock insurance, agricultural and farm property insurance and other agricultural insurance risks.

1.3.2 Liability Insurance

The essence of liability insurance is that insureds pay a premium to insurers and, thus, transfer the risks of damages of third parties from themselves to insurers. In case of insurance event, the latter is liable for damages to third parties if those damages caused material liability. Liability insurance is of double importance. First of all, it protects insureds from losses. It compensates for the damage to the victims as a result of their actions. Secondly, it guarantees compensation to victims for the damage caused to third parties. For this reason, liability insurance in many countries is compulsory, since indemnification to the third parties should be guaranteed.

The main fact of liability insurance for motor owners is that vehicles, first of all, automobiles, are a major source of danger to their surroundings. Statistical data shows that the damage caused by road accidents of motor vehicles is too high. In all countries of the world there are relevant legislative norms, which state that the damages to property, life or health of third parties is fully subject to compensation by the person who has caused the damage. Therefore, it is important to apply the most effective mechanism for the regulation of legal relations arising in the process of compensating the damage caused by vehicle accidents. The solution to this problem is primarily through the use of civil liability insurance of vehicle owners. The essence of this type of insurance is that the car owner insures his/her civil liability to third parties, in which the insurer pays the damage caused to the property, life or health of the injured party in case of an accident by the insured.

In accordance with the insurance agreements of motor third party liability, insurers compensate the damage caused to third parties within the insurance sum, which was caused by the insured as a result of the use of vehicle.

The bonus-malus scheme is used in many countries. Discounting and rebates systems are used in the calculation of each insurance policy depending on the number of accidents occurred by the insureds during the previous years. Discounts are applied for long periods of no claims, and on the contrary, in case of frequent claims, the insurance premium can increase. Each insured is assigned to specific level depending on which coefficients are established that increase or decrease the amount of the insurance premium. For example, the premium of the first no-claim year can be reduced by 10%, second year by 20%, third year by 30%, and so on. At the same time, the insurance premium will increase by 10%, 20% and 30% as a result of the first, second and third accidents. For the 1-st time of registering the vehicle, the insured is assigned to the initial level, according to which no discounts or rebates are applied. In case of no accidents in the first year, the insured is assigned to the first level that gets a discount, a second level in the second year, and a third level in the third year. Thus, for each year without accident another level is added and, in case of an accident, the level of the insured is usually reduced by one. This system works in Italy and is based on the following criteria: in case of registering the vehicle for the first time, the insured will be assigned to the default class that will determine conditions for his/her policy - class 14, in a range from class 1 (the most virtuous) to class 18 (the least virtuous drivers). Otherwise, the insurance company will consider the accidents that the policyholder have been involved in recent five years.[4]

Freight liability insurance is another type of liability insurance and is classified according to the type of vehicle and category of persons to which the insured is liable for damages. Based on the type of vehicle, the insurance is divided water transport carriers' liability insurance (shipowners), air transport carriers' liability insurance, motor transport carriers' liability insurance, rail transport carriers' liability insurance.

Professional liability insurance is a type of liability insurance that provides insurance coverage to people who are frustrated by providing professional services to their customers and inadvertently causing any malfunctions, mistakes, unforeseen defects in the performance of their professional responsibilities, although they were originally trusted by customers because of their professional skills. In other words, the essence of this insurance is that the insurer takes on the responsibility of the insured in respect of compensation for damage caused by insured errors in the course of professional activity. It should be noted that this form of insurance protects the property interests of the professionals who have higher education, possess special knowledge and have received special qualifications. Professional activities are characterized by the intellectual work that has been based on higher education and relevant years of work experience and professional ethics and other ethical principles.

The property interests of the insured as well as the obligation of the insured to compensate for the damage caused to the third parties and to the environment as a result of the environmental hazard are the object of the environmental damage or environmental liability insurance. It should be noted that the liability insurance of enterprises that danger environment should be based on the liability legislation.

Liability insurance of goods manufacturers and sellers is the next type of liability insurance. Consumers can be damaged due to poor quality products. The causes of the damage can be the product design, errors or omissions during production process, the use of poor quality raw materials, materials, semi-finished products, unclear, stringent

[4] According to https://europa.eu

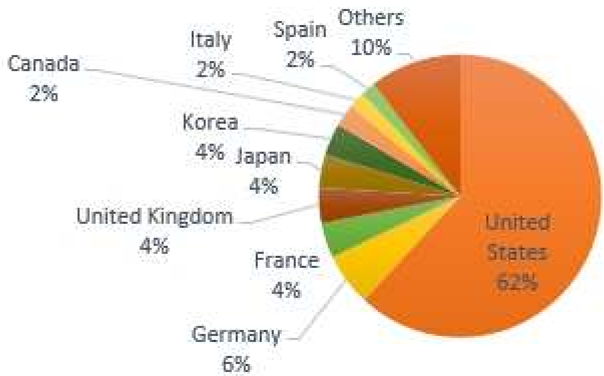


Figure 1.3.1: Non-life insurance market shares in OECD countries, gross insurance premium, 2015.

instructions on the use of the product and so on. In such cases, consumers have rights to make a claim for occurred damages to the manufacturers or distributors of the product. Thus, manufacturers and sellers have the risk of unforeseen costs associated with the need to compensate consumers for their products. An effective way of mitigating these risks is the insurance of liability for damages caused by the use of their products.

Health insurance is one of the forms of social protection that directly stems from the need to maintain the health of the population. It is associated with unforeseen accidents or diseases that damage the health. This insurance covers insurance risk that is related to the cost of medical care resulting from the accident.

Accident insurance is one of the most important types of private insurance that provides with the payment of insurance sum for the unexpected damage to health or in case of insureds death. Accident insurance covers adults, children, schoolchildren, road passengers, and individual groups of workers in more hazardous working conditions. Insurance can be carried out on the basis of common rules for almost all citizens and the rules that take into account the specific characteristics of the population (children, passengers, etc.). In case of accident insurance, the old and sick persons' insurance is subject to certain restrictions.

Non-life insurance market is also continuing to develop. In 2015, the gross insurance premiums in OECD countries were almost 1933.04 bn USD which is lower than life insurance gross premiums. 44% of these premiums were health and accident premiums. But only 12% of these money was for health insurance. The second largest premiums of 24% were for motor vehicle insurance. 14% were for fire and other property damage insurance. In 2016, the gross insurance premiums in OECD countries increased and were almost 2007.03 bn USD. The same groups of insurance were again majority and were respectively 45%, 23% and 14% in the gross insurance premiums.

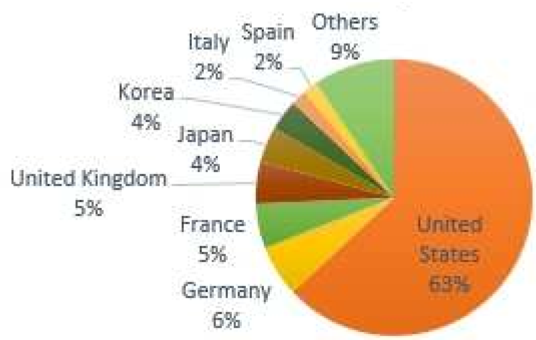


Figure 1.3.2: Non-life insurance market shares in OECD countries, gross insurance premium, 2016.

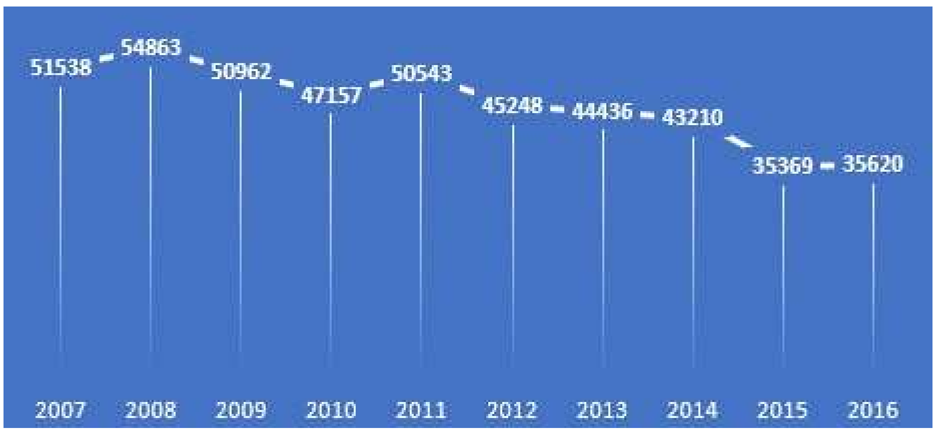


Figure 1.3.3: Non-life insurance gross premiums in Italy, 2007-2016.

Market shares of OECD countries in non-life insurance gross premiums for 2015 are illustrated in fig. 1.3.1. US has the main part in here with 62% followed by Germany with 6%. For 2016, the shares are illustrated in fig. 1.3.2. The picture is almost the same as in 2015.

In Italy, the major groups of insurances in this group have always been vehicle insurance, accident and health insurance and fire and other property damage insurance. But interestingly enough vehicle insurance always takes almost the half or more than half of the premium volumes (fig. 1.3.4).

1.4 The legislative framework

Insurance firms in EU rely on The Solvency II Directive 2009/138/EC as a prudential framework. It was introduced in 2009, but came into effect only in 2016. The aim of this Directive is to unify and harmonize the EU insurance regulation and establish an adequate solvency margin.

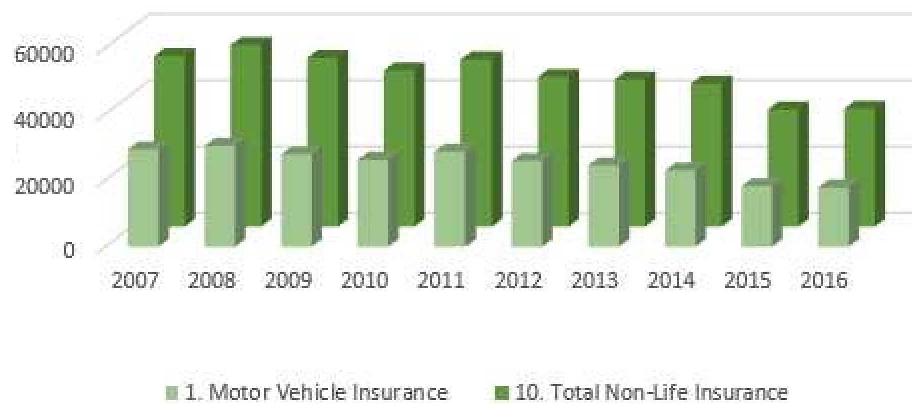


Figure 1.3.4: Vehicle insurance shares in total non-life insurance, 2007-2016.

The existing solvency margin requirements were established in 1973 under the First Non-Life Directive (73/239/EEC) and in 1979 under the First Life Directive (79/267/EEC).

International Conferences on Insurance Solvency were held in 1986 and 1988. In 1988 D'Arcy gave the definition of risk for actuaries: *that is the uncertainty, which is important for reinsurance and solvency purposes, but could be ignored in pricing and reserving.* Meanwhile financial economists divide risks into diversifiable and systematic.

Solvency is a term that describes the state of having more assets than liabilities. The purpose of Solvency I was to revise and update the current solvency system. Solvency II covers wider range. Solvency capital requirement is the main control factor in this case. Solvency I had realistic minimum capital requirements, but does not reflect the true risk faced by insurance companies. Besides, Solvency I cannot tackle the range of insurance risks.

According to ec.europa.eu, Solvency II promotes transparency, comparability and competitiveness in the sector. Solvency II is based on 3 pillar approach like Basel II. 1-st pillar includes quantitative requirements such as minimum capital and solvency capital requirements. Solvency capital is calculated based on either European Standard Formula or companies can use their own internal models. It guarantees that the insurance company has enough finances to withstand the hard times. The minimum capital requirement is lower requirement and a threshold below which interventions are required by national regulators.

The 2-nd pillar includes qualitative requirements such as adequate and transparent governance and regular risk management system.

3-rd pillar includes supervisory reporting and public disclosure. Firms disclose some information publicly and report more information to the supervisor.[5]

The detailed requirements for applying Solvency II are described in Regulation (EU) 2015/35. It includes asset and liability valuation, the ways of managing insurance companies, the level of capital for investment assets, etc.

Italian Association of Insurance Companies (ANIA) has initiated a project to support insurance undertakings in drafting their first Solvency and Financial Condition Report under Solvency II rules.[6]

[5]Sandström, A. (2010), p. 21.

[6]ANIA. *Italian Insurance (2016-2017),* p. 19.

Chapter 2

Individual Risk Prediction in Vehicle Insurance: Bonus-Malus Systems

We are living in the era where the various types of risks need to be managed. As a risk based civilization, the need of protection has become more pronounced. The result is the need of a financial security against possible losses. The development of the insurance business is related to the urgent need to protect the individuals and their assets against possible losses caused by specific events. The process of insurance consists of offering an equitable method of transferring the risk of an uncertain loss for some payment.

As we see insurance itself is a technique to hedge the risks, but risk management decisions will help us to understand deeply the functions of insurance. The category of risk, in fact, is of fundamental importance, since only if there is a risk for an insured person, it is possible to build one or another insurance model.

Automobile insurance is one of the most large-scale sectors of the insurance market. Insurance company takes the responsibility to reimburse the costs to the policyholder on the basis of the signed contract. These expenses are compensated in case of the damage and loss of the insured vehicle as a whole or just certain parts of it. These situations can occur as a result of fire, an accident, a natural disaster, etc. Auto insurance holds an increased interest because it is required to manage a large number of situations (both the number of insured vehicles and the accidents) with a wide set of risks.[1]

The relevance of the topic is expressed by the fact that the current development of the voluntary insurance market of vehicles is caused by the quantity of purchased vehicles.

[1]David, M., D.V. Jemna (2015), p. 152.

2.1 Vehicle Insurance

In this thesis we will consider any event associated with a random phenomenon that has more than one possible outcome as a risk which is *quantifiable* and *calculable*. The latter is the 1-st characteristic of risk in insurance and is a crucial point, because it means that insurance is radically different from lottery. For the event to be considered as a risk in insurance it is necessary to evaluate its probability. Insurance has a dual basis: the statistical table with the regularity of certain events and the calculus of probabilities applied to that statistic. This helps to evaluate the chances of that class of event actually occurring.

The 2-nd characteristic of risk in insurance is the *risk collectivity*. It means that the accident affects the whole population. Risk is calculable when it is spread over the population. Insurance covers groups.

The 3-rd characteristic of risk in insurance is that risk itself is a *capital*. What is insured is the capital against which loss the insurer offers a guarantee.

From these 3 characteristics of risk as "the actual value of a possible damage in a determined unit of time" we can define insurance as "the compensation of effects of chance through mutuality organized according to the statistics law".[2] Any risk is associated with an individual who is either bearing the risk or is risky to another individual. The ways to model risk can be parametric, nonparametric and semi-parametric. The parametric approach is based on estimating the specific set of parameters of a specific distribution. The nonparametric approach is based on the analysis of generic parameters, such as means, variances, covariances and quantiles. The semi-parametric approach is the mixture of parametric and nonparametric approaches.

Risk can be viewed from different perspectives such as the occurrence of the loss event (if the loss occurred or no), the frequency of the loss event (the quantity of the recorded losses in the accounting year which is, of course, a nonnegative integer), the time of the loss event occurrence (the interval of time measured in reference to the beginning of the contract) and the severity of the loss event (in currency units how much money is spent to cover the losses).

According to this classification, risk variables can be dichotomous qualitative, count, duration or continuous variable.

Dichotomous qualitative variables are variables which have only two levels (e.g., in case of gender, we categorize someone as either "male" or "female"). In case of risk management in automobile insurance, the dichotomous qualitative variable indicates if road accidents

[2] Ewald, F., (1991). pp. 201-205.

were reported to the insurance company, or if any claims on the automobile insurance were filed in the given year. There are two possible outcomes: either “yes” or “no”. To quantify these responses dummy variables 1 and 0 are assigned to each of them respectively. Depending on the variable choices, coverage choice of the policyholder can also be a dichotomous explanatory variable. For example, Hsu et al. (2016), take coverage of the policyholder as a dichotomous variable, where 1 indicates the high value of coverage and 0 indicates the low value of coverage.

As we know count data is a type of data, where observations take only non-negative integer values and these integer values arise from counting. Count variable is the individual piece of count data. Count variable gives the number of claims filed on the automobile insurance in the accounting year. Count variables are the basis of individual risks. Those risks need to be predicted when calculating the pure premium or updating the premium of policyholders. As a count variable we will consider the number of claims.

Duration variable can be the time between the issue of the insurance policy and the moment of the first claim incidence or the time from the incidence of a claim to the time of its report to the insurance company, the time from the claim report to the settlement.

Continuous variable is the amount paid by the insurer for the claim settlement or the total costs of claims filed in the accounting year.

A good measure for individual risk is the score. *Score* is a quantified measure of individual risk based on individual characteristics.[3]

Now, we will discuss each of risk variables separately.

2.2 Dichotomous Risk Variables

Van Hoorde et al. (2013), illustrate how simple dichotomous updating improve the validity of multinomial prediction models. Dichotomous qualitative models are widely studied using non-Bayesian techniques (Bermudez et al., (2008)).

Let denote the loss event by L. The following random variable is the indicator of that event:

$$\mathbb{1}_{\mathbb{L}} = \begin{cases} 1 & \text{if } L \text{ occurs} \\ 0 & \text{if } L \text{ does not occur.} \end{cases} \tag{2.2.1}$$

We define the random variable Y as follows:

[3]Gourieroux, C., J. Jasiak (2007), p. 2

$$Y = 1 - \mathbb{1}_{\mathbb{L}} = \begin{cases} 0 & \text{if } L \text{ occurs} \\ 1 & \text{if } L \text{ does not occur.} \end{cases} \tag{2.2.2}$$

Both $\mathbb{1}_{\mathbb{L}}$ and Y take only two values, so they are dichotomous qualitative variables. Y is a better setup for risk analysis, because it will allow us to define the score as a decreasing function of risk. To assess the probability of a loss event for fixed time horizon, we should find the probability that Y is equal to 0. Therefore, Y is the risk variable. The probability of Y is equal to 0 or 1 is the risk prediction.

So, how should we proceed with predicting individual risks? First of all in order to predict the risks we need to consider the characteristics of an individual such as the gender, age, marital status, car characteristics etc. One of the ways to do it is computing the conditional probability of a loss event. For any discrete variable the conditional (marginal) probabilities of outcomes form a conditional (marginal) probability function.

Suppose for the individual i we have the risk variable of the characteristic denoted by Y_i. Considering that X is the vector of individual covariates and Y is the dichotomous risk variable, we have the joint distribution of (Y, X). Y takes only two values 0 and 1 as mentioned above.

We will denote the conditional probability function of Y given $X = x$ as follows:

$$p_1(x) = P[Y = 1|X = x] \tag{2.2.3}$$

and

$$p_0(x) = 1 - p_1(x) = P[Y = 0|X = x]. \tag{2.2.4}$$

The marginal distribution of Y is denoted by

$$p_0 = P[Y = 0] \tag{2.2.5}$$

and

$$p_1 = P[Y = 1]. \tag{2.2.6}$$

We denote the conditional probability density functions of X given $Y = 0$ and $Y = 1$ respectively by $f_0(x)$ and $f_1(x)$.

If we have $f(x) = p_1 f_1(x) + p_2 f_2(x)$, then for the marginal and conditional probabilities we will have the following relationships:

$$p_0 = \int p_0(x) f(x) dx \tag{2.2.7}$$

and

$$p_1 = \int p_1(x) f(x) dx. \tag{2.2.8}$$

p_1 and p_0 are the initial beliefs about the probabilities of not occurrence of the loss event and the occurrence of it respectively (*a priori* beliefs). When new information comes out, we update our beliefs about those probabilities according to the Bayes rule and we get our *a posteriori* beliefs which are expressed as follows:

$$p_1(x) = p_1 \frac{f_1(x)}{f(x)} \tag{2.2.9}$$

and

$$p_0(x) = p_0 \frac{f_0(x)}{f(x)}. \tag{2.2.10}$$

At the end of the accounting period we will know if the loss event actually occurred or not. This will lead to the final step of updating our beliefs:

$$P[Y = 1 | Y, X] = Y. \tag{2.2.11}$$

Another way of risk prediction is calculating the expected value of Y.

The conditional (marginal) expectation of Y is equal to the conditional (marginal) probability that the individual will be a "good" risk:

$$E[Y] = p_1 \tag{2.2.12}$$

and

$$E[Y | X = x] = p_1(x). \tag{2.2.13}$$

We can also write that $E[Y] = E[E(Y|X)]$.[4]

In order to assess our prediction accuracy we have to look at the variance of random variable Y. The marginal (conditional) variance provides a measure of dispersion of a random variable about the marginal (conditional) mean:

$$Var[Y] = p_1(1 - p_1) \tag{2.2.14}$$

[4] Gourieroux, C., J. Jasiak (2007),pp. 8-10.

and

$$Var[Y|X = x] = p_1(x)(1 - p_1(x)). \tag{2.2.15}$$

On average, the conditional expectation provides more accurate prediction of Y than the marginal expectation because $Var[Y] \geq E[Var(Y|X)]$ (from the law of total variance).

In a very large number of repeated experiments, on average, the conditional expectation provides a better prediction of risk than the marginal expectation.

We will try to find the relationship between dichotomous qualitative models and count data models.

2.3 Premium Calculation in Insurance

Assume we have n individuals who set up a "pool" to pay money to the member who had a loss because of the risk event. Let $\tau^{(j)}$ be the random loss event. We define the random loss as follows

$$X^{(j)} = \begin{cases} x^{(j)} & \text{if } \tau^{(j)} \\ 1 & \text{if } \bar{\tau}^{(j)} \qquad j = 1, 2, ...n. \end{cases} \tag{2.3.1}$$

$\tau^{(j)}$ and $X^{(j)}$ are independent. $p^{(j)}$ is the probability of suffering a loss. We also assume that $X^{(j)}, \quad j = 1, 2, ...n$, are independent and identically distributed.

Let

$$X^{(P)} = \sum_{j=1}^{n} X^{(j)} \tag{2.3.2}$$

$X^{(P)}$ is called outgo of the pool. From the independence hypothesis we have

$$E[X^{(P)}] = \sum_{j=1}^{n} E[X^{(j)}] = \sum_{j=1}^{n} x^{(j)} p^{(j)} \tag{2.3.3}$$

and

$$Var[X^{(P)}] = \sum_{j=1}^{n} Var[X^{(j)}] = \sum_{j=1}^{n} (x^{(j)})^2 p^{(j)} (p^{(j)}). \tag{2.3.4}$$

If we assume that we have homogeneous probabilities and amounts of loss, then

$$x^{(j)} = \bar{x}, p^{(j)} = \bar{p}. \tag{2.3.5}$$

From the *iid* assumption it follows that

$$E[X^{(j)}] = \bar{x}\bar{p}, \quad E[X^{(P)}] = n\bar{x}\bar{p}$$

and

$$Var[X^{(j)}] = \bar{x}^2\bar{p}(1-\bar{p}) \quad Var[X^{(P)}] = n\bar{x}^2\bar{p}.$$

The calculation of pure premiums is very important for the insurance companies as it somehow illustrates the health of the company. For this we consider short-term policies and we can ignore the index of time. We denote by $\bar{X}$ the benefit if the event τ that causes financial loss occurs with probability p

$$\bar{X} = \begin{cases} \bar{x} & \text{if } \tau \\ 1 & \text{if } \bar{\tau}. \end{cases} \tag{2.3.6}$$

The expected benefit will be

$$E(\bar{X}) = \bar{x}\bar{p}, \tag{2.3.7}$$

which is equal to the expected premium

$$\bar{P} = \bar{x}\bar{p}.$$

It is obvious that if there is an accident, the insurance company will suffer a loss equal to the difference of $\bar{P}$ and $\bar{X}$. But if there is no accident, insurance company will gain amount equal to $\bar{P}$.

The gains of the company can be expressed by M

$$M = \bar{P} - \bar{X}. \tag{2.3.8}$$

The expected gains will be

$$E(M) = \bar{P} - E(\bar{X}) = \bar{x}\bar{p} - \bar{x}\bar{p} = 0. \tag{2.3.9}$$

$\bar{P}$ that is calculated in this way is called equivalence premium.

In case we have a pool with n risks and k number of claims, the equilibrium will be achieved if

$$n\bar{P} = k\bar{x}. \tag{2.3.10}$$

From where it follows that

$$n\bar{x}\bar{p} = k\bar{x} \tag{2.3.11}$$

or

$$n\bar{p} = k. \tag{2.3.12}$$

(2.3.12) means that in equilibrium the number of claims is equal to the expected number of claims.

When $\bar{p}$ and n are both very small, $n\bar{p}$ can be a non-integer, so the prefect balance will never be achieved. If $k > n\bar{p}$, insurance company will suffer losses. For that case, the insurer should apply a higher level of premium (P_h):

$$P_h = \bar{P} + h, \tag{2.3.13}$$

where h illustrates the *safety loading* as it shows how the insurance company is playing safe in order to avoid possible big losses.

(2.3.8) and (2.3.9) will respectively become

$$M = P_h - \bar{X}, \tag{2.3.14}$$

$$E(M) = P_h - E(\bar{X}) = h. \tag{2.3.15}$$

In the latter case P_h is charged directly from the contract.

If $k = n\bar{p}$, the total safety loading nm illustrates the profit of managing the pool or the profit margin. Safety loading can be included in the calculations of premium in explicit and implicit ways. They both show how the safety loading is connected to the quantitative features of the contract.

$$P_h = (1+\alpha)E(\bar{X}) = E(\bar{X}) + \alpha E(\bar{X}) = E(\bar{X}) + h = \bar{x}\bar{p} + \alpha\bar{x}\bar{p} = (1+\alpha)\bar{x}\bar{p}. \tag{2.3.16}$$

In this way when we choose $\alpha > 0$ to calculate $\alpha\bar{P}$, we adopt the explicit approach. The example of explicit approach is the introduction of the variance of $\bar{X}$ in the calculation of P_h.[5] Otherwise, the insurance company can "increase" the probability of the loss

$$p' = (1+\alpha)\bar{p}. \tag{2.3.17}$$

[5]Olivieri, A., E. Pitacco (2011). pp. 58-60.

Then

$$P_h = \bar{x}p'. \tag{2.3.18}$$

This is the implicit approach of the premium calculation. In this case

$$h = P_h - P = \bar{x}(p' - \bar{p}). \tag{2.3.19}$$

2.4 Poisson and Negative Binomial Regression Models

Poisson regression model is one of the ways to predict the number of claims. The probability function of number of the count variable Y for the given time interval will help us to predict the number of the insurance claims.

This problem can be solved through individual and collective ways. In case of individual approach, we take the portfolio that consists of individual policies. Each of them has a certain probability of claim. The total number of claims is the sum of the contributions from the individual policies. The probabilities are derived by means of the addition theorem of probability calculus from the primary probabilities. In case of collective approach-this was adopted by F. Lundberg. Portfolio of policies is considered as a whole. A "process" is considered where only times and number of claims are recorded. This process takes Poisson form.

The main disadvantage of Poisson distribution is that while applying it to the real data, we get poor fit and the mean-variance equality is not often satisfied. In order to have additional degree of freedom, we introduce an unobservable heterogeneity factor which is a random variable. And as a random variable it has a *gamma distribution*. Heterogeneity factors include the effects of all the relevant variables that are not in the explanatory variables' list. The Poisson regression model with gamma heterogeneity factor is called negative binomial regression model.

Based on these approach, Gourieroux, C. and J. Jasiak (2007) proposed another way of premium calculations that will be introduced in this section.

2.4.1 Poisson Regression

Let us consider the claim process arrivals. $Y(t)$ is the number of events in the time interval $(0, t]$. $Y(0) = 0$. The collection of random variables $[Y(t) : t \geq 0]$ is a stochastic process of number of events over time. For $s > 0$ the increment $Y(t+s) - Y(t)$ is the number of events in time interval $(t, t+s]$.

We assume that the claim number process satisfies the independence of increments, their stationarity and the exclusion of multiple events conditions.

Independence of increments means that events that occur in disjoint time intervals are independent. For $n = 2, 3, ...$ the numbers of events in k disjoint intervals given by increments $Y(t) : t \geq 0$ are independent.

This means that, for example, a car accident will not give a rise to another accident. This is not always true because the accident can spread from one risk to another. This condition is satisfied when we define the combination of some events as one risk unit (e.g. the ship and its cargo are considered one risk unit).

Stationarity of increments means that the number of events in a specific time interval depends only on the length of that interval. For all $m > 0$ and $t \geq 0$, the distribution of increments $Y(t+m) - Y(t)$ depends only on m and not t.

Usually in insurance the intensity of claims is constant. This means that the portfolio of policies is so large that the exit of individual policies and the entry of new policies can't affect the collective flow of the events to a significant degree.

This, however, is not applied to all the situations. For example, if there is a seasonal variation in claim intensities or the risk intensities are changing, we can divide the time interval into subintervals in a way that corresponding subprocesses approximately have constant intensities. Thus, they are Poisson processes. As we know the sum of Poisson variables is a Poisson variable. So the total number of claims will have a Poisson distribution. So we can ignore the seasonal variations in claims.

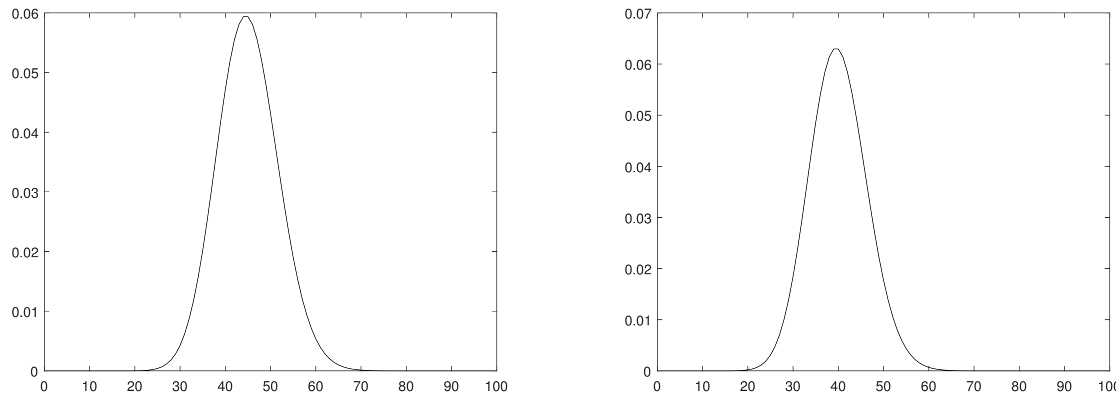
Exclusion of multiple events condition does not always hold. For example, there might be an accident where two cars are involved. This issue can be solved by regarding the case of two vehicles as one claim.[6]

As we see the assumptions of the basic Poisson regression model are not always satisfied. For this reason, different extensions to the Poisson regression model have been introduced.

There is also another problem with Poisson distribution. We can not apply it in case of heterogeneous insureds who have idiosyncratic risks related to their individual policies. We solve this problem by taking this heterogeneity as a distinct unobserved random variable. When the distribution of heterogeneity is unspecified, we have a semi-parametric model. And if the heterogeneity has gamma distribution, we have a negative binomial model with observable covariates.

The negative binomial model is used in automobile insurance in order to update the

[6] Beard, R.E., T. Pentikainen and E. Pesonen (1984), pp. 19-22.



(a) $\lambda = 45$ (b) $\lambda = 40$

Figure 2.4.1: Poisson probability mass function for different λ.

policy premiums (e.g. bonus-malus scheme that will be discussed later).

For now we will discuss Poisson distribution of number of claims.

The Poisson family of distributions has one parameter denoted by λ. λ is the expected number of claims in per unit time or the mean of Poisson distribution. If Y has Poisson distribution ($Y \sim Poi(\lambda)$), The probability mass function then is given by

$$P(Y = y) = e^{-\lambda} \frac{\lambda^y}{y!}, \quad y = 0, 1, 2.... \tag{2.4.1}$$

The probability generating function of Poisson distribution is $G_Y(n) = E(n^Y)$, where

$$G_Y(n) = \exp\left(\lambda(n-1)\right). \tag{2.4.2}$$

The moment generating function is

$$M_Y(t) = \exp(\lambda(e^t - 1)). \tag{2.4.3}$$

The mean and variance of Poisson distribution are equal to each other:

$$E[Y] = Var[Y] = \lambda. \tag{2.4.4}$$

The skewness of Poisson distribution is $\frac{1}{\sqrt{\lambda}}$. As λ intensity decreases, the coefficient of skewness increases and the distribution becomes less symmetrical (figure 2.4.1). λ depends on the values of observable covariates for each individual i:

$$\lambda_i = \exp(z_i' \beta), \tag{2.4.5}$$

where z_i is the vector of different transformations of x_i. The latter is a characteristic of individual i. β is the vector of unknown parameters. The intensity parameter λ is always positive.

It is obvious that

$$Y_i|x_i \sim Poi(\exp(z_i'\beta)). \tag{2.4.6}$$

$Y_1, Y_2, ..., Y_n$ are risk count variables. They are independent and conditional on covariates. As the conditional distribution on x_i is also Poisson, we will have the following:

$$E(Y_i|x_i) = Var(Y_i|x_i) = \exp(z_i'\beta). \tag{2.4.7}$$

$S_i = z_i'\beta$ is the score that are used to rate the individuals in the sample with respect to the risk. The high value of the score shows bad risk as the higher the score, the higher the expected number of claims and its variance.

2.4.2 Maximum Likelihood Estimation of Poisson Regression

The log-likelihood function of Poisson regression model is the following:

$$L(\beta) = \log(\prod_{i=1}^{n} \exp(-\exp(z_i'\beta)\frac{\exp(y_i z_i'\beta)}{y_i!})) = \sum_{i=1}^{n}(y_i z_i'\beta - \exp(z_i'\beta) - \log(y_i!)). \tag{2.4.8}$$

First order conditions (FOCs) of the maximum likelihood function are

$$\frac{\partial L(\hat{\beta}_n)}{\partial \beta} = 0, \tag{2.4.9}$$

$$\sum_{i=1}^{n}(y_i - exp(z_i'\hat{\beta}_n))z_i = 0. \tag{2.4.10}$$

The residuals of the model for individual i are

$$\hat{u}_i = y_i - exp(z_i'\hat{\beta}_n). \tag{2.4.11}$$

When the covariates include the constant, the sum of residuals is 0.

Second order conditions (SOCs) of the maximum likelihood function are

$$\frac{\partial^2 L(\beta)}{\partial \beta \partial \beta'} = -\sum_{i=1}^{n} z_i z_i' \exp(z_i'\beta_n). \tag{2.4.12}$$

The MLE $\hat{\beta}_n$ is consistent and asymptotically normal. The estimated variance matrix of $\hat{\beta}_n$ is

$$\hat{Var}(\hat{\beta}_n) = \left(- \frac{\partial^2 L(\hat{\beta}_n)}{\partial\beta\partial\beta'} \right)^{-1} = \left(\sum_{i=1}^{n} z_i z_i' \exp(z_i'\hat{\beta}_n) \right)^{-1}. \tag{2.4.13}$$

Now we turn to the discover the connection of dichotomous risk and count variables. We can define the number of claims as follows:

$$Y^* = \begin{cases} 1 & \text{if } Y > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.4.14}$$

As Y^* is the occurrence of at least one claim, $Y^* = 0$ is for good risk individuals, and $Y^* = 1$ for bad risk individuals. The conditional distribution of good risk given x_i is

$$P(Y_i^* = 0|x_i) = \exp(-\exp(z_i'\beta) \tag{2.4.15}$$

and the conditional distribution of bad risk given x_i is

$$P(Y_i^* = 1|x_i) = 1 - \exp(-\exp(z_i'\beta)). \tag{2.4.16}$$

The latter is the Gompit regression model for dichotomous risk.

In order to find MLE, the function to be maximized is

$$\max_{\beta} \sum_{i=1}^{n} (y_i^* \log(1 - \exp(-\exp(z_i'\beta))) - (1 - y_i^*)\exp(z_i'\beta)). \tag{2.4.17}$$

The MLE found in this case is less accurate than in case of count data variables, because we have less qualitative information.

2.4.3 Negative Binomial Distribution

The negative binomial distribution has 2 parameters denoted by m and p, where $m > 0$ and $0 < p < 1$. If Y is a random variable that has negative binomial distribution ($Y \sim nb(m,p)$), then the probability mass function of Y is given as

$$P(Y = y) = \frac{\Gamma(m+y)}{\Gamma(y+1)\Gamma(m)} = \binom{k+y-1}{y} p^m q^y, \tag{2.4.18}$$

where $y = 0, 1, 2, ...$ and $q = 1 - p$.

Γ is the gamma function:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}, \tag{2.4.19}$$

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \tag{2.4.20}$$

$$\Gamma(n) = (n-1)! \quad \text{for} \quad n = 1, 2, 3, ... \tag{2.4.21}$$

When m is an integer, calculation of the probability mass function is straightforward as it can be expressed in terms of factorials.[7]

The possible outcomes can have two outcomes, "success" and "failure". When m is an integer, the distribution models the number of failures before the m-th "success" occurs in a series of independent Bernoulli trials.[8] And

$$P(success) = p. \tag{2.4.22}$$

We can now express the probability mass function as

$$P(Y = y) = \frac{(k+y-1)!}{y!(k-1)!}p^k q^y, \quad y = 0, 1, 2, ... \tag{2.4.23}$$

There is an alternative way to calculate the probability mass function, regardless if m is an integer or not, is

$$P(Y = y+1) = \frac{m+y}{y+1}qP(Y = y), \quad y = 0, 1, 2, ... \tag{2.4.24}$$

The probability generating function exists for $|n| < 1$ and is equal to

$$G_Y(n) = \left(\frac{p}{1-qn}\right)^m. \tag{2.4.25}$$

When $t < -\log q$, there exists a finite moment generating function:

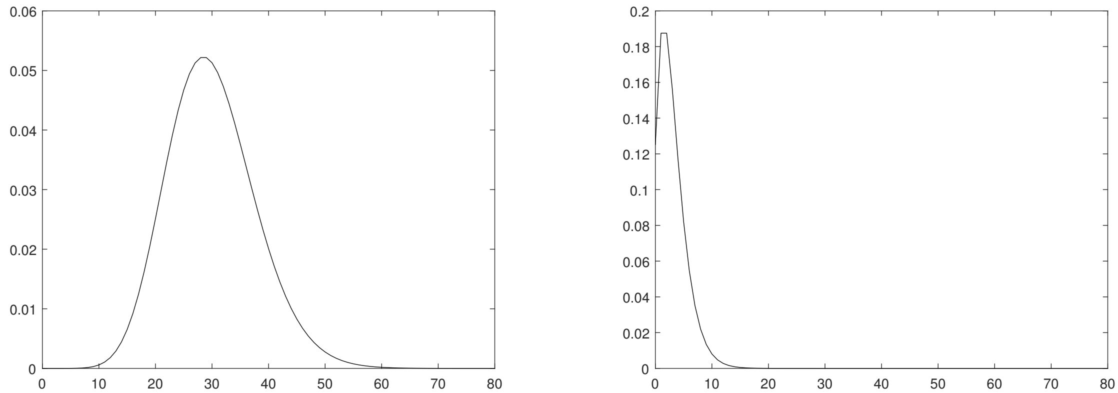
$$M_Y(t) = \left(\frac{p}{1-qe^t}\right)^m. \tag{2.4.26}$$

Respectively, the mean and the variance of the distribution are:

$$E(Y) = \frac{mq}{p}, \tag{2.4.27}$$

$$Var(Y) = \frac{mq}{p^2}. \tag{2.4.28}$$

[7]Dickson, D.C.M. (2016), pp. 3-4.
[8]Gray, R.J., S.M. Pitts, (2012), pp. 16-17.



(a) $m = 30, p = 1/2$ (b) $m = 3, p = 1/2$

Figure 2.4.2: Negative binomial probability mass function for different λ.

$Var(Y) > E(Y)$ as $p < 1$. This means that the tails of negative binomial distribution are heavier than the tails of Poisson distribution in case of equal means. This is why the negative binomial distribution fits better to the real data than the Poisson distribution. If we decrease parameter m and keep p fixed, the distribution will become less symmetrical (figure 2.4.2).

This model is used in automobile insurance for bonus-malus scheme. If we take into account the heterogeneity, we will have the following form for the individual intensity:

$$\lambda_i = \exp\left(z_i'\beta + \epsilon_i\right) = \mu_i exp(z_i'\beta). \tag{2.4.29}$$

μ_i or ϵ_i is the heterogeneity factor.

In order to model the negative binomial regression, we first assume that

$$Y_i|x_i, \mu_i \sim Poi(\mu_i exp(z_i'\beta)). \tag{2.4.30}$$

Then we specify

$$\mu_i|x_i \sim \gamma(a, a) \tag{2.4.31}$$

and

$$f(\mu) = a^a \mu^{a-1} \frac{\exp(-a\mu)}{\Gamma(a)}, \tag{2.4.32}$$

where a is a scalar and measure the amount of heterogeneity and

$$\Gamma(a) = \int_0^\infty \exp -\mu\mu^{a-1} d\mu. \tag{2.4.33}$$

Given that the explanatory variables include the constant term, we will introduce the new not too restrictive constraint $E\mu = 1$. Then we will have

$$Var(\mu) = 1/a. \tag{2.4.34}$$

For the conditional mean by the law of total expectations we have

$$E(Y_i|x_i) = E(E(Y_i|x_i, \mu_i)|x_i) = E(\mu_i exp(z_i'\beta)) = exp(z_i'\beta). \tag{2.4.35}$$

For the conditional variance by the variance decomposition formula we have

$$\begin{aligned} Var(Y_i|x_i) = E(Var(Y_i|x_i, \mu_i)|x_i) + Var(E(Y_i|x_i, \mu_i)) \\ = E(\mu_i exp(z_i'\beta)) + Var(\mu_i exp(z_i'\beta)) = exp(z_i'\beta) + 1/ae^{2z_i'\beta}. \end{aligned} \tag{2.4.36}$$

If a increases we have less heterogeneity in the population, and $Var(Y_i|x_i)/E(Y_i|x_i)$ ratio, which is called overdispersion, decreases. We will have $Var(Y_i|x_i)/E(Y_i|x_i) = 1$ (Poisson regression model) if $a \to +\infty$ so the distribution degenerates to the point $\mu = 1$.

The conditional density of negative binomial distribution is

$$\begin{aligned} f(y_i|x_i) &= \int_0^\infty \exp\left(-\mu \exp\left(-z_i'\beta\right)\right)\mu^{y_i} \frac{\exp\left(y_i z_i'\beta\right)}{y_i!} a^a \mu^{a-1} \frac{\exp -a\mu}{\Gamma(a)} d\mu \\ &= \frac{a^a \exp\left(y_i z_i'\beta\right)}{y_i!\Gamma(a)} \int_0^\infty \exp\left(-\mu\mu^{y_i+a-1}(a + exp(-z_i'\beta))\right) d\mu \\ &= \frac{\Gamma(y_i + a)}{\Gamma(y_i+1)\Gamma(a)} \frac{((1/a)\exp\left(z_i'\beta\right))^{y_i}}{(1 + (1/a)\exp\left(z_i'\beta\right))^{y_i+a}}. \end{aligned} \tag{2.4.37}$$

In order to find the MLE of this model, we just need to derive the log-likelihood of this function with respect to a and β (find FOCs).

2.5 The Bonus-Malus Scheme

Third party liability vehicle insurance is compulsory within EU countries. Insurance in one EU country is valid in another EU country although the calculation of motor insurance premiums may differ from one country to another. Some EU countries adopted premium assessment system which takes into account the claim histories or so called bonus-malus scheme.

There are different designs for BMS. One example includes -1/top scale system that has 6 levels from 0 to 5 and the starting level is 5. In case of claims-free year, the policyholder is rewarded by 1 bonus class. In case of a claim, all discounts are vanished and the policyholder transfers to 5-th level. Another examples include -1/+2 and -1/+3 systems. The discounts for claim-free year work as in the case of -1/top scale system. If n_t claims are reported during the year t, then the policyholder goes up by respectively $2n_t$ and $3n_t$ levels.

One way of calculating bonus-malus premiums is proposed by Gourieroux, C. and J. Jasiak (2007). It is based on computing the relative increment of premium $\frac{P_{i,T+1}}{P_{i,T}}$ at the end of period.

$$\frac{P_{i,T+1}}{P_{i,T}} = \frac{C_{i,T+2} exp(z_{i,T+2}{}'\beta)\hat{\mu}_{i,T+1}}{C_{i,T+1} exp(z_{i,T+1}{}'\beta)\hat{\mu}_{i,T}}. \tag{2.5.1}$$

$\hat{\mu}_{i,T}$ is the expected heterogeneity factor at the end of period, $\frac{C_{i,T+2}}{C_{i,T+1}}$ is the inflation rate. In bonus case, the formula means that if the observed number of accidents $Y_{i,T+1}$ is lower than the expected one $exp(z_{i,T+1}{}'\beta)$, the increment of heterogeneity factors will be lower than 1.

This is not the only way of updating bonus-malus premiums. The ratemaking or premium determination process is very important to insurance companies. For this purpose the observable characteristics of insureds should be taken into account. This will lead to the initial segmentation of risk classes and *a priori* premiums determination. But the problem here is that unlike life insurance, in vehicle insurance there will still be some residual heterogeneity left among those risk classes. So, insurance companies need to determine *a posteriori* premiums. This is the basis of bonus-malus scheme where the criteria used for assigning an individual to an initial risk class is the number of claims. Discounts are applied for good drivers with no accidents and claims and bad drivers who reported accidents are punished. In other words, bonus-malus is *a posteriori* ratemaking scheme that comes to complement *a priori* rates. Bonus-malus (BM) ratemaking consists of 3 very important parts which include the levels, relativities and transition rules of the system.

The levels are the classes to which policyholders are assigned. As I already stated in chapter 1, in Italy there are 18 BM levels: New drivers enter into this system with a pre-specified level. In our notation let l be the level of the BM scheme ($l = 1, 2, ..., s$).

The level l is associated with relativity r_l%. That relativity is a premium adjustment coefficient. The meaning is that the policyholder of level l should pay $r_l \times$ base premium in order to be covered by the company. Base premium is the *a priori* premium that was determined on the basis of policyholder's observable characteristics.

Depending on their claim history, policyholders move among the levels according to some transitivity rules. Let $t_{ij}(k)$ be the transition rule if k claims are reported:

$$t_{ij}(k) = \begin{cases} 1 & \text{if the policy is transferred from level } \quad i \text{ to level } \quad j \\ 0 & \text{otherwise.} \end{cases} \tag{2.5.2}$$

Then $T(k)$ defined as follows will be a matrix of 0 and 1, with a unique 1 in each row.

$$T(k) = \begin{bmatrix} t_{00}(k) & t_{01}(k) & ... & t_{0s}(k) \\ t_{10}(k) & t_{11}(k) & ... & t_{1s}(k) \\ \vdots & \vdots & \ddots & \vdots \\ t_{s0}(k) & t_{s1}(k) & ... & t_{ss}(k) \end{bmatrix}. \tag{2.5.3}$$

Let $\Big\{L_1, L_2, ...\Big\}$ be the trajectory of the policyholder in BM system. They are random variables and take values from $\Big\{0, 1, ..., s\Big\}$. L_k is the level occupied by the policyholder in the time interval of $(k, k+1)$. We assume that the current level and current year claims number is enough to determine the next level and that annual claim numbers are independent. The trajectory across the BM levels may be represented by a memoryless Markov chain.

Let υ be the mean claim frequency and Y_k be the number of reported claims in the time interval of $(k, k+1)$. It is obvious that the trajectory will be dependent on the claim frequency and we will have $\Big\{L_1(\upsilon), L_2(\upsilon), ...\Big\}$. We will also assume that $Y_1, Y_2, ...$ are independent and have $Poi(\upsilon)$ distribution. Let $p_{l_1 l_2}(\upsilon)$ be the probability that the policyholder will move from level l_1 to l_2, where υ is the mean annual claim frequency. Let $P(\upsilon)$ be one step transition matrix:

$$P(\upsilon) = \Big\{p_{l_1 l_2}(\upsilon)\Big\}, \quad l_1, l_2 = 0, 1, 2, ...s, \tag{2.5.4}$$

where

$$P_{l_1 l_2}(\upsilon) = Pr(L_{k+1}(\upsilon) = l_2 | L_k(\upsilon) = l_1), \quad l_1, l_2 \in 0, 1, ..., s, \tag{2.5.5}$$

and

$$p_{l_1 l_2}(\upsilon) \geq 0 \quad \forall \quad l_1, l_2, \tag{2.5.6}$$

$$\sum_{l_2=0}^{s} p_{l_1 l_2}(\upsilon) = 1. \tag{2.5.7}$$

Since Y_{k+1} and $L_k(\upsilon)$ the transition probabilities can be expressed

$$p_{l_1l_2}(\upsilon) = \sum_{n=0}^{+\infty} Pr(L_{k+1}(\upsilon) = l_2|Y_{k+1} = n, L_k(\upsilon) = l_1)Pr(Y_{k+1} = n|L_k(\upsilon) = l_1)$$
$$= \sum_{n=0}^{\infty} \frac{\upsilon^n}{n!} exp(-\upsilon t_{l_1l_2}(n))). \tag{2.5.8}$$

We know that

$$P(\upsilon) = \begin{bmatrix} p_{00}(\upsilon) & p_{01}(\upsilon) & ... & p_{0s}(\upsilon) \\ p_{10}(\upsilon) & p_{11}(\upsilon) & ... & p_{1s}(\upsilon) \\ \vdots & \vdots & \ddots & \vdots \\ p_{s0}(\upsilon) & p_{s1}(\upsilon) & ... & p_{ss}(\upsilon) \end{bmatrix}. \tag{2.5.9}$$

It follows that in a matrix form we can write

$$P(\upsilon) = \sum_{k=0}^{\infty} \frac{\upsilon^k}{k!} exp(-\upsilon)T(k). \tag{2.5.10}$$

Since $Y_1, Y_2, ...$ are independent and identically distributed, each policyholder will eventually stabilize around the equilibrium level which corresponds to υ. The policy remains unchanged after claim-free period. This is the equilibrium in the long-run or the stationary equilibrium. The *stationary* distribution is $\pi(\upsilon) = (\pi_1(\upsilon), \pi_2(\upsilon), ..., \pi_s(\upsilon))^T$, where $\pi_{l_2}(\upsilon)$ is the stationary probability for the policyholder with mean claim frequency υ to be in level l_2.

$$\pi_{l_2}(\upsilon) = \lim_{n \to +\infty} p^n_{l_1l_2}(\upsilon). \tag{2.5.11}$$

$\pi(\upsilon)$ is the unique solution of the following equation systems:

$$\pi_j(\upsilon) = \sum_{l=0}^{s} \pi_l(\upsilon)p_{lj}(\upsilon), \quad j = 0, 1, ..., s, \tag{2.5.12}$$

or in a matrix notation

$$\begin{cases} \pi^T(\upsilon) = \pi^T(\upsilon)P(\upsilon) \\ \pi^T(\upsilon)e = 1, \end{cases} \tag{2.5.13}$$

where $e = (1, 1, ..., 1)^T$.

According to Rolski–Schmidli–Schmidt–Teugels formula if E is $(s+1)$x$(s+1)$ matrix with all elements equal to 1 and $P(\upsilon)$ is a regular matrix then the stationary distribution can be find as

$$\pi^T = e^T(\mathbb{1} - P(\upsilon) + E)^{-1}. \tag{2.5.14}$$

First of all with the help of *a priori* mean claim frequency the relativities should be determined. Those *a priori* frequencies must be as distinguishable as possible in order to prevent the further reinforcement of *a priori* risk classification at the BM steady state. This BM levels are achieved by combining *a priori* and *a posteriori* ratemakings. The purpose of *a priori* ratemaking is to put the policyholders into different tariff classes according to *a priori* risk classification. The policyholders who belong to the same tariff classes pay the same *a priori* premium. Then BM is used as *a posteriori* ratemaking to tackle the residual heterogeneity.

Let Θ_i be the random parameter that captures the residual heterogeneity of i-th policyholder. As I mentioned before Y_i are independent and identically distributed with mean $\lambda_i\theta$ given $\Theta_i = \theta$.

$$P(Y_i = k|\Theta_i = \theta) = \exp(-\lambda_i\theta)\frac{(\lambda_i\theta)^k}{k!}, \quad k = 0, 1, 2, ... \tag{2.5.15}$$

where $\Theta_1, \Theta_2, ...$ are independent and follow Γ distribution:

$$f(\theta) = \frac{1}{\Gamma(a)} a^a \theta^{a-1} \exp(-a\theta), \quad \theta > 0. \tag{2.5.16}$$

From here it follows that the number of claims Y_i has a negative binomial distribution with parameters $E(\Theta_i) = 1$ and $Var(\Theta_i) = \frac{1}{a}$. The problem then becomes the estimation of a.[9] The consistent estimator of a is given by

$$\frac{1}{\hat{a}} = \frac{\sum_{i=1}^n \left\{(n_i - \hat{\lambda}_i)^2 - n_i\right\}}{\sum_{i=1}^n \hat{\lambda}_i^2}. \tag{2.5.17}$$

The main purpose is to pick such r_l that it will be as close as possible to Θ. For that purpose mainly quadratic and exponential loss functions are used:

$$E\Big((\Theta - r_L)^2\Big) \tag{2.5.18}$$

and

$$E(\exp(-c(\Theta - r_L))), \tag{2.5.19}$$

where L is the level of a random policyholder that will be picked after reaching the

[9] υ from the previous notation here becomes $(\lambda\theta)$.

steady state.

The solution of quadratic loss function is given by

$$
\begin{aligned}
r_l^q &= E(\Theta|L=l) = E\Big(E(\Theta|L=l,\Lambda)|L=l\Big) \\
&= \sum_k E(\Theta|L=l,\Lambda=\lambda_k)Prob(\Lambda=\lambda_k|L=l) \\
&= \sum_k \int_0^\infty \theta \frac{Prob(L=l|\theta=\theta,\Lambda=\lambda_k)w_k}{Prob(L=l,\Lambda=\lambda_k)} dF_\Theta(\theta) \frac{Prob(\Lambda=\lambda_k, L=l)}{Prob(L=l)} \\
&= \frac{\sum_k w_k \int_0^\infty \theta\pi_l(\lambda_k\theta)dF_\Theta(\theta))}{\sum_k w_k \int_0^\infty \pi_l(\lambda_k\theta)dF_\Theta(\theta))}
\end{aligned}
\tag{2.5.20}
$$

and for the exponential loss function it is

$$
r_L^{exp} = 1 + \frac{1}{c}(E\Big[lnE(\exp(-c\Theta)|L)\Big] - lnE\Big[(\exp(-c\Theta)|L)\Big]), \tag{2.5.21}
$$

where w_k is the weight assigned to the k-th risk class whose annual expected claim frequency is λ_k. If Λ is the *a priori* mean claim frequency of randomly selected policyholder, then the actual annual mean claim frequency of this policyholder will be $\Lambda\Theta$. c is the severity parameter of BMS.[10] It follows that

$$
Prob(\Lambda = \lambda_k) = w_k. \tag{2.5.22}
$$

The proportion of policyholders in l (the distribution of L) will be

$$
P(L=l) = \sum_k w_k \int_0^\infty \pi_l(\lambda_k\theta)f(\theta)d\theta, \quad l = 0, 1, ..., s. \tag{2.5.23}
$$

In addition,

$$
\pi_l(\lambda_k\theta) = Prob(L=l|\Theta=\theta,\Lambda=\lambda_k). \tag{2.5.24}
$$

When *a priori* ratemaking is not taken into account, we will have the following formula for calculations of relativities taking $E(\Lambda) = \bar{\lambda}$:

$$
r_l = \frac{\int_0^\infty \theta\pi_l(\bar{\lambda}\theta)dF_\Theta(\theta)}{\int_0^\infty \pi_l(\bar{\lambda}\theta)dF_\Theta(\theta)}. \tag{2.5.25}
$$

The main purpose of bonus-malus is to deal with the heterogeneity within risk classes. The insurer uses *a priori* variables to obtain λ_is. Unlike life insurance where those variables can describe the estimated results very well, in vehicle insurance some heterogeneity will

[10]Denuit, M., X. Marechal, S. Pitrebois, J.F. Walhin (2007). Chapter 4.

be left. There will be some unobservable variables that have not been considered in *a priori* ratemaking. Heterogeneity between risk groups should be treated as given and be included in the computation of optimal relativities. The transition levels are applied to all policyholders unconditionally of their *a priori* mean claim frequency. Those transition rules are applied in the majority of BMS, apart from old Belgian and Portuguese systems. In both of those cases systems are dependent on the number of consecutive claim-free years which violates Markovian property. In these cases fictitious levels are introduced (for example, as Lemaire (1995) proposed splitting certain levels into sub-levels) to recover the Markovian property.

In case of quadratic loss function the relationship between *a priori* and *a posteriori* is described as follows:

$$\begin{aligned} E(\Lambda|L=l) &= \sum_k \lambda_k Prob(\Lambda=\lambda_k|L=l) = \sum_k \lambda_k \frac{Prob(L=l|\Lambda=\lambda_k)w_k}{Prob(L=l)} \\ &= \frac{\sum_k \lambda_k w_k \int_0^\infty \pi_l(\lambda_k\theta)dF_\Theta(\theta)}{\sum_k w_k \int_0^\infty \pi_l(\lambda_k\theta)dF_\Theta(\theta)}. \end{aligned} \tag{2.5.26}$$

First of all the insured with higher λ_k is expected to be on a higher level of BMS. The formula expresses the idea that policyholders who were *a priori* given discounts, are being discounted *a posteriori* as well, and the opposite, policyholders who are at the highest ranks on the system *a priori* will be penalized *a posteriori* as well.

2.6 The Efficiency of Bonus-Malus System

The elasticity of BMS shows the system's response to the change of mean claim frequency. In order to assess the elasticity, Loimaranta's efficiency is given by the formula is used. If $\bar{r}(\upsilon) = \sum_{l=0}^{s} \pi_l(\upsilon) r_l$ is the average relativity in the steady state with annual mean claim frequency υ, then Loimaranta's efficiency is given by the formula

$$Eff(\upsilon) = \frac{\frac{d\bar{r}(\upsilon)}{\bar{r}(\upsilon)}}{\frac{d\upsilon}{\upsilon}}. \tag{2.6.1}$$

$Eff(\upsilon)$ is a positive number, as it is obvious that the increase in the expected annual claim frequency will increase the average relativity. The value of $Eff(\upsilon)$ close to 0 implies that BMS does not modify the structure of policyholders by classes when there is a change in the annual mean claim frequency. The value close to 1 will describe the ideal bonus-malus system.

In order to calculate Loimaranta's efficiency we use the stationary probabilities $\pi_l(\theta)$. We first need to solve the following equation system with respect to $\frac{d\pi_(\upsilon)}{d\upsilon}$:

$$\begin{cases} \frac{d\pi^T(\upsilon)}{d\upsilon} = \frac{d\pi^T(\upsilon)}{d\upsilon}P(\upsilon) + \pi^T(\upsilon)\frac{dP(\upsilon)}{d\upsilon} \\ \sum_{l=0}^{s} \frac{d\pi^T(\upsilon)}{d(\upsilon)} = 0. \end{cases} \tag{2.6.2}$$

Then using $\frac{d\pi_(\upsilon)}{d\upsilon}$, we get the derivatives of $\bar{r}(\upsilon)$.

$$\frac{d\bar{r}(\upsilon)}{d\upsilon} = \sum_{l=0}^{s} \frac{d\pi^T(\upsilon)}{d(\upsilon)} r_l. \tag{2.6.3}$$

As we can see Loimaranta efficiency is a function of mean annual claim frequency. This is a good tool for analyzing the efficiencies of different bonus-malus systems adopted by insurance companies and compare them.

Chapter 3

Italian Bonus-Malus System and The Calculation of Relativities

The new Italian bonus-malus system was adopted in 1991. The old system consisted of 13 classes and the levels were $70, 70, 70, 75, 80, 85, 92, 100, 115, 132, 152, 175, 200$. The starting level was at 115. Every claim free year was rewarded by going up with 1 in the levels. Every claim was punished by going down by 1 in the levels. After 1991, the number of classes increased and the punishment became more severe. The new system consists of 18 classes with the levels $50, 53, 56, 59, 62, 66, 70, 74, 78, 82, 88, 94, 100, 115, 130, 150, 175, 200$. Starting level is the same as in the old system. If no claims occurred in the respective year, the policyholder will be discounted and move up by 1 position. If there is one accident, the policyholder will be moved down by one position. If k_t is the number of claims during the respective year and it is bigger than 1, the policyholder will be moved down by $3k_t$ levels. This means that after 6-th claim, no matter what the level of policyholder is, all his or her rewards will be canceled and he or she will be moved to 18-th level.

3.1 Transition Rules in Italian Bonus-Malus System

The transition rules for the Italian bonus-malus system are illustrated in table 3.1.1.[1]

[1] based on Lemaire, J., Z. Hongmin (1994).

Table 3.1.1: Transition rules for Italian bonus-malus system.

Class	Premium	Class after						
		0	1	2	3	4	5	≥ 6
		claims						
18	200	17	18	18	18	18	18	18
17	175	16	18	18	18	18	18	18
16	150	15	18	18	18	18	18	18
15	130	14	17	18	18	18	18	18
14	115	13	16	18	18	18	18	18
13	100	12	15	18	18	18	18	18
12	94	11	14	17	18	18	18	18
11	88	10	13	16	18	18	18	18
10	82	9	12	15	18	18	18	18
9	78	8	11	14	17	18	18	18
8	74	7	10	13	16	18	18	18
7	70	6	9	12	15	18	18	18
6	66	5	8	11	14	17	18	18
5	62	4	7	10	13	16	18	18
4	59	3	6	9	12	15	18	18
3	56	2	5	8	11	14	17	18
2	53	1	4	7	10	13	16	18
1	50	1	3	6	9	12	15	18

According to (2.5.3) we will have

$$T(0) = \begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0
\end{pmatrix}.$$

For $k = 1, 2, 3, 4, 5$ we will respectively have

$$T(1) = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \qquad
T(2) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

$$T(3) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

$$T(4) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \qquad
T(5) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

For all $k \geq 6$

$$T(k) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

From (2.5.10) it follows that for the Italian bonus-malus system transition probabilities $P(v)$ are the following:

$$
\begin{pmatrix}
e^{-\upsilon} & 0 & \upsilon e^{-\upsilon} & 0 & 0 & \frac{\upsilon^2}{2!}e^{-\upsilon} & 0 & 0 & \frac{\upsilon^3}{3!}e^{-\upsilon} & 0 & 0 & \frac{\upsilon^4}{4!}e^{-\upsilon} & 0 & 0 & \frac{\upsilon^5}{5!}e^{-\upsilon} & 0 & 0 & 1-\Sigma_1 \\
e^{-\upsilon} & 0 & 0 & \upsilon e^{-\upsilon} & 0 & 0 & \frac{\upsilon^2}{2!}e^{-\upsilon} & 0 & 0 & \frac{\upsilon^3}{3!}e^{-\upsilon} & 0 & 0 & \frac{\upsilon^4}{4!}e^{-\upsilon} & 0 & 0 & \frac{\upsilon^5}{5!}e^{-\upsilon} & 0 & 1-\Sigma_2 \\
0 & e^{-\upsilon} & 0 & 0 & \upsilon e^{-\upsilon} & 0 & 0 & \frac{\upsilon^2}{2!}e^{-\upsilon} & 0 & 0 & \frac{\upsilon^3}{3!}e^{-\upsilon} & 0 & 0 & \frac{\upsilon^4}{4!}e^{-\upsilon} & 0 & 0 & \frac{\upsilon^5}{5!}e^{-\upsilon} & 1-\Sigma_3 \\
0 & 0 & e^{-\upsilon} & 0 & 0 & \upsilon e^{-\upsilon} & 0 & 0 & \frac{\upsilon^2}{2!}e^{-\upsilon} & 0 & 0 & \frac{\upsilon^3}{3!}e^{-\upsilon} & 0 & 0 & \frac{\upsilon^4}{4!}e^{-\upsilon} & 0 & 0 & 1-\Sigma_4 \\
0 & 0 & 0 & e^{-\upsilon} & 0 & 0 & \upsilon e^{-\upsilon} & 0 & 0 & \frac{\upsilon^2}{2!}e^{-\upsilon} & 0 & 0 & \frac{\upsilon^3}{3!}e^{-\upsilon} & 0 & 0 & \frac{\upsilon^4}{4!}e^{-\upsilon} & 0 & 1-\Sigma_5 \\
0 & 0 & 0 & 0 & e^{-\upsilon} & 0 & 0 & \upsilon e^{-\upsilon} & 0 & 0 & \frac{\upsilon^2}{2!}e^{-\upsilon} & 0 & 0 & \frac{\upsilon^3}{3!}e^{-\upsilon} & 0 & 0 & \frac{\upsilon^4}{4!}e^{-\upsilon} & 1-\Sigma_6 \\
0 & 0 & 0 & 0 & 0 & e^{-\upsilon} & 0 & 0 & \upsilon e^{-\upsilon} & 0 & 0 & \frac{\upsilon^2}{2!}e^{-\upsilon} & 0 & \frac{\upsilon^3}{3!}e^{-\upsilon} & 0 & 0 & 0 & 1-\Sigma_7 \\
0 & 0 & 0 & 0 & 0 & 0 & e^{-\upsilon} & 0 & 0 & \upsilon e^{-\upsilon} & 0 & 0 & \frac{\upsilon^2}{2!}e^{-\upsilon} & 0 & 0 & \frac{\upsilon^3}{3!}e^{-\upsilon} & 0 & 1-\Sigma_8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\upsilon} & 0 & 0 & \upsilon e^{-\upsilon} & 0 & 0 & \frac{\upsilon^2}{2!}e^{-\upsilon} & 0 & 0 & \frac{\upsilon^3}{3!}e^{-\upsilon} & 1-\Sigma_9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\upsilon} & 0 & 0 & \upsilon e^{-\upsilon} & 0 & 0 & \frac{\upsilon^2}{2!}e^{-\upsilon} & 0 & 0 & 1-\Sigma_{10} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\upsilon} & 0 & 0 & \upsilon e^{-\upsilon} & 0 & 0 & \frac{\upsilon^2}{2!}e^{-\upsilon} & 0 & 1-\Sigma_{11} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\upsilon} & 0 & 0 & \upsilon e^{-\upsilon} & 0 & 0 & \frac{\upsilon^2}{2!}e^{-\upsilon} & 1-\Sigma_{12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\upsilon} & 0 & 0 & \upsilon e^{-\upsilon} & 0 & 0 & 1-\Sigma_{13} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\upsilon} & 0 & 0 & \upsilon e^{-\upsilon} & 0 & 1-\Sigma_{14} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\upsilon} & 0 & 0 & \upsilon e^{-\upsilon} & 1-\Sigma_{15} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\upsilon} & 0 & 0 & 1-e^{-\upsilon} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\upsilon} & 0 & 1-e^{-\upsilon} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\upsilon} & 1-e^{-\upsilon}
\end{pmatrix},
$$

where Σ_j is the sum of the rest of elements of j-th row.

Because of the private data that insurance companies do not give away, we will use the estimated mean claim frequency by Denuit et al. (2007). In order to demonstrate the long-term steady state of the Italian system, the maximum likelihood estimate of λ or υ of Poisson distribution equal to 0.1462 is taken (obtained from the data of a Belgian motor third party liability insurance portfolio observed during the year 1997, Appendix A).

$$
\begin{pmatrix}
0.864 & 0 & 0.126 & 0 & 0 & 0.009 & 0 & 0 & 0.0004 & 0 & 0 & 0.00001 & 0 & 0 & 0.0000 & 0 & 0 & 0.0000 \\
0.864 & 0 & 0 & 0.126 & 0 & 0 & 0.009 & 0 & 0 & 0.0004 & 0 & 0 & 0.00001 & 0 & 0 & 0.0000 & 0 & 0.0000 \\
0 & 0.864 & 0 & 0 & 0.126 & 0 & 0 & 0.009 & 0 & 0 & 0.0004 & 0 & 0 & 0.00001 & 0 & 0 & 0.0000 & 0.0000 \\
0 & 0 & 0.864 & 0 & 0 & 0.126 & 0 & 0 & 0.009 & 0 & 0 & 0.0004 & 0 & 0 & 0.00001 & 0 & 0 & 0.0000 \\
0 & 0 & 0 & 0.864 & 0 & 0 & 0.126 & 0 & 0 & 0.009 & 0 & 0 & 0.0004 & 0 & 0 & 0.00001 & 0 & 0.0000 \\
0 & 0 & 0 & 0 & 0.864 & 0 & 0 & 0.126 & 0 & 0 & 0.009 & 0 & 0 & 0.0004 & 0 & 0 & 0.00001 & 0.0000 \\
0 & 0 & 0 & 0 & 0 & 0.864 & 0 & 0 & 0.126 & 0 & 0 & 0.009 & 0 & 0 & 0.0004 & 0 & 0 & 0.0000 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.864 & 0 & 0 & 0.126 & 0 & 0 & 0.009 & 0 & 0 & 0.0004 & 0 & 0.0000 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.864 & 0 & 0 & 0.126 & 0 & 0 & 0.009 & 0 & 0 & 0.0004 & 0.0000 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.864 & 0 & 0 & 0.126 & 0 & 0 & 0.009 & 0 & 0 & 0.0005 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.864 & 0 & 0 & 0.126 & 0 & 0 & 0.009 & 0 & 0.0005 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.864 & 0 & 0 & 0.126 & 0 & 0 & 0.009 & 0.0005 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.864 & 0 & 0 & 0.126 & 0 & 0 & 0.0097 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.864 & 0 & 0 & 0.126 & 0 & 0.0097 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.864 & 0 & 0 & 0.126 & 0.0097 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.864 & 0 & 0 & 0.136 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.864 & 0 & 0.136 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.864 & 0.136
\end{pmatrix},
$$

After 20 years and 30 years respectively, the transition probabilities will be $P^{(20)}(0.1462)$ and $P^{(30)}(0.1462)$ as illustrated in table 3.1.2 and 3.1.3. As we can see even after 30 years, the system is not stable and has not reached to its steady state yet. If we apply Rolski–Schmidli–Schmidt–Teugels formula, the system will converge to (0.6498, 0.1023, 0.1184, 0.0420, 0.0337, 0.0217, 0.0120, 0.0079, 0.0047, 0.0029, 0.0018, 0.0011, 0.0007,

Table 3.1.2: Transition probabilities of Italian bonus-malus system after 20 years.

Class, 20 years

Class

0.6478	0.1014	0.1170	0.0403	0.0327	0.0206	0.0110	0.0073	0.0042	0.0024	0.0015	0.0009	0.0005	0.0003	0.0002	0.0001	0.0001	0.0000
0.6478	0.0998	0.1186	0.0403	0.0316	0.0217	0.0110	0.0068	0.0047	0.0024	0.0014	0.0010	0.0005	0.0003	0.0002	0.0001	0.0001	0.0000
0.6478	0.0998	0.1143	0.0447	0.0316	0.0193	0.0133	0.0068	0.0037	0.0034	0.0014	0.0007	0.0008	0.0003	0.0002	0.0002	0.0001	0.0000
0.6392	0.1084	0.1143	0.0375	0.0387	0.0193	0.0098	0.0103	0.0037	0.0020	0.0027	0.0007	0.0004	0.0007	0.0002	0.0001	0.0001	0.0000
0.6392	0.0938	0.1289	0.0375	0.0287	0.0294	0.0098	0.0058	0.0082	0.0020	0.0011	0.0023	0.0004	0.0002	0.0006	0.0001	0.0001	0.0001
0.6392	0.0938	0.1061	0.0603	0.0287	0.0167	0.0225	0.0058	0.0030	0.0073	0.0011	0.0006	0.0021	0.0002	0.0002	0.0005	0.0001	0.0001
0.6051	0.1279	0.1061	0.0315	0.0575	0.0167	0.0078	0.0205	0.0030	0.0016	0.0068	0.0006	0.0004	0.0020	0.0002	0.0002	0.0004	0.0001
0.6051	0.0800	0.1540	0.0315	0.0232	0.0510	0.0078	0.0045	0.0190	0.0016	0.0010	0.0064	0.0004	0.0004	0.0018	0.0002	0.0002	0.0003
0.6051	0.0800	0.0887	0.0969	0.0232	0.0127	0.0461	0.0045	0.0023	0.0183	0.0010	0.0008	0.0061	0.0004	0.0004	0.0015	0.0002	0.0003
0.5213	0.1638	0.0887	0.0221	0.0980	0.0127	0.0055	0.0451	0.0023	0.0016	0.0177	0.0008	0.0009	0.0056	0.0004	0.0006	0.0012	0.0003
0.5213	0.0570	0.1956	0.0221	0.0159	0.0948	0.0055	0.0037	0.0438	0.0016	0.0017	0.0169	0.0009	0.0011	0.0050	0.0006	0.0007	0.0008
0.5213	0.0570	0.0619	0.1558	0.0159	0.0091	0.0913	0.0037	0.0032	0.0423	0.0017	0.0022	0.0156	0.0011	0.0015	0.0041	0.0007	0.0008
0.3755	0.2028	0.0619	0.0129	0.1589	0.0091	0.0056	0.0895	0.0032	0.0040	0.0401	0.0022	0.0029	0.0139	0.0015	0.0018	0.0031	0.0008
0.3755	0.0315	0.2333	0.0129	0.0123	0.1558	0.0056	0.0075	0.0854	0.0040	0.0058	0.0366	0.0030	0.0038	0.0116	0.0018	0.0020	0.0020
0.3755	0.0315	0.0365	0.2097	0.0123	0.0129	0.1487	0.0075	0.0106	0.0789	0.0058	0.0078	0.0318	0.0038	0.0045	0.0089	0.0020	0.0020
0.2008	0.2063	0.0365	0.0130	0.2092	0.0129	0.0150	0.1413	0.0107	0.0135	0.0713	0.0078	0.0091	0.0267	0.0045	0.0046	0.0064	0.0020
0.2008	0.0316	0.2113	0.0130	0.0379	0.1843	0.0150	0.0345	0.1177	0.0136	0.0235	0.0559	0.0091	0.0121	0.0191	0.0047	0.0047	0.0037
0.2009	0.0316	0.0366	0.1877	0.0379	0.0385	0.1609	0.0345	0.0339	0.0975	0.0235	0.0218	0.0433	0.0121	0.0105	0.0134	0.0047	0.0037

0.0004, 0.0002, 0.0002, 0.0001, 0.0001).

3.2 Calculation of Italian Bonus-Malus System Relativities

Further we will calculate the relativities taking into account the heterogeneity. Again, because of the luck of information, we will take already estimated results of λs and the heterogeneity factor a as in the book by Denuit et al. (2007), table 2.7 (Appendix B). For the stationary distributions we will use Rolski–Schmidli–Schmidt–Teugels formula. And quadratic loss function will be assumed. $1 - P(v) + E$ matrix is illustrated in 3.2.1.

Calculations are done by MatLab, generating gamma distribution for θ with parameters $(\hat{a}, \hat{a})$. The long-term stationarity probabilities for each value of $\lambda_k, \quad k = 1, 2, ..., 23$ will be 23x18 matrix. The final distribution of policyholders will be calculated based on these steady state transition probabilities according to (2.5.23). From formula we see that heterogeneity factor is also included in the calculation of level distributions.

Table 3.1.3: Transition probabilities of Italian bonus-malus system after 30 years.

Class, 30 years

Class

0.6418	0.1005	0.1161	0.0409	0.0326	0.0207	0.0114	0.0074	0.0043	0.0027	0.0016	0.0009	0.0006	0.0003	0.0002	0.0001	0.0001	0.0000
0.6415	0.1009	0.1161	0.0406	0.0329	0.0207	0.0113	0.0076	0.0043	0.0026	0.0017	0.0009	0.0006	0.0004	0.0002	0.0001	0.0001	0.0000
0.6415	0.0999	0.1171	0.0406	0.0322	0.0214	0.0113	0.0072	0.0047	0.0026	0.0015	0.0011	0.0006	0.0003	0.0002	0.0001	0.0001	0.0001
0.6415	0.0999	0.1151	0.0426	0.0322	0.0203	0.0124	0.0072	0.0042	0.0031	0.0015	0.0009	0.0007	0.0003	0.0002	0.0002	0.0001	0.0001
0.6381	0.1032	0.1151	0.0396	0.0352	0.0203	0.0109	0.0088	0.0042	0.0025	0.0022	0.0009	0.0005	0.0005	0.0002	0.0001	0.0001	0.0001
0.6381	0.0978	0.1205	0.0396	0.0312	0.0243	0.0109	0.0069	0.0061	0.0025	0.0015	0.0016	0.0005	0.0003	0.0004	0.0001	0.0001	0.0001
0.6381	0.0978	0.1123	0.0479	0.0312	0.0193	0.0158	0.0069	0.0039	0.0046	0.0015	0.0009	0.0013	0.0003	0.0002	0.0003	0.0001	0.0001
0.6264	0.1095	0.1123	0.0375	0.0416	0.0193	0.0102	0.0125	0.0039	0.0023	0.0038	0.0009	0.0006	0.0011	0.0002	0.0002	0.0002	0.0001
0.6264	0.0930	0.1289	0.0375	0.0292	0.0317	0.0102	0.0064	0.0100	0.0023	0.0015	0.0032	0.0006	0.0004	0.0009	0.0002	0.0002	0.0001
0.6265	0.0930	0.1061	0.0603	0.0292	0.0179	0.0240	0.0064	0.0038	0.0086	0.0015	0.0010	0.0028	0.0004	0.0004	0.0007	0.0002	0.0001
0.5974	0.1221	0.1061	0.0340	0.0556	0.0179	0.0095	0.0210	0.0038	0.0025	0.0076	0.0010	0.0008	0.0024	0.0004	0.0003	0.0005	0.0002
0.5975	0.0843	0.1440	0.0340	0.0267	0.0469	0.0095	0.0065	0.0183	0.0025	0.0020	0.0067	0.0008	0.0008	0.0020	0.0003	0.0003	0.0003
0.5977	0.0844	0.0960	0.0821	0.0267	0.0170	0.0394	0.0065	0.0045	0.0164	0.0020	0.0018	0.0058	0.0008	0.0008	0.0015	0.0003	0.0003
0.5429	0.1394	0.0961	0.0312	0.0777	0.0170	0.0105	0.0354	0.0045	0.0040	0.0144	0.0018	0.0018	0.0048	0.0008	0.0008	0.0011	0.0003
0.5431	0.0745	0.1610	0.0312	0.0268	0.0680	0.0106	0.0091	0.0309	0.0040	0.0040	0.0122	0.0018	0.0018	0.0038	0.0008	0.0008	0.0007
0.5434	0.0746	0.0863	0.1061	0.0268	0.0200	0.0587	0.0092	0.0082	0.0269	0.0040	0.0040	0.0101	0.0019	0.0018	0.0028	0.0008	0.0007
0.4754	0.1429	0.0863	0.0413	0.0918	0.0200	0.0209	0.0471	0.0082	0.0103	0.0206	0.0040	0.0047	0.0073	0.0018	0.0018	0.0018	0.0007
0.4755	0.0747	0.1546	0.0413	0.0369	0.0751	0.0209	0.0180	0.0373	0.0104	0.0089	0.0157	0.0047	0.0039	0.0052	0.0018	0.0014	0.0011

$$
\begin{matrix}
2-e^{-\upsilon} & 1 & 1-\upsilon e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^2}{2!}e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^3}{3!}e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^4}{4!}e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^5}{5!}e^{-\upsilon} & 1 & 1 & \Sigma_1 \\
1-e^{-\upsilon} & 2 & 1 & 1-\upsilon e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^2}{2!}e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^3}{3!}e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^4}{4!}e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^5}{5!}e^{-\upsilon} & 1 & \Sigma_2 \\
1 & 1-e^{-\upsilon} & 2 & 1 & 1-\upsilon e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^2}{2!}e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^3}{3!}e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^4}{4!}e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^5}{5!}e^{-\upsilon} & \Sigma_3 \\
1 & 1 & 1-e^{-\upsilon} & 2 & 1 & 1-\upsilon e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^2}{2!}e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^3}{3!}e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^4}{4!}e^{-\upsilon} & 1 & 1 & \Sigma_4 \\
1 & 1 & 1 & 1-e^{-\upsilon} & 2 & 1 & 1-\upsilon e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^2}{2!}e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^3}{3!}e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^4}{4!}e^{-\upsilon} & 1 & \Sigma_5 \\
1 & 1 & 1 & 1 & 1-e^{-\upsilon} & 2 & 1 & 1-\upsilon e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^2}{2!}e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^3}{3!}e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^4}{4!}e^{-\upsilon} & \Sigma_6 \\
1 & 1 & 1 & 1 & 1 & 1-e^{-\upsilon} & 2 & 1 & 1-\upsilon e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^2}{2!}e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^3}{3!}e^{-\upsilon} & 1 & 1 & \Sigma_7 \\
1 & 1 & 1 & 1 & 1 & 1 & 1-e^{-\upsilon} & 2 & 1 & 1-\upsilon e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^2}{2!}e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^3}{3!}e^{-\upsilon} & 1 & \Sigma_8 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1-e^{-\upsilon} & 2 & 1 & 1-\upsilon e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^2}{2!}e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^3}{3!}e^{-\upsilon} & \Sigma_9 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1-e^{-\upsilon} & 2 & 1 & 1-\upsilon e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^2}{2!}e^{-\upsilon} & 1 & 1 & \Sigma_{10} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1-e^{-\upsilon} & 2 & 1 & 1-\upsilon e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^2}{2!}e^{-\upsilon} & 1 & \Sigma_{11} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1-e^{-\upsilon} & 2 & 1 & 1-\upsilon e^{-\upsilon} & 1 & 1 & 1-\frac{\upsilon^2}{2!}e^{-\upsilon} & \Sigma_{12} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1-e^{-\upsilon} & 2 & 1 & 1-\upsilon e^{-\upsilon} & 1 & 1 & \Sigma_{13} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1-e^{-\upsilon} & 2 & 1 & 1-\upsilon e^{-\upsilon} & & \Sigma_{14} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1-e^{-\upsilon} & 2 & 1 & 1-\upsilon e^{-\upsilon} & \Sigma_{15} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1-e^{-\upsilon} & 2 & 1 & e^{-\upsilon} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1-e^{-\upsilon} & 2 & e^{-\upsilon} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1-e^{-\upsilon} & 1+e^{-\upsilon}
\end{matrix}
$$

Figure 3.2.1: $\mathbb{1} - P(\upsilon) + E$ matrix of Italian BMS.

Table 3.2.1: The share of policyholders occupying bonus-malus levels in Italian BMS.

level	the percentage of policyholders	level	the percentage of policyholders
1	45.07%	10	1.86%
2	5.55%	11	1.61%
3	7.04%	12	1.4%
4	4.14%	13	1.3%
5	4.39%	14	1.37%
6	4.35%	15	1.63%
7	4.34%	16	2.19%
8	2.8%	17	3.27%
9	2.41%	18	5.37%

It is clear that after *a priori* ratemaking 45.07% of policyholders occupied the 1-st level. 3-rd level is occupied by 7.04% policyholders followed by the 2-nd level occupied by 5.55% policyholders. The rest of policyholders (42.34%) are divided among the rest of the levels. The default level is occupied only by 1.37% of policyholders.

When calculating the relativities with *a posteriori* ratemaking taking into account *a priori* ratemaking, we use (2.5.20).

Table 3.2.2: Relativities with *a priori* ratemaking.

level	r_l with *a priori* ratemaking	level	r_l with *a priori* ratemaking
1	79.6%	10	264.3%
2	164.9%	11	273.6%
3	176.3%	12	286%
4	221.7%	13	302.2%
5	233.9%	14	319.4%
6	248%	15	333.4%
7	261.2%	16	343.1%
8	255%	17	353%
9	261%	18	369.6%

When *a priori* ratemaking is not recognized, we use (2.5.25) for the calculations of relativities and get table 3.2.3. As the mean value of λ, the estimate from the book Denuit, M., et al. (2007) is taken. That is equal to 0.1474 and the heterogeneity factor $\hat{a} = 0.889$. The gamma distribution of θ is generated with parameters $\hat{a}$, $\hat{a}$.

Table 3.2.3: Relativities without *a priori* ratemaking.

level	r_l without *a priori* ratemaking	level	r_l without *a priori* ratemaking
1	40.56%	10	227%
2	90.8%	11	251%
3	95.9%	12	279.1%
4	133%	13	311.8%
5	146.4%	14	337.4%
6	167.5%	15	344.6%
7	189.6%	16	327.7%
8	193.2%	17	304.4%
9	212.3%	18	285.4%

The connection between *a priori* and *a posteriori* ratemakings is shown in the table 3.2.4.

Table 3.2.4: Connection between *a priori* and *a posteriori* ratemakings for Italian BMS.

level	$E(\Lambda\|L=l)$	level	$E(\Lambda\|L=l)$
1	14.18%	10	13.75%
2	14.46%	11	14.03%
3	14.33%	12	14.57%
4	13.8%	13	15.32%
5	13.51%	14	16.21%
6	13.15%	15	17.28%
7	12.8%	16	18.46%
8	13.32%	17	19.72%
9	13.36%	18	20.96%

In table 3.2.4 we see that in general *a priori* expected mean claim frequency increases with the occupied level.

In Italian system the policyholders who paid high without *a priori* ratemaking tend to pay high with *a priori* ratemaking as well.

In the theory, *a posteriori* corrections are weaker when applying *a priori* ratemaking. In our case it is correct only for 13-15 classes. The lowest percentage of relativities 45.07% without *a priori* ratemaking becomes 79.6% with *a priori* ratemaking. Thus in this example *a priori* ratemaking is strict to policyholders with low claims and high claims.

These deviations from the theory are because of the absence of data and the differences between Italian system and the Belgian bonus-malus systems. We take the estimated parameters of Belgian system and apply it to Italian system. In Belgian system the premium is dependent on claim history as a policyholder with no-claim history for 4 years cannot be in a class above 14.

Another problem can be the fact that without *a priori* ratemaking, the steady state probabilities for higher levels of the system are very low. Interestingly enough *a priori* ratemaking relativities are decreasing after the levels higher than the default level. This means that in a steady state the policyholders should pay almost 2 times higher than their *a priori* premiums. Meanwhile the rest of policyholders should pay more than 2 times of their *a priori* premiums.

Besides, with this approach we consider that the number of claims is enough for predictions, but some studies show that taking into account the severity of those claims will give more realistic results.

Conclusion

While gross insurance premiums are quite high, over 2015-2016 those premiums decreased in OECD countries. Despite this, the main contributor to the world insurance funds is USA and it is hardly going to change in the near future. USA owns almost half of both life and non-life insurance premiums. The main part of the non-life premiums are occupied by health and accident insurance in OECD countries.

Life insurance gross premiums are increasing in Italy over the recent year, thus, leading to the average increase in gross insurance premiums. On the contrary non-life insurance gross premiums are drastically decreasing. Interestingly enough, vehicle insurance premiums are the highest among non-life insurance gross premiums in Italy. They were always close to the half of non-life insurance premiums. This is the reason why it is good to know what to expect in vehicle insurance in sense of premiums and risks.

The range of risks in vehicle insurance is very wide. But it is always good to have good prediction tools for the risks in this insurance to avoid huge losses and to generate profits. Those tools include the fact that the claim frequency in vehicle insurance has a Poisson distribution and can be easily predicted by Poisson regression. But the latter has a problem with treating the heterogeneity of risk characteristics of policyholders. That is when another very important tool called negative binomial regression helps us. the advantage of this tool is that we take into account the heterogeneity and assume that it has a gamma distribution. But the problem with those tools is that the absence of micro data limits the prediction accuracy. With proper data the maximum likelihood method can be applied and the estimates of mean claim frequency and heterogeneity factor can be discussed.

As we saw, there are few ways of calculating the premiums in insurance and vehicle insurance. One includes the "safety loading" method, where insurance companies set higher premium than the expected one. Another method is the calculation of premium increment taking into account the expected number of claims and the inflation factor. The last one, based on which our analysis was done, is the calculation of relativities of bonus-malus system. It can also be used in another insurance systems with bonus-malus factor and not only in vehicle insurance.

When we don't take into account the heterogeneity, the steady state transition probabilities are not reached even after 20 years in Italian bonus-malus system.

But when we calculate transition probabilities taking into account the heterogeneity of policyholders, we see that we get contradictory results for the Italian BMS. We saw that according to steady state transition probabilities, the policyholders on the higher levels of the system without *a priori* ratemaking have lower relativities in case of with *a priori* ratemaking than those who occupied lower levels of the system without *a priori* segmentation. This problem can be caused both by the absence of data, that would help us to get more realistic estimates of mean claim frequency and heterogeneity factor and by the fact that maybe the premiums that are initially attached to the higher level occupants are too high. Another reason can be that in the steady state the transition probabilities of occupying high levels are less possible. This is because initially the majority of policyholders are the lowest level of the system. And having more than 6 claims in a year is not very probable.

But of course, applying Belgian system estimates could bring to incorrect results as the systems of both countries are different and in case of Belgian system the Markovian property is not satisfied. This can cause problems with the use of parameters for the Italian system.

Appendix A

Data on Observed Claimed Distribution

Number of claims	Number of policies	Total exposure in years
0	12962	10545.94
1	1369	1187.13
2	157	134.66
3	14	11.08
4	3	2.52
Total	14505	11881.35

Appendix B

The Estimated Mean Claim Frequencies and Heterogeneity Factor

$\hat{\lambda}$	*weights*	$\hat{a}$
0.1176	0.1049	1.65
0.1408	0.1396	
0.1897	0.0398	
0.2272	0.0705	
0.1457	0.0076	
0.1746	0.0122	
0.2351	0.0013	
0.2816	0.0014	
0.1761	0.0293	
0.2109	0.0299	
0.284	0.0152	
0.3402	0.0242	
0.2182	0.0007	
0.2614	0.0009	
0.352	0.0002	
0.0928	0.1338	
0.1112	0.1973	
0.1498	0.0294	
0.1794	0.0661	
0.1151	0.0372	
0.1378	0.0517	
0.1856	0.0025	

Appendix C

Matlab Codes for Poisson and Negative Binomial Mass Functions

```
clc;
clear all;
%draws Poisson pdf with lambda=45
x = 0:100;
y = poisspdf(x,45);
plot(x,y,'k')
title('Poisson Distribution with lambda=45')
%%decrease lambda=40, pdf is less symmetric
% x = 0:100;
% y = poisspdf(x,40);
% plot(x,y,'k')
```

```
clc;
clear all;
%draws negative binomial pmf with lambda=45
x = 0:80;
y = nbinpdf(x,30,1/2);
plot(x,y,'k')
%title('Negative Binomial Distribution with k=3 and p=1/2')
%%increase k=30 leaving p fixed, pdf is less symmetric
%x = 0:80;
%y = nbinpdf(x,30,1/2);
%plot(x,y,'k')
%title('Negative Binomial Distribution with k=30 and p=1/2')
```

Appendix D

Matlab Code for Italian Bonus-Malus System's 20 and 30 Years Transition Probabilities Calculation

```
clc;
clear all;
A=xlsread('stst'); %(filename in brackets, resolution .xls or .xlsx)
row=A(1:end,1);
col=A(1,1:end);
prob=A(2:end,2:end);
C=prob^20 %20 years later
D=prob^30  %30 years later
% converges to
A1=xlsread('lambda');
B=[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1];
Pi=B*inv(A1)
```

Appendix E

Matlab Code for Calculation of Relativities

```
clc;
clear all;
a=1.065;
a1=0.889;
lambda=0.1457;
%theta=gamrnd(a,a,23,1); %generation of Gamma distributed
    heterogeneity
%factor
%heta1=gamrnd(a1,a1,23,1);
A0=xlsread('lambda and weights');
A00=xlsread('theta1');
A1=xlsread('lambda1');%(f1lename 1n brackets, resolut1on .xls or
    .xlsx)
A2=xlsread('lambda2'); % with a priori
A3=xlsread('lambda3');
A4=xlsread('lambda4');
A5=xlsread('lambda5');
A6=xlsread('lambda6');
A7=xlsread('lambda7');
A8=xlsread('lambda8');
A9=xlsread('lambda9');
A10=xlsread('lambda10');
A11=xlsread('lambda11');
A12=xlsread('lambda12');
A13=xlsread('lambda13');
A14=xlsread('lambda14');
```

```
A15=xlsread('lambda15');
A16=xlsread('lambda16');
A17=xlsread('lambda17');
A18=xlsread('lambda18');
A19=xlsread('lambda19');
A20=xlsread('lambda20');
A21=xlsread('lambda21');
A22=xlsread('lambda22');
A23=xlsread('lambda23');
D1=xlsread('mean1');%without a priori
D2=xlsread('mean2');
D3=xlsread('mean3');
D4=xlsread('mean4');
D5=xlsread('mean5');
D6=xlsread('mean6');
D7=xlsread('mean7');
D8=xlsread('mean8');
D9=xlsread('mean9');
D10=xlsread('mean10');
D11=xlsread('mean11');
D12=xlsread('mean12');
D13=xlsread('mean13');
D14=xlsread('mean14');
D15=xlsread('mean15');
D16=xlsread('mean16');
D17=xlsread('mean17');
D18=xlsread('mean18');
D19=xlsread('mean19');
D20=xlsread('mean20');
D21=xlsread('mean21');
D22=xlsread('mean22');
D23=xlsread('mean23');
B= [1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1] % steady state probability calculation
pi1=B*inv(A1);
pi2=B*inv(A2);
```

```
pi3=B*inv(A3);
pi4=B*inv(A4);
pi5=B*inv(A5);
pi6=B*inv(A6);
pi7=B*inv(A7);
pi8=B*inv(A8);
pi9=B*inv(A9);
pi10=B*inv(A10);
pi11=B*inv(A11);
pi12=B*inv(A12);
pi13=B*inv(A13);
pi14=B*inv(A14);
pi15=B*inv(A15);
pi16=B*inv(A16);
pi17=B*inv(A17);
pi18=B*inv(A18);
pi19=B*inv(A19);
pi20=B*inv(A20);
pi21=B*inv(A21);
pi22=B*inv(A22);
pi23=B*inv(A23);
pi=[pi1; pi2; pi3; pi4; pi5; pi6; pi7; pi8; pi9; pi10; pi11;
    pi12; pi13; pi14; pi15; pi16; pi17; pi18; pi19; pi20; pi21;
    pi22; pi23]
P1=A0(:,2)'*pi(:,1)
P2=A0(:,2)'*pi(:,2)
P3=A0(:,2)'*pi(:,3)
P4=A0(:,2)'*pi(:,4)
P5=A0(:,2)'*pi(:,5)
P6=A0(:,2)'*pi(:,6)
P7=A0(:,2)'*pi(:,7)
P8=A0(:,2)'*pi(:,8)
P9=A0(:,2)'*pi(:,9)
P10=A0(:,2)'*pi(:,10)
P11=A0(:,2)'*pi(:,11)
P12=A0(:,2)'*pi(:,12)
```

```
P13=A0(:,2)'*pi(:,13)
P14=A0(:,2)'*pi(:,14)
P15=A0(:,2)'*pi(:,15)
P16=A0(:,2)'*pi(:,16)
P17=A0(:,2)'*pi(:,17)
P18=A0(:,2)'*pi(:,18)
%P=P1+P2+P3+P4+P5+P6+P7+P8+P9+P10+P11+P12+P13+P14+P15+P16+P17+
    P18
r_w1=A0(:,4)'*pi(:,1)/P1 % relativities with a priori ratemaking
r_w2=A0(:,4)'*pi(:,2)/P2
r_w3=A0(:,4)'*pi(:,3)/P3
r_w4=A0(:,4)'*pi(:,4)/P4
r_w5=A0(:,4)'*pi(:,5)/P5
r_w6=A0(:,4)'*pi(:,6)/P6
r_w7=A0(:,4)'*pi(:,7)/P7
r_w8=A0(:,4)'*pi(:,8)/P8
r_w9=A0(:,4)'*pi(:,9)/P9
r_w10=A0(:,4)'*pi(:,10)/P10
r_w11=A0(:,4)'*pi(:,11)/P11
r_w12=A0(:,4)'*pi(:,12)/P12
r_w13=A0(:,4)'*pi(:,13)/P13
r_w14=A0(:,4)'*pi(:,14)/P14
r_w15=A0(:,4)'*pi(:,15)/P15
r_w16=A0(:,4)'*pi(:,16)/P16
r_w17=A0(:,4)'*pi(:,17)/P17
r_w18=A0(:,4)'*pi(:,18)/P18
r_w=[r_w1; r_w2; r_w3; r_w4; r_w5; r_w6; r_w7; r_w8; r_w9; r_w10
    ; r_w11; r_w12; r_w13; r_w14; r_w15; r_w16; r_w17; r_w18]
pr1=B*inv(D1); % probabilities without a prori ratemaking
pr2=B*inv(D2);
pr3=B*inv(D3);
pr4=B*inv(D4);
pr5=B*inv(D5);
pr6=B*inv(D6);
pr7=B*inv(D7);
pr8=B*inv(D8);
```

```
pr9=B*inv(D9);
pr10=B*inv(D10);
pr11=B*inv(D11);
pr12=B*inv(D12);
pr13=B*inv(D13);
pr14=B*inv(D14);
pr15=B*inv(D15);
pr16=B*inv(D16);
pr17=B*inv(D17);
pr18=B*inv(D18);
pr19=B*inv(D19);
pr20=B*inv(D20);
pr21=B*inv(D21);
pr22=B*inv(D22);
pr23=B*inv(D23);
pr=[pr1; pr2; pr3; pr4; pr5; pr6; pr7; pr8; pr9; pr10; pr11;
    pr12; pr13; pr14; pr15; pr16; pr17; pr18; pr19; pr20; pr21;
    pr22; pr23]
prob1=sum(pr(:,1));
prob2=sum(pr(:,2));
prob3=sum(pr(:,3));
prob4=sum(pr(:,4));
prob5=sum(pr(:,5));
prob6=sum(pr(:,6));
prob7=sum(pr(:,7));
prob8=sum(pr(:,8));
prob9=sum(pr(:,9));
prob10=sum(pr(:,10));
prob11=sum(pr(:,11));
prob12=sum(pr(:,12));
prob13=sum(pr(:,13));
prob14=sum(pr(:,14));
prob15=sum(pr(:,15));
prob16=sum(pr(:,16));
prob17=sum(pr(:,17));
prob18=sum(pr(:,18));
```

```
r_wo1=A00(:,1)'*pr(:,1)/prob1 % relativities without a prori
    ratemaking
r_wo2=A00(:,1)'*pr(:,2)/prob2
r_wo3=A00(:,1)'*pr(:,3)/prob3
r_wo4=A00(:,1)'*pr(:,4)/prob4
r_wo5=A00(:,1)'*pr(:,5)/prob5
r_wo6=A00(:,1)'*pr(:,6)/prob6
r_wo7=A00(:,1)'*pr(:,7)/prob7
r_wo8=A00(:,1)'*pr(:,8)/prob8
r_wo9=A00(:,1)'*pr(:,9)/prob9
r_wo10=A00(:,1)'*pr(:,10)/prob10
r_wo11=A00(:,1)'*pr(:,11)/prob11
r_wo12=A00(:,1)'*pr(:,12)/prob12
r_wo13=A00(:,1)'*pr(:,13)/prob13
r_wo14=A00(:,1)'*pr(:,14)/prob14
r_wo15=A00(:,1)'*pr(:,15)/prob15
r_wo16=A00(:,1)'*pr(:,16)/prob16
r_wo17=A00(:,1)'*pr(:,17)/prob17
r_wo18=A0(:,1)'*pr(:,18)/prob18
r_wo=[r_wo1; r_wo2; r_wo3; r_wo4; r_wo5; r_wo6; r_wo7; r_wo8;
    r_wo9; r_wo10; r_wo11; r_wo12; r_wo13; r_wo14; r_wo15; r_wo16;
     r_wo17; r_wo18]
E1=A0(:,5)'*pi(:,1)
E2=A0(:,5)'*pi(:,2)
E3=A0(:,5)'*pi(:,3)
E4=A0(:,5)'*pi(:,4)
E5=A0(:,5)'*pi(:,5)
E6=A0(:,5)'*pi(:,6)
E7=A0(:,5)'*pi(:,7)
E8=A0(:,5)'*pi(:,8)
E9=A0(:,5)'*pi(:,9)
E10=A0(:,5)'*pi(:,10)
E11=A0(:,5)'*pi(:,11)
E12=A0(:,5)'*pi(:,12)
E13=A0(:,5)'*pi(:,13)
E14=A0(:,5)'*pi(:,14)
```

```
E15=A0(:,5)'*pi(:,15)
E16=A0(:,5)'*pi(:,16)
E17=A0(:,5)'*pi(:,17)
E18=A0(:,5)'*pi(:,18)
L1=E1/P1 % Connection between a priori and a posteriori
    corrections
L2=E2/P2
L3=E3/P3
L4=E4/P4
L5=E5/P5
L6=E6/P6
L7=E7/P7
L8=E8/P8
L9=E9/P9
L10=E10/P10
L11=E11/P11
L12=E12/P12
L13=E13/P13
L14=E14/P14
L15=E15/P15
L16=E16/P16
L17=E17/P17
L18=E18/P18

L=[L1; L2; L3; L4; L5; L6; L7; L8; L9; L10; L11; L12; L13; L14;
    L15; L16; L17; L18]
```

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