Ca' Foscari
University
of Venice

Master's Degree programme

in Economics and Finance

"Second Cycle (D.M. 270/2004)"

Final Thesis

Optimal non-proportional reinsurance

under VaR:

the cedant point of view

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Academic Year
2016 / 2017

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Introduction

With the work of Broch (1960) a huge interest started among actuaries and academics in the field of optimal reinsurance. This interest was and is driven by the potential risk management effects of reinsurance as a technique for mitigating and managing insurer risk exposure. This exposure is the sum of the retained risk by the insurer and the premium paid to the reinsurer, which represents a trade-off: the more risk is retained the lower will be the premium owed, on the other hand if the insurer wishes to transfer more risk to the reinsurer, the premium will increase. Therefore, the optimal reinsurance is of fundamental importance to determine the right balance between the risk retention and the risk transfer.

The entrance into force of Solvency II from first January 2016, aims to harmonies the regulatory insurance sector environment within the European Union. It applies to both insurance and reinsurance undertakings operating in the European Union. The Commission delegated regulation (EU) 2015/35 and the quantitative impact study five (QIS5) recognize reinsurance as a risk mitigating technique, so that this risk management tool is becoming more and more important for insurance companies.

The studies of Chi and Tan (2013) propose an optimal reinsurance model from the perspective of an insurance company by minimizing its total risk exposer under value at risk, assuming a large class of premium principles for the calculation of the reinsurance premium that satisfy three basic axioms: distribution invariance, risk loading and stop-loss order preserving.

The work of Tan and Weng (2014), studies an empirical approach to design the optimal reinsurance, assuming the variance as risk measure to minimize the total risk exposer of the insurance undertaking. In this paper, an empirical approach will be applied on the model proposed by Chi and Tan (2013), to study the performance and the results on different loss distributions which are characteristic for insurance losses, and on real data provided by Copenhagen Reinsurance, assuming the expected value premium principle.

The first chapter describes reinsurance, underlying its functions and defining the main possible arrangements. In particular, the difference between facultative and treaty reinsurance is specified and the distinction between proportional and non-proportional forms is explained. In the last part of the first chapter, five principles are reported as a technical guidance for the recognition of reinsurance as a risk mitigation technique, under Solvency II standard formula.

In the second chapter, the underlying assumption of the optimal reinsurance model of Chi and Tan (2013) are illustrated and the derivation of the model is explained in detail, providing the definition of the value at risk and the limited stop-loss reinsurance form.

The final chapter briefly introduces the most common loss distributions that model the insurance claim sizes and thereafter it reports an empirical analysis of the optimal reinsurance model proposed in the second chapter, applying the expected value premium principle. The analysis is performed using the Matlab software, assuming the Monte Carlo method for the expected value calculation for the simulations. In particular, the behaviours and the impacts of the premium principle safety loading, the dispersion of the distribution and the ruin probability on the priority, capacity and value at risk are studied.

In other words, this paper describes the important reinsurance advantages, underlying the principles that recognise reinsurance as a risk mitigating technique under Solvency II. After a deep analysis of the optimal reinsurance model under value at risk studied by Chi and Tan (2013), an empirical analysis is applied on this model to determine the behaviour of the variables of the limited stop-loss reinsurance contract. This empirical analysis is performed using loss distributions that represent real insurance losses and finally the model is tested on real losses of the Danish fire loss data.

I Chapter

1.1. What is reinsurance?

Essentially, reinsurance is insurance for insurance companies. It is a transaction in which an insurance company (the "reinsurer") agrees to indemnify another insurance company (the "reinsured", "cedant" or "primary insurer") for a specified share of specified type of insurance claim of a policy or policies it has issued, in exchange of a premium. The original policyholder is not involved in the reinsurance transaction.

The purpose of insurance is to reduce the financial cost of individuals, companies and other entities emerging from the possible occurrence of contingent events. The insurer selling an insurance policy is committed to indemnify the policyholder for part of the losses arising from these contingent claims. In such a way, individuals, companies and other entities can perform riskier activities, increasing competition, efficiency and innovation. The purpose of reinsurance is similar. It reduces the financial costs of an insurance company, increasing competition, innovation and efficiency in the market. Reinsurance helps protect insurance companies against extraordinary and unpredictable losses by allowing them to spread their risks. Finally, in the same way, also reinsurers can transfer part of their risk to other reinsurers, such a cession is called "retrocession".

The following figure describes how risks are transferred from individuals and corporations to reinsurers, passing through the primary insurer. The last step in the figure 1.1., represents the retrocession, in which the reinsurer enters into a reinsurance agreement with another reinsurer.

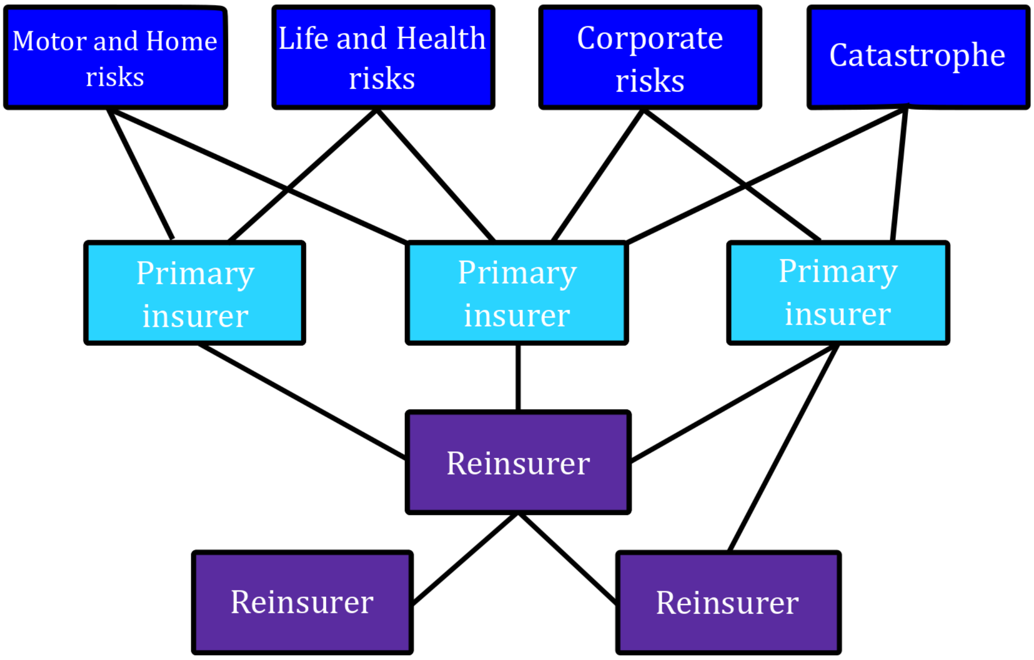

Figure 1.1. The risk transfer from policyholders to reinsurer

The providers of reinsurance are direct writers, brokers, reinsurance departments of the insurer and pools or associations. The writes contract the reinsurance relationship with the primary insurer. Brokers work as intermediaries between the insurance company and the reinsurer, providing the production or sale support. Usually, they represent the cedant and are compensated in form of a commission or fee (generally paid by the reinsurer). Often, they also collect premiums and manage the claim payments. The reinsurance departments of an insurer assume the reinsurance business of an insurance company, typically reinsuring subsidiaries and affiliates. Finally, pools or associations are unions of insurance companies that increase their underwriting capacity, premium capacity or to cover risks which are not insurable in conventional ways. Pools or associations assume a predetermined and fixed interest of the risks jointly underwriting for insurance or reinsurance operations. They are generally run by a separate company, which administrate, underwrite and manage the loss.

1.2. Functions of Reinsurance

Reinsurance does not change the nature of an insurance coverage. In the long run, it cannot transform bad business into good. However, it can provide direct assistance to the insurer. The main reasons for purchasing reinsurance are the following:

- Capacity relief: purchasing reinsurance coverage allows the reinsured to write higher policy limits maintaining manageable risk level. Thus, smaller insurers can write policies beyond their capacity.

- Stabilization: reinsurance can smooth the cedant's underwriting and operating results from year to year and protects the cedant's surplus against shocks from larger unpredictable losses. Usually, the smaller and predictable claims are retained by the reinsured, whereas the reinsurer protects against shares of larger and infrequent claims.

- Surplus relief: in a growing period, an insurance company can have a stressed surplus, by ceding part of its liabilities to the reinsurer it can make use of the reinsurer's surplus. Basically, it is a loan of surplus, so that the cedant can use the reinsurer's surplus until the cedant's surplus is large enough to support the new business.

- Catastrophe protection: reinsurance can provide protection to the reinsured against a large single, catastrophic loss or multiple large losses. This decreases the cedant's probability of financial ruin.

- Expertise and experience: reinsurers have the knowledge and ability to advise their clients, the cedants. This informal consulting service is of self-interest and includes the review of the reinsured's operations to be able to offer advice. The service provided includes assistance on underwriting, marketing, pricing, loss prevention, risk management, claims handling, reserving investment and personnel issues. The reinsurer's contacts with many similar insurance companies provide experience in the pricing of potential high loss policies and the handling of large and rare claims.

- Market entrance: reinsurers help spread the risk of lines of business until the premiums portfolio reaches a certain maturity. Often, reinsurance is taken into consideration when the business is in a new area, where claim history and data are not available.

1.3. Types of reinsurance arrangements and forms

Essentially, there exist two type of reinsurance arrangements: facultative reinsurance and treaty reinsurance. These two types are now described in detail.

1.3.1. Facultative reinsurance

Under facultative reinsurance, the insurance company negotiates a contract for each insurance policy or each single risk (in this case risk is considered as the object under reinsurance protection), it wishes to reinsure. In other words, the reinsurer underwrites individually each contract accepting the risk. Basically, it is the same idea as primary insurance, in which individual risks are underwritten between the insurance company and the policyholder. As the word facultative implies, the reinsurer has the right to accept or reject the individual risk that has been offered by the primary insurance company. The main function is to underwrite large and specific risks to provide additional capacity, usually because the primary insurer is either unwilling or unable to retain all the risk on its own. Often, for the primary insurer it is useful to get facultative reinsurance assistance when it has no experience with a particular risk. This type of arrangement is expensive for the primary insurer, mainly because it allows some degree of adverse selection for the reinsurer, and it is reasonable only if the risks are few. Usually, the duration of the facultative reinsurance depends on the duration of the original policy of the individual risk.

1.3.2. Treaty reinsurance

In treaty reinsurance, also called obligatory reinsurance, the primary insurer purchases reinsurance coverage for a specific portfolio and all risks thereof are automatically ceded, within the terms and conditions of the treaty. The cedant has not to decide to cede each individual risk but he commits to cede part of the portfolio. A treaty reinsurance is a more stable contractual relationship between the cedant and the reinsurer, in which type, terms and conditions of reinsurance are agreed in advance. The risk exposure is usually defined by the annual statement line of business, some variant or subset thereof. The reinsurer does not analyse each individual risk and must cover all risks within the treaty, without the possibility to refuse a specific one. For these reasons, treaty reinsurance is easier to operate and administer and less expensive than the facultative method. The cedant is provided with expertise and services by the reinsurer; usually, the two parties establish a close and long-term working partnership, which makes adverse selection less likely to occur. The duration of a treaty reinsurance depends on the line of business. Usually, non-life reinsurance treaties are renewed annually. For life and health policies (like medical insurance or personal accident insurance), treaty reinsurance is renewed on an annual or five-year basis; whereas for many life insurance products (like term or endowment insurance), the life treaty reinsurance has a duration up to 30 years, depending on the original contracts.

In other words, the reinsurance of a single risk is undertaken arranging facultative reinsurance, whereas treaty reinsurance is preferred for the reinsurance of an entire portfolio.

The characteristics of these two arrangements are summarized in the following table:

Facultative reinsurance **(Individual Risk)**	**Treaty reinsurance** **(Book of Business)**
– Individual risk review – Right to accept or reject each risk on its own merit – A profit is expected by the reinsurer in the short and long term, and depends primarily on the reinsurer's risk selection process – Adapts to short-term ceding philosophy of the insurer – A facultative certificate is written to confirm each transaction – Can reinsure a risk that is otherwise excluded from a treaty – Can protect a treaty from adverse underwriting results	– No individual risk acceptance by the reinsurer – Obligatory acceptance by the reinsurer of covered business – A long-term relationship in which the reinsurer's profitability is expected, but measured and adjusted over an extended period of time – Less costly than "per risk" reinsurance – One treaty contract encompasses all subject risks

Table 1.3.2.1.: Characteristics of facultative and treaty reinsurance (Source: Munich Re (2010))

Besides the choice of the reinsurance arrangement, the insurance company has to decide between proportional and non-proportional reinsurance, which differ in how the premiums and the potential losses are shared. Both facultative and treaty reinsurance can be written on either a proportional or non-proportional basis. Proportional reinsurance involves the reinsurer taking a pre-agreed percentage share of the original premiums and liabilities of the individual risk or portfolio. Whereas, non-proportional reinsurance pays for losses above a fixed amount of the individual risk or portfolio. Usually, it involves a fixed limit up to which the primary reinsurer accounts for all losses on its own, called deductible, retention or priority, and a maximum limit up to which the reinsurer pays the part in excess of the deductible, called capacity or layer.

Now, proportional and non-proportional reinsurance are described in detail.

1.3.3. Proportional reinsurance

The proportional reinsurance contract states the ratio at which premiums and liabilities are shared. The reinsurance company's share of liabilities is directly proportional to the amount of premium received. The reinsurer pays a reinsurance commission to the primary insurer, which are costs related to the acquisition and administration of original policies. Reinsurance commissions are usually a percentage of the optimal premium; however, they can be increased or reduced depending on the quality of the individual risk or portfolio written by the insurer.

There exist two types of proportional reinsurance: quota share reinsurance and surplus share reinsurance.

Quota share is the simplest form of proportional reinsurance. The reinsurer assumes a pre-agreed percentage or quota of the individual policy or policies written by the insurer within the terms of the contract. The retained portion of premiums is a fixed percentage of each policy's premium and the remaining part is ceded to the reinsurer. Losses are shared at the same ratio. This form is particularly suitable for homogenous portfolios, like in the case of motor and household insurance, in which the written risks are all similar.

Quota share reinsurance is usually used by:

- Young and fast-growing insurance companies.
- Insurance companies which enter a new line of business.
- Primary insurance companies which are seeking capital relief due to solvency capital requirements or protection against random fluctuations.

However, this form of reinsurance does not protect against extreme losses, like a catastrophe because if the loss is large the retained proportion is also large, and since it is not flexible the insurer can retain too much or cede too much, at the expense of profitability.

Surplus share is the most used proportional reinsurance form. The primary insurer retains all risks up to a specific amount for each policy in the portfolio, called retention. On the other side, the reinsurer accepts the amount which is in excess of the retention. Usually, it participates to each loss up to a limit amount which is a multiple of the retention. Also in the surplus share reinsurance, the reinsurer pays commissions to the primary reinsurer.

In the past, the commissions were supposed to cover administration and acquiring costs, but since the market place became more competitive, the remaining original premium was not sufficient to cover the total losses incurred. Therefore, many reinsurers adopt the procedure of reimbursing only the original premium that is not paid out for losses to the primary insurer. Surplus share reinsurance is more flexible than quota share reinsurance since it allows the primary insurer to better calibrate reinsurance setting the proper retention limit depending on the type of risk, the size of risk and the overall company's risk appetite. However, it is more complicated and more expensive, since there are significant administration costs. This form of proportional reinsurance is a particularly suitable tool for balancing the reinsurer's portfolio by ceding part of the exposure to single large risks in its portfolio. Nevertheless, such attitude results in possible adverse selection from the point of view of the reinsurer. Moreover, surplus share reinsurance provides the reinsured with capacity to underwrite larger risks, stabilize results and it can minimize the exposure to large losses and catastrophic events.

1.3.4. Non-proportional reinsurance

Conversely to proportional reinsurance, in which the value of the sum insured is considered to determine shares of premiums and liabilities, for non-proportional reinsurance the amount of loss is of primary importance. As the name implies, there is no proportional relationship between the original premium and the premium paid by the primary insurer to the reinsurer. The latter is calculated individually and negotiated by the two parties. The reinsurance premium is influenced by many factors: primary insurer's prior loss experience, potential loss and premium estimates from the book of business, geographic area of business and desired retention level.

As has been seen before, the primary insurer participates to the loss up to the deductible or priority, and the reinsurer contributes to the loss up to the capacity in excess of the priority. The total random loss function can be defined as follows:

$$X^{[P]} = X^{[ret]} + X^{[ced]} \tag{1.1}$$

where $X^{[P]}$ is the total random loss insured by the primary insurer, $X^{[ret]}$ is the retained random loss and $X^{[ced]}$ is the ceded random loss. According to non-proportional reinsurance, the retained loss $X^{[ret]}$ is defined as:

$$X^{[ret]} = \begin{cases} x & if & x \leq d \\ d & if & d < x < d + c \\ x - c & if & x \geq d + c \end{cases} \tag{1.2}$$

where x is the occurred loss, d is the priority and c is the capacity. Whereas the ceded random loss is defined as:

$$X^{[ced]} = \begin{cases} 0 & if & x \leq d \\ x - d & if & d < x < d + c \\ c & if & x \geq d + c \end{cases} \tag{1.3}$$

The advantages of non-proportional reinsurance are:

- The priority can limit the liabilities reflecting the capacity and the risk appetite of the insurer.
- Potential earnings are greater because the premium paid is lower, since small losses are retained by the insurer and are not ceded to the reinsurer.
- The administration is less complicated and less expensive for both parties.
- The reinsurer is able to define the price of risk on its own and it does not depend on the original premium.

The main forms of non-proportional reinsurance are:

- Excess of loss cover, which includes: per risk working excess of loss (in short WXL/R) and catastrophe excess of loss (in short CatXL) or per event excess of loss.
- Stop-loss cover.

Under insurance cover, the claim payment depends on a defined insured loss event having taken place. The loss occurrence and the amount may vary and depend on the insured peril and on the line of business. Therefore, an excess of loss reinsurance cover must be designed differently depending on the various types of losses that can take place in a line of business. For example, a building destroyed by a fire is a huge loss for the insurance industry, as well as many small losses caused by a windstorm. Therefore, the design of the right excess of loss reinsurance is fundamental. Now, to better understand the differences, the different forms of non-proportional reinsurance are described in detail.

The ***per risk working excess of loss*** is used by insurers whenever they seek to limit losses on any one risk. The term “working” indicates that the cover is triggered by a loss on a single risk. In other words, the cedant is indemnified up to a certain limit (capacity), against the amount of loss in excess of the priority, with respect to the risk involved in each loss. It protects against large losses involving any one risk and it is very effective as risk mitigation tool (often used in fire reinsurance). The WXL/R is not effective for frequent and cumulative losses, where many policies are triggered by the same event, such as a natural catastrophe (Bumann (1997)).

The ***excess of loss per event*** is designed to provide protection against accumulation of losses affecting several risks, independently from the number of risks. The term “loss event” defines the number of risks affected by the loss that trigger the treaty. This term must be well defined in each excess of loss reinsurance treaty.

It is important to notice that this cover must be negotiated by the insurer in such a way that the trigger is not a single loss on a single risk. The essential difference with the per risk excess of loss is that the unit of loss is not the individual loss of each policy but the aggregate loss of the portfolio caused by and event, defined in the reinsurance treaty.

The CatXL is an effective instrument for risk mitigation of large catastrophe losses, made up by the sum of hundreds of thousands of relative small losses caused by the same event (Brahin et al (2013)).

Finally, ***the stop-loss reinsurance*** is a less frequent form of treaty reinsurance, in which the reinsurer covers any loss of the total annual loss which exceed the priority (often defined as a percentage of the annual premium). Primary insurers use this form of coverage to protect itself against large claims fluctuations at the expense of potential earning, therefore it is not used to guarantee profits. Usually, it is chosen when the insurer's claims and administration cost has been higher than premiums (called technical loss). Stop-loss reinsurance allows the most comprehensive protection compared to other non-proportional reinsurance forms.

The following table summarizes the different types of reinsurance depending on facultative or treaty reinsurance and on proportional and non-proportional reinsurance.

	Proportional	**Non-proportional**
Facultative reinsurance (per risk)	Quota share Surplus share	WXL/R
Treaty reinsurance (per portfolio)		CatXL Stop-loss

Table 1.2.: Reinsurance forms and arrangements (Source: Olivieri and Pitacco (2011))

1.4. Recognition of reinsurance under Solvency II

Actually, many insurance companies use risk mitigation techniques to reduce capital requirements and to stabilize their earnings. Under Solvency II, reinsurance is recognized as a risk mitigation instrument. Art. 208 of Commission Delegated Regulation (EU) 2015/35 states that insurance and reinsurance undertakings, using reinsurance or special purpose vehicles to transfer risks, can benefit from these risk mitigation effects, allocating them to the scenario-based calculation in a manner that captures the economic effect of the protection provided, without double-counting (meeting the requirements of art. 209, art. 211 and art. 213 of this regulation).

Under the Solvency II standard formula, the following principles can be considered as a technical guidance, for the recognition of reinsurance as a risk mitigation technique (Swiss Re (2011)):

- If a transaction from one company to another is recognised in a legal form or accounting treatment (for example, IFRS), but there is no or just little risk transfer from an economical point of view, the risk transfer is not considered as a reduction in risk and hence it gives no or very little capital relief. Moreover, the additional risk of the transfer should be taken into consideration, as well as the basic risk, when entering into a reinsurance transaction. This principle assures the protection of policyholders and that risks are valued on a real economic basis.

- The contractual arrangement between the two parties and the risk transfer must be clearly defined, legally effective and enforceable in all relevant jurisdictions. This principle protects policyholder as the previous one and avoid that the economic effects of the risk transfer are disputable.

- The risk transfer from the primary insurer to the reinsurer should be valued using sound economic principles and at real market value of asset and liabilities. This principle assures a true picture of the insurer's balance sheet value, allowing a proper cash flow if the reinsurance contract is triggered.

- The reinsurer must have a solvency ratio higher than 100% and at least a BBB credit rating. It guaranties that the primary insurer is entering into a transaction with a creditworthy party and that the policyholders are protected. The credit rating influences the capital relief: the higher it is the higher the potential capital relief, and vice versa. Moreover, a higher credit rating of an individual reinsurer has a higher weight for the primary insurer than many diversified reinsurers, holding other things constant.
- The last principle covers only financial risk mitigating techniques, as ILSs. It defines when they can be used as capital relief. It requires the reinsured to have a direct claim on the reinsurer, the clauses of the contract cannot be outside the control of the primary insurer and the terms and conditions of the cover must be clearly defined. The principle assures the protection of the policyholders and the avoidance of disputes for the economic effect of the transfer.

Given the new framework of Solvency II, in which insurance and reinsurance undertakings have to operate from the 1st January 2016, and the new restrictive solvency capital requirements it causes, the recognition of reinsurance as a risk mitigation technique is very important to reduce these limits. Therefore, the optimization of reinsurance is of fundamental importance for insurance and reinsurance companies that seek such risk mitigation technique.

II Chapter

Due to the important rule of reinsurance as a recognised risk mitigation technique, in this chapter a model will be presented to find the optimal reinsurance under value-at-risk. The purposed model has been studied by Chi and Tan (2013), which allows the application of eight premium principles (see chapter three, section 3.2.). Therefore, it is a more general model, whereas Chi and Tan (2007), Chi and Tan (2011) and Cai et all (2008) consider a specific premium principle, the expected value premium principle, for the optimal reinsurance under value at risk.

2.1. Preliminaries and Assumptions

Let X denote the (aggregate) amount of loss initially assumed by the insurer (i.e. before underwriting a reinsurance contract with the reinsurer). The amount X represents the set of the possible outcomes that the insurer is obliged to pay to the beneficiary, for this reason it is possible to assume that X is a non-negative random variable. It is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with cumulative distribution function (cdf) $F_x(x) = \mathbb{P}(X \leq x)$, survival function $S_X(\mathrm{x}) = \left(1 - F_x(x)\right) = \mathbb{P}(X > x)$ and expected value $0 < \mathbb{E}[X] < \infty$ (Cai et al. (2008)). The cdf is a right-continuous and non-decreasing[1] function with:

$$F_X(0) = \lim_{x \to 0} F_X(x) = \lim_{x \to 0} \mathbb{P}(\mathrm{X} \leq \mathrm{x}) = 0 \qquad (2.1)$$

$$F_X(\infty) = \lim_{x \to 0} F_X(x) = \lim_{x \to \infty} \mathbb{P}(\mathrm{X} \leq \mathrm{x}) = 1. \qquad (2.2)$$

[1] In this paper, increasing and decreasing mean non-decreasing and non-increasing respectively.

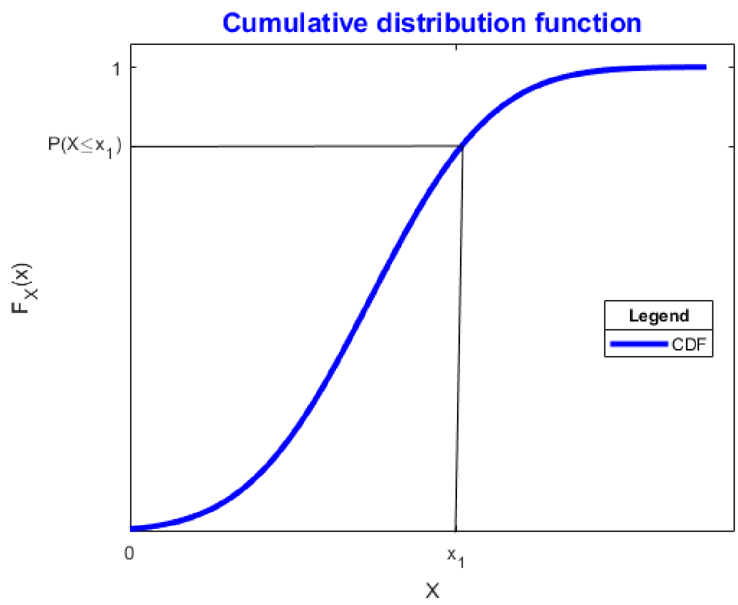

Figure 2.1.: The cumulative distribution function

Under a reinsurance agreement, the insurer (the cedant) transfers part of its loss exposure to another insurer (the reinsurer). Thus, the loss X is split between the two parties:

$$X = R_f(X) + f(X) \tag{2.3}$$

where the $R_f(X)$ is the residual loss retained by the insurer and $f(X)$ is the loss ceded to the reinsurer, satisfying $0 \leq f(X) \leq X$. Consequently, $R_f(x)$ is the retained loss function and $f(x)$ is known as the ceded loss function, satisfying $0 \leq f(x) \leq x$. The optimal reinsurance problem is regarded with the optimal proportioning of the loss X between $R_f(X)$ and $f(X)$. More precisely, the choices of the constraints on the ceded loss function as well as the risk measure show interesting insights on the optimal design of reinsurance treaties. The latter will be considered in the next pages, whereas for what concerns the former, it is possible to consider three feasible classes (Chi and Tan (2011)):

- $f(x)$ is an increasing convex function.
- Both $f(x)$ and $R_f(x)$ are increasing functions.
- $R_f(x)$ is an increasing and left continuous function.

Since an insurance contract is an agency relationship with possible asymmetry of information, there exists moral hazard, which can be seen as the change of an individual's behaviour after entering into a contract. The second mentioned class partially avoids moral hazard, so that as the loss X increases both parties are obligated to pay more. Formally, the set of admissible ceded loss functions $\mathcal{C}$ is defined as:

$$\mathcal{C} \triangleq \{0 \leq f(x) \leq x : both\ R_f(x)\ and\ f(x)\ are\ increasing\ functions\}. \quad (2.4)$$

The ceded loss function has the property of been not only increasing but also Lipschitz continuous; this means that:

$$0 \leq f(x_a) - f(x_b) \leq x_a - x_b \quad \forall\ 0 \leq x_a \leq x_b. \quad (2.5)$$

In addition, the non-proportional reinsurance equation (1.1) is contained in $\mathcal{C}$ and as shown in Chi and Tan (2011) this set of admissible ceded loss functions contains the set of increasing convex ceded loss functions.

As it has been seen in the first chapter, by underwriting a reinsurance contract, the cedant incurs in an additional cost payable to the reinsurer. The premium principle used $\pi(.)$ is a function from the set of non-negative random variables $\mathcal{Z}$ to the set of non-negative real numbers $\mathbb{R}^+$. In this case the reinsurance premium $\pi\big(f(X)\big)$ is function of the ceded loss function $f(X)$. It is defined as follows:

$$\pi: f(X) \to \mathbb{R}^+. \quad (2.6)$$

According to Chi and Tan (2013), the premium principle must satisfy three weak but necessary axioms: distribution invariance, risk loading and stop-loss order preserving. They are defined as:

- Distribution invariance: For $Y \in \mathcal{Z}$, $\pi(Y)$ depends only on the cdf $F_Y(y)$. It is an implicit assumption in actuarial science.
- Risk loading:

  $$\pi(Y) \geq \mathbb{E}[Y], \forall \text{ all } Y \in \mathcal{Z}. \quad (2.7)$$

  It is applied to assure the safety of the reinsurance company, otherwise it will go bankrupt.
- Stop-loss ordering preserving: For $Y, X \in \mathcal{Z}$, $\pi(Y) \leq \pi(X)$, if Y is smaller than X in the stop-loss order, that is if:

  $$\mathbb{E}\big[\max\big((Y-d), 0\big)\big] \leq \mathbb{E}\big[\max\big((X-d), 0\big)\big]\ \forall\ d \in \mathbb{R} \quad (2.8)$$

  provided that the expectations exit. It is denoted as $Y \leq_{sl} X$.

Under the presence of reinsurance, the insurer total risk exposure $T_f(X)$ becomes:

$$T_f(X) = R_f(X) + \pi\big(f(X)\big). \tag{2.9}$$

This equation shows that the insurer is no longer exposed to the whole risk X, but to the sum of the premium paid to the reinsurer $\pi\big(f(X)\big)$ and the residual loss retained $R_f(X)$. This new risk exposure of the insurer clearly shows a trade-off:

- If the ceded loss $f(X)$ is small, then the premium paid to the reinsurer $\pi\big(f(X)\big)$ will be relatively low, but the residual loss retained $R_f(X)$ will be relatively high.
- On the other hand, if the insurer will decrease the residual loss retained exposure $R_f(X)$, the ceded loss function $f(X)$ increases as well as the premium to be paid to the reinsurer $\pi\big(f(X)\big)$.

For this reason, a criterion should be chosen to determine the optimal ceded loss function f^* in the total risk exposure of the insurer $T_f(X)$. The criterion consists in choosing an appropriate risk measure φ that minimizes the total risk exposure of the insurer $T_f(X)$:

$$\varphi\left(T_{f^*}(X)\right) = \min_{f \in \mathcal{C}} \varphi\left(T_f(X)\right). \tag{2.10}$$

The risk measure used for this purpose is the Value at Risk (VaR):

Definition 2.1. The Value at Risk of a non-negative random variable X at confidence level $1-\alpha$, where $0 < \alpha < 1$, is defined as follows:

$$VaR_\alpha(X) \triangleq \inf\{x \geq 0 : \mathbb{P}(X > x) \leq \alpha\} \triangleq \inf\{x \geq 0 : F_X(x) \leq 1-\alpha\}. \tag{2.11}$$

The value-at-risk is also defined as the quantile risk measure, since $VaR_\alpha(X)$ is exactly the $(1-\alpha) - quantile$ of the random variable X.

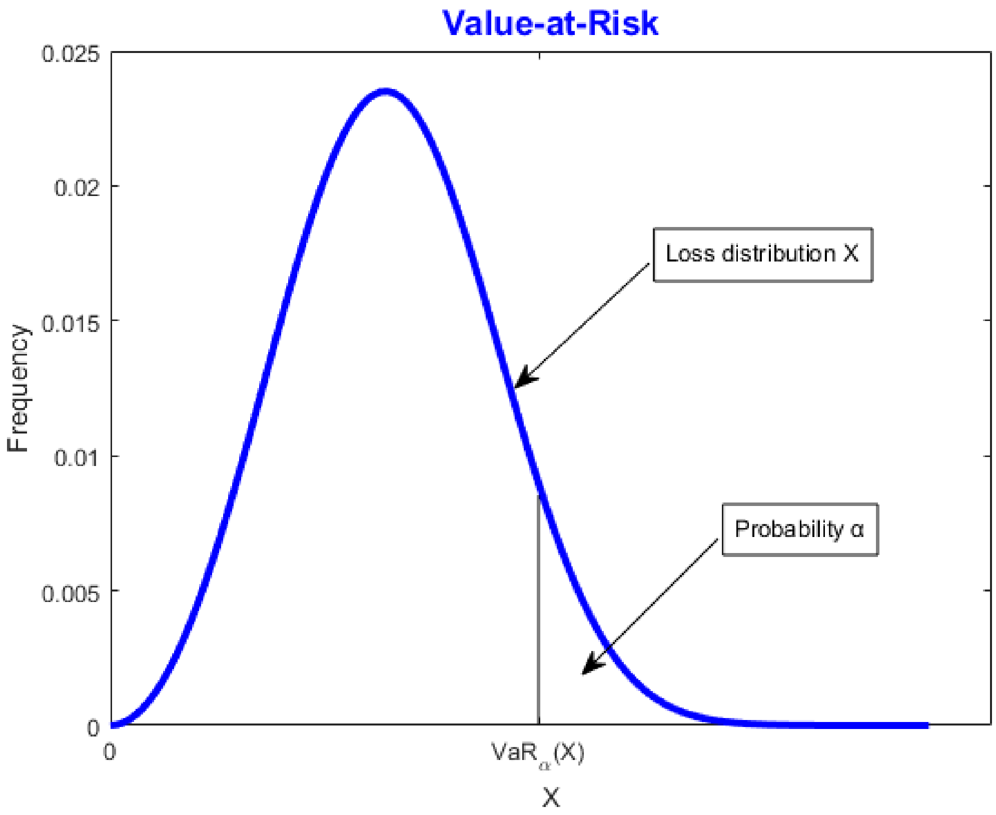

Figure 2.2.: The value-at-risk

From the definition (2.11), it follows that:

$$VaR_\alpha(X) \leq x \Leftrightarrow S_X(x) \leq \alpha \;\; \forall x \in \mathbb{R}^+ \tag{2.12}$$

$$VaR_\alpha(X) \leq x \Leftrightarrow \mathbb{P}(X > x) \leq \alpha \; \forall x \in \mathbb{R}^+. \tag{2.13}$$

This means that if $x = 0$ then the $VaR_\alpha(X) = 0$ for $\alpha \geq S_X(0)$. In order to avoid the discussion of trivial cases, it is assumed that the parameter α is restricted to $0 < \alpha < S_X(0)$.

For the purpose of this project, the VaR has another important property, discussed in the Dhaene et all (2002). It is possible to prove that for any given increasing continuous function ψ,

$$VaR_\alpha\big(\psi(X)\big) = \psi\big(VaR_\alpha(X)\big). \tag{2.14}$$

In this paper, the VaR is taken into consideration for several reasons:

- Huge interest among practitioners and academicians for this risk measure.
- Widely used within the insurance and banking industry for quantifying market risk, portfolio optimization, setting capital adequacy and so forth.
- It has become the benchmark risk measure in the financial world because regulators accept this model as the basis to set capital requirements.

The concept of VaR answers the following question: how much is it expected to lose in a specified period of time with a given probability. For example, if a portfolio of an insurer has a yearly $VaR_{0.005} = € \, 10 \, million$ then a loss of $€ \, 10 \, milion$ or more is expected every 200 years.

2.2. Value-at-risk optimal reinsurance model

Summarizing, the optimal reinsurance model subject to the feasible set of ceded loss functions $\mathcal{C}$ becomes:

$$VaR_\alpha\left(T_{f^*}(X)\right) = \min_{f \in \mathcal{C}} VaR_\alpha\left(T_f(X)\right). \tag{2.15}$$

To solve this model, a limited stop-loss reinsurance treaty is constructed for any given ceded loss function $f \in \mathcal{C}$, such that it is better than f, in the sense of minimizing the VaR of the total exposure of the reinsurer. This limited stop-loss reinsurance treaty denoted by $h_f(x)$, is defined as follows:

$$h_f(x) \triangleq \min\{\max[x - VaR_\alpha(X) - f(VaR_\alpha(X)), 0], f(VaR_\alpha(X))\}, with\ x \geq 0 \tag{2.16}$$

or equivalently,

$$h_f(x) \triangleq \begin{cases} 0 & if \quad x \leq VaR_\alpha(X) - f(VaR_\alpha(X)) \\ x - VaR_\alpha(X) - f(VaR_\alpha(X)) & if \quad VaR_\alpha(X) - f(VaR_\alpha(X)) < x < VaR_\alpha(X) \\ f(VaR_\alpha(X)) & if \quad x \geq VaR_\alpha(X) \end{cases} \tag{2.17}$$

with $x \geq 0$, where $VaR_\alpha(X) - f(VaR_\alpha(X))$ is the deductible and $f(VaR_\alpha(X))$ is the upper limit.

The retained loss function purchasing insurance coverage becomes:

$$R_{h_f}(x) = \begin{cases} x & if \quad x \leq VaR_\alpha(X) - f\big(VaR_\alpha(X)\big) \\ VaR_\alpha(X) - f\big(VaR_\alpha(X)\big) & if \quad VaR_\alpha(X) - f\big(VaR_\alpha(X)\big) \leq x \leq VaR_\alpha(X) \\ x - f\big(VaR_\alpha(X)\big) & if \quad x \geq VaR_\alpha(X). \end{cases} \quad (2.18)$$

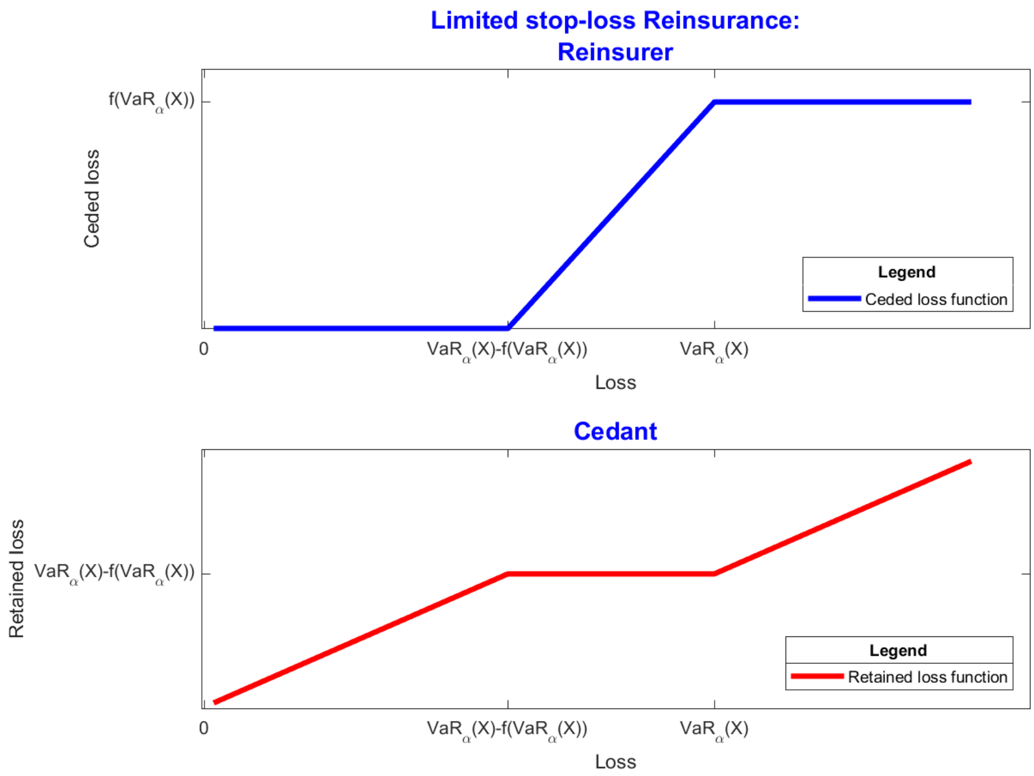

Figure 2.3.: The limited stop-loss reinsurance

Moreover, if $x = VaR_\alpha(X)$ then

$$h_f\big(VaR_\alpha(X)\big) = f\big(VaR_\alpha(X)\big) \quad (2.19)$$

and

$$h_f(x) \in \mathcal{C}. \quad (2.20)$$

Now it is possible to write the total risk exposure of the insurer with the corresponding ceded loss function $h_f(x)$ as follows:

$$T_{h_f}(X) = R_f(X) + \pi\left(h_f(X)\right). \quad (2.21)$$

Comparing the $VaR_\alpha\left(T_f(X)\right)$ and $VaR_\alpha\left(T_{h_f}(X)\right)$ the following important result can be proven:

Theorem 2.1. For the VaR-based optimal reinsurance model (2.15), the limited stop-loss reinsurance $h_f(x)$ is optimal in the sense that:

$$VaR_\alpha\left(T_f(X)\right) \geq VaR_\alpha\left(T_{h_f}(X)\right) \quad \forall\, f \in \mathcal{C} \tag{2.22}$$

Proof:

First of all it is possible to prove that for any ceded loss function $f \in \mathcal{C}$, the limited stop-loss reinsurance treaty $h_f(x)$ is always smaller or equal:

$$f(x) \geq h_f(x)\ \forall\, x \geq 0. \tag{2.23}$$

Given that $f(x) \leq x$ and if $x \leq VaR_\alpha(X)$, it implies that:

$$f(x) \leq VaR_\alpha(X). \tag{2.24}$$

Since the ceding loss function $f \in \mathcal{C}$ is Lipschitz-continuous and non-negative, it is possible to state:

$$f\big(VaR_\alpha(X)\big) - f(x) \leq VaR_\alpha(X) - x \tag{2.25}$$

$$-f(x) \leq VaR_\alpha(X) - x - f\big(VaR_\alpha(X)\big) \tag{2.26}$$

$$f(x) \geq x - VaR_\alpha(X) + f\big(VaR_\alpha(X)\big). \tag{2.27}$$

Recalling again that $x \leq VaR_\alpha(X)$, the right-hand side can be rewritten as:

$$f(x) \geq \max\big(x - VaR_\alpha(X) + f\big(VaR_\alpha(X)\big), 0\big). \tag{2.28}$$

Because for $0 \leq x \leq VaR_\alpha(X)$ the value of the ceded loss function is non-negative, and hence $\max\big(x - VaR_\alpha(X) + f\big(VaR_\alpha(X)\big), 0\big)$. Since from the definition of $h_f(x)$, in this partition of the domain the $\max\big(x - VaR_\alpha(X) + f\big(VaR_\alpha(X)\big), 0\big)$ coincide with $h_f(x)$:

$$h_f(x) = \max\big(x - VaR_\alpha(X) + f\big(VaR_\alpha(X)\big), 0\big)\ \forall\, 0 \leq x \leq VaR_\alpha(X). \tag{2.29}$$

Hence:

$$f(x) \geq h_f(x)\ \forall\, 0 \leq x \leq VaR_\alpha(X). \tag{2.30}$$

On the other side, given $x \geq VaR_\alpha(X)$ and due to the increasing property of $f(x)$, it implies:

$$f(x) \geq f\big(VaR_\alpha(X)\big). \tag{2.31}$$

Hence:

$$f(x) \geq f\big(VaR_\alpha(X)\big) = h_f(x),\ \forall\, x \geq VaR_\alpha(X). \tag{2.32}$$

It has been proven that $h_f(X) \leq f(X)$ in the usual stochastic order $\forall\, x$. This means that $h_f(X)$ is less likely than $f(X)$ to take on larger values and this is true for all values of x (Denuit et al (2005)). Furthermore, since the premium function $\pi(.)$ maintains the stop-loss order property, it follows that $\pi\left(h_f(X)\right) \leq \pi\big(f(X)\big)$.

The value at risk of the total risk exposure of the insurer $T_f(X)$, due to the translation invariance property of VaR, can be written as:

$$VaR_\alpha\left(T_f(X)\right) = VaR_\alpha\left(R_f(X)\right) + \pi\big(f(X)\big). \tag{2.33}$$

From the property (2.14), the $VaR_\alpha\left(R_f(X)\right) = R_f\big(VaR_\alpha(X)\big)$, substituting this equality into the equation (2.33), it follows

$$VaR_\alpha\left(T_f(X)\right) = R_f\big(VaR_\alpha(X)\big) + \pi\big(f(X)\big). \tag{2.34}$$

Recalling that $R_f(X) = X - f(X)$, it is possible to write $R_f\big(VaR_\alpha(X)\big) = VaR_\alpha(X) - f\big(VaR_\alpha(X)\big)$ and it is substituted into the equation:

$$VaR_\alpha\left(T_f(X)\right) = VaR_\alpha(X) - f\big(VaR_\alpha(X)\big) + \pi\big(f(X)\big). \tag{2.35}$$

Since the $h_f\big(VaR_\alpha(X)\big) = f\big(VaR_\alpha(X)\big)$, the equation can be rewritten as:

$$VaR_\alpha\left(T_f(X)\right) = VaR_\alpha(X) - h_f\big(VaR_\alpha(X)\big) + \pi\big(f(X)\big). \tag{2.36}$$

Since the insurance company purchase the limited stop-loss reinsurance treaty, the premium to be paid to the reinsurer is the limited stop-loss reinsurance premium $\pi\left(h_f(X)\right)$ and since $\pi\left(h_f(X)\right) \leq \pi\big(f(X)\big)$, it implies that:

$$VaR_\alpha\left(T_f(X)\right) \geq VaR_\alpha(X) - h_f\big(VaR_\alpha(X)\big) + \pi\left(h_f(X)\right) \tag{2.37}$$

$$VaR_\alpha\left(T_f(X)\right) \geq VaR_\alpha\left(T_{h_f}(X)\right). \tag{2.38}$$

Hence, the limited stop-loss reinsurance of the form (2.16) is optimal.

Now, setting $d = VaR_\alpha(X) - f\big(VaR_\alpha(X)\big)$, where d represents the deductible, it follows that:

$$0 \leq d \leq VaR_\alpha(X). \tag{2.39}$$

As reductio ab absurdum, if $d > VaR_\alpha(X)$ it would imply that:

$$VaR_\alpha(X) - h_f\big(VaR_\alpha(X)\big) > VaR_\alpha(X) \tag{2.40}$$

$$-h_f\big(VaR_\alpha(X)\big) > VaR_\alpha(X) - VaR_\alpha(X) \tag{2.41}$$

$$h_f\big(VaR_\alpha(X)\big) < 0. \tag{2.42}$$

This can never be true since $h_f(x) \in \mathcal{C}$ and so it assumes only non-negative value.

The VaR-based optimal reinsurance model becomes:

$$\min_{f\in\mathcal{C}_v} VaR_\alpha\left(T_f(X)\right) = \min_{0\leq d\leq VaR_\alpha(X)} \{d + \pi(min\{max\{X-d,0\}, VaR_\alpha(X) - d\})\}. \tag{2.43}$$

where

$$\mathcal{C}_v \triangleq \{\min\{max\{X-d,0\}, VaR_\alpha(X) - d\} :\ 0 \leq d \leq VaR_\alpha(X)\}. \tag{2.44}$$

This set $\mathcal{C}_v$ contains all admissible limited stop-loss reinsurance treaties $h_f(x)$ and satisfies the condition: $\mathcal{C}_v \subseteq \mathcal{C}$.

III Chapter

In this chapter, an empirical analysis will be provided of the theoretical results of the previous chapter, taking into consideration different possible distributions to model the insurance's losses (called loss distributions) and the impact of the different parameters will be analysed. Finally, using the Danish fire loss data provided by Copenhagen Reinsurance the different limited stop-loss reinsurance variables of the model (2.43) will be analysed.

Considering insurance losses from a single policy or a portfolio, the possible random values are non-negative and the distribution that best fit these losses are positively skewed and very often they have high probabilities in the right-hand tails. These distributions can be described as long tailed or heavy (fat) tailed (Gray and Pitts, 2012). Distributions with fat tails are suitable to model claim sizes, since they allow losses to take very high values.

The distributions that meet these conditions and are representative for insurance losses are:

- Lognormal.
- Pareto.
- Weibull.
- Exponential.
- Gamma.

These models are informative to the insurance and reinsurance companies and provide them tools to make decisions on premium loading, expected profits, reserves and the impact of reinsurance and priorities (Achieng, 2010). Moreover, according to Packová and Brebera (2015), the Pareto distribution is often used for modelling insurance losses and it plays a central role in quoting non-proportional reinsurance.

In chapter 2, the VaR-based optimal reinsurance model has been provided (2.43); it is a minimization problem in one variable (d the priority), which will be solved using Matlab. Matlab (Math laboratory) is a software environment for engineers and scientists developed by MathWorks, which is used for matrix and array computations, to develop and run algorithms, for data visualization and for many other issues such as optimization and graphical representations useful for the purpose of this project.

3.1. Loss distribution

This section will briefly analyse the five different distributions in order to better understand their characteristics for the following analysis.

The probability distribution can be characterized by different parameters depending on the particular distribution. These parameters are:

- Location parameter: it shifts the distribution to the right or left without changing the shape or the volatility.
- Scale parameter: it quantifies the dispersion of the random variable and its inverse quantifies the precision of it.
- Shape parameter: it is any parameter that is not changed by the changes of the location or scale parameters. It describes the shape of the graph for particular distributions. Often the skewness or tail weight of a distribution can be specified by the shape parameters (Ruppert (2010)).

3.1.1. Exponential distribution

It is described by one parameter, the scale one $\lambda(> 0)$, and it is considered a sub-family of the gamma distribution (for $\alpha = 1$, see below gamma distribution). It is denoted as:

$$X \sim Exp(\lambda)$$

The probability density function of the exponential family is defined as:

$$f(x) = \lambda e^{-\lambda \cdot x}, \quad x > 0 \tag{3.1}$$

The expected value and the variance are defined as follows:

$$\mathbb{E}(X) = \frac{1}{\lambda} \tag{3.2}$$

and

$$\mathbb{V}ar(X) = \frac{1}{\lambda^2}. \tag{3.3}$$

3.1.2. Gamma distribution

A gamma distribution is characterized by a shape parameter $\alpha(> 0)$ and a scale parameter $\sigma(> 0)$. It is denoted as:

$$X \sim G(\alpha, \sigma).$$

It is considered a gamma distribution if the probability distribution function is equal to:

$$f(x) = \frac{\sigma^{\alpha}}{\Gamma(\alpha)} \cdot x^{\alpha-1} \cdot e^{-\sigma \cdot x}, \quad x > 0 \tag{3.4}$$

where $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} \cdot e^{-x} dx$. This is the gamma function.

The parameter α determines the skewness and kurtosis of the distribution: the more the it increases the more the random variable becomes symmetrical and tend to a normal distribution. For $\alpha > 1$, the density decreases more slowly than the exponential distribution and due to the second parameter, it is more flexible in modelling data than a one parameter distribution like the exponential one.

The expected value and the variance of this distribution are defined as:

$$\mathbb{E}(X) = \sigma \cdot \alpha \quad and \quad \mathbb{V}ar(X) = \sigma^2 \cdot \alpha. \qquad \text{(3.5) and (3.6)}$$

3.1.3. Lognormal distribution

This family of distributions is described by two parameters, the location parameter μ and scale parameter σ, which are sometimes called "mean log" and "standard deviation log", respectively. It is denoted as:

$$X \sim lognomal\ (\mu, \sigma)$$

with probability density function:

$$f(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot \frac{1}{x} \cdot \exp\left\{-\frac{1}{2} \cdot \left(\frac{log(x) - \mu}{\sigma}\right)^2\right\}, \quad x > 0. \tag{3.7}$$

The name of this distribution derives from the fact that:

$$X \sim lognormal(\mu, \sigma) \Leftrightarrow Y = \log(X) \sim \mathrm{N}(\mu, \sigma^2). \tag{3.8}$$

This means that $\log(X)$ is normally distributed with mean μ and standard deviation σ.

The expected value and the variance of the lognormal distribution can be defined as:

$$\mathbb{E}(X) = e^{\mu+\frac{\sigma^2}{2}} \quad and \quad \mathbb{V}ar(X) = \left(e^{\sigma^2} - 1\right) \cdot \left(e^{2\mu+\sigma^2}\right). \qquad \text{(3.9) and (3.10)}$$

3.1.4. Pareto distribution

The Pareto family has several sub-families. The one with shape parameter $\alpha(> 0)$ and scale parameter $\sigma(> 0)$, sometimes called "American Pareto", is widely used in modelling losses in general reinsurance. It is denoted as:

$$X \sim Pa(\alpha, \sigma).$$

The probability density function of the Pareto distribution is defined as:

$$f(x) = \frac{\alpha \cdot \sigma^{\alpha}}{(\sigma + x)^{\alpha+1}}, \quad x > 0. \qquad (3.11)$$

The exponential, gamma and lognormal distributions tail off faster than the Pareto one.

Here, the expected value and the variance of the Pareto distribution can be defined as:

$$\mathbb{E}(X) = \frac{\alpha \cdot \sigma}{\alpha - 1}, \quad with\ \alpha > 1 \qquad (3.12)$$

and

$$\mathbb{V}ar(X) = \frac{\alpha \cdot \sigma^2}{(\alpha-1)^2 \cdot (\alpha-2)}, \quad with\ \alpha > 2. \qquad (3.13)$$

3.1.5. Weibull distribution

It is a family of distributions characterized by two parameters: shape parameter $\alpha(> 0)$ and scale parameter $\sigma(> 0)$. If $\sigma = 1$, the Weibull distribution is equal to the exponential one. It is denoted as:

$$X \sim Wei(\alpha, \sigma).$$

The probability density function is defined as follows:

$$f(x) = \alpha \cdot \sigma \cdot x^{\sigma-1} \cdot e^{-\alpha \cdot x^{\sigma}}, \quad x > 0. \qquad (3.14)$$

The expected value of this distribution is defined as:

$$\mathbb{E}(X) = \Gamma\left(1 + \frac{1}{\sigma}\right) \cdot \alpha^{-\frac{1}{\sigma}}. \qquad (3.15)$$

Whereas, the variance is defined as:

$$\mathbb{V}ar(X) = \Gamma\left(1+\frac{2}{\sigma}\right)\cdot\alpha^{-\frac{2}{\sigma}} - \Gamma\left(1+\frac{1}{\sigma}\right)\cdot\alpha^{-\frac{1}{\sigma}}. \tag{3.16}$$

3.2. Empirical analysis

For the empirical analysis of the VaR-based optimal reinsurance model (2.43), a premium principle for the calculation of the reinsurance premium must be assumed. According to Chi and Tan (2013), the premium principles that satisfy the three basic axioms analysed in the previous chapter are: net, expected value, exponential, proportional hazard, principle of equivalent utility, Wang's, Swiss, and Dutch's. These eight premium principles have different assumptions and the used of one instead of another could lead to a different optimal solution.

In this paper, the expected value premium principle will be assumed, since it has remained the most fundamental and widely used (Cai et al (2008)). It is defined as follows:

$$\pi(X,\omega) = (1+\omega)\cdot\mathbb{E}(X) \tag{3.17}$$

where $\omega\ (>0)$ is the so-called safety loading. It is a parameter that increases the premium and so the potential profit of the insurance or reinsurance undertaking, leading the basic axiom of risk loading to satisfy $\pi(Y) > E(Y)\ for\ all\ Y \in \mathcal{Z}$, stronger condition than (2.7). It is one of the simplest premium principles.

Substituting this premium principle (3.17) in the equation (2.43), the optimization problem becomes:

$$\min_{f\in\mathcal{C}_v} VaR_\alpha\left(T_f(X)\right) = \min_{0\le d\le VaR_\alpha(X)}\{d + (1+\omega)\cdot\mathbb{E}_X(min\{max\{X-d,0\}, VaR_\alpha(X)-d\})\}. \tag{3.18}$$

The implementation of the optimization problem in Matlab follows the Monte Carlo method, which is very often used for the optimization and for generation of draws from a probability distribution. The expected value can be seen as:

$$\frac{1}{N}\cdot\sum_{i=1}^{N} f\left(X^i\right) \xrightarrow{N\longrightarrow\infty} \mathbb{E}_X\left[f(X)\right],\ for\ X^i \sim^{iid} D(.) \tag{3.19}$$

where $D(.)$ Is a given distribution.

Moreover, the optimization problem (3.18) can be seen as:

$$\min_{f \in \mathcal{C}_v} VaR_\alpha\left(T_f(X)\right) = \min_{0 \le d \le VaR_\alpha(X)} \{d + (1+\omega) \cdot \mathbb{E}_X[g(X^i, d)]\} \quad (3.20)$$

where,

$$g(X^i, d) = min\{max\{X^i - d, 0\}, VaR_\alpha(X) - d\}. \quad (3.21)$$

Substituting (3.19) in the optimization model (3.18), it is possible to write:

$$\min_{f \in \mathcal{C}_v} VaR_\alpha\left(T_f(X)\right) \simeq \min_{0 \le d \le VaR_\alpha(X)} \left\{d + (1+\omega) \cdot \frac{1}{N} \cdot \sum_1^N g(X^i, d)\right\} \quad (3.22)$$

for $X^i \sim^{iid} D\,(.)\ \forall\ i = 1, \dots, N$. Substituting the equation (3.21) into (3.22):

$$\min_{f \in \mathcal{C}_v} VaR_\alpha\left(T_f(X)\right) \simeq \min_{0 \le d \le VaR_\alpha(X)} \left\{d + (1+\omega) \cdot \frac{1}{N} \cdot \sum_{i=1}^N min\{max\{X^i - d, 0\}, VaR_\alpha(X) - d\}\right\}. \quad (3.23)$$

Now, choosing the probability level α, the safety loading ω and a probability distribution of X it is possible to determine the optimal priority, with corresponding value-at-risk and capacity, and the premium payment.

The probability distributions are created randomly using Matlab, changing the parameters of the distributions it is possible to obtain distributions with different characteristics. The probability levels α are chosen differently depending on the particular focus of the analysis; sometimes the ruin probability is kept fixed and sometimes it is the parameter let free to vary.

In the following subsections, the impact of each parameter on the optimization problem will be analysed, reporting data from simulations and providing graphics, holding the other ones fixed.

3.2.1. Safety loading $\boldsymbol{\omega}$

The safety loading is set by the reinsurer and since the cedant point of view is taken into consideration, the safety loading ω is given. However, it is worth to mention that the magnitude of this parameter has a direct impact on the variable subject to optimization. If the safety loading increases, the optimal priority increases and the capacity decreases, for a given probability level α and for a particular distribution. In other words, if the price of reinsurance increases, holding the $VaR_\alpha(X)$ constant (due to the probability level α) the only parts that can change are the capacity and the priority, simply noticeable from the relationship $VaR_\alpha(X) = d + f\big(VaR_\alpha(X)\big)$.

However, the equation (3.17) on the reinsurance contract (2.16) , it is not possible to state that $\pi\big(h_f(X), \omega_1\big) \leq \pi\big(h_f(X), \omega_2\big) \;\; for \; \omega_1 \leq \omega_2$, because the change in the risk sharing amounts (capacity and priority) can compensate the increase in safety loading, so that the premium does not necessarily increase.

Considering $0 < \omega < 1$, the probability level equal to 0.5% and the Pareto loss distribution with shape and a scale parameter 20 and 1800, respectively, the results of a simulation are provided for 10 values of the safety loading in table 3.2.1.1. (see appendix A.1. for Matlab algorithm). The mean and the standard deviation are constant because the simulation has been performed on the same distribution, as the value at risk since the probability level α has been fixed; the priority increases with consequent decrease in the capacity. This means that if the reinsurer charges a higher safety loading, the optimal limit stop-loss reinsurance coverage parameters change, in terms of lower capacity and higher priority. In this case, the premium tends to increase, however from 55% to 65% of safety loading it does not. However, the capacity decrease and the premium increase for an increase in safety loading are difficult to analyse simply looking at the figure: the ROL (Rate On Line) is a ratio representing the amount that has to be paid by the insurer to receive coverage, expressed in %. It is the ratio between the premium and reinsurance recoverable in case of losses, defined as:

$$ROL = \frac{Premium}{Capacity} \tag{3.24}$$

If the ratio increases the cedant has to pay more for reinsurance coverage. It is possible to notice that the ROL increases with the increase in safety loading. This means that the relative cost of reinsurance increases with increasing safety loading.

Omega	Priority	Capacity	$VaR_{0.5\%}$	Premiums	Mean	St. Dev.	ROL
5%	94.0	570.1	664.1	91.1	181.4	99.1	16%
15%	101.8	562.2	664.1	91.6	181.4	99.1	16.3%
25%	109.5	554.5	664.1	91.5	181.4	99.1	16.5%
35%	116.6	547.5	664.1	91.5	181.4	99.1	16.7%
45%	123.0	541.1	664.1	91.6	181.4	99.1	16.9%
55%	129.0	535.1	664.1	91.8	181.4	99.1	17.2%
65%	135.1	529.0	664.1	91.4	181.4	99.1	17.3%
75%	139.4	524.6	664.1	92.4	181.4	99.1	17.6%
85%	144.2	519.9	664.1	92.7	181.4	99.1	17.8%
95%	147.7	516.4	664.1	94.2	181.4	99.1	18.2%

Table 3.2.1.1.: The impact of the safety loading on the optimization problem

This can be seen graphically in the following pictures:

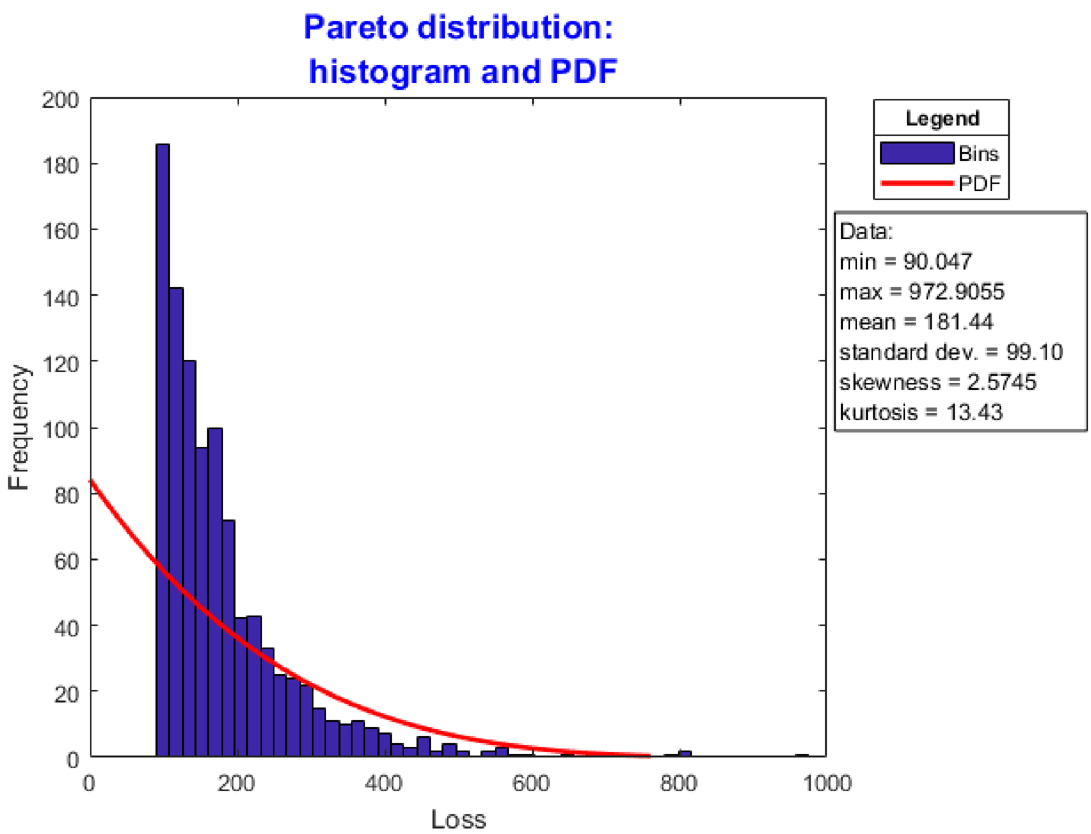

Figure 3.2.1.1.: The Gamma distribution with $X \sim Pa(20,1800)$

Figure 3.2.1.1. shows the simulated loss distribution (Pareto distribution), with different measures describing it. The distribution is very far from a normal one due to a high kurtosis equal to 13.43 and high positive skewness of 2.57. The loss is in the interval between 90.05 and 972.91. For this loss distribution, two separated graphs are provided, showing the ceded loss function and the retained loss function at the probability level $\alpha = 0.005$ for a safety loading $\omega_1 = 0.15$ and $\omega_2 = 0.65$.

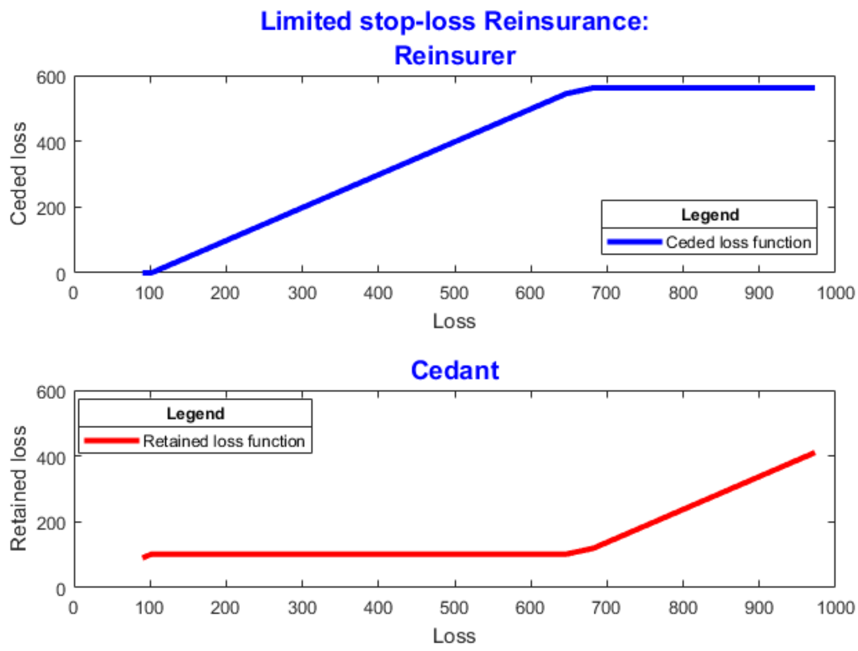

Figure 3.2.1.2. (a): Comparison and impact of different safety loadings with $\omega_1 = 0.15$

Figure 3.2.1.2. (b): Comparison and impact of different safety loadings with $\omega_2 = 0.65$

It is possible to notice that the first graph (figure 3.2.1.2. (a)) and the second one (figure 3.2.1.2. (b)) have both the same $VaR_{0.005} = 664.1$ (meaning that the loss will be expected to be greater than 664.1 every 200 years, if the period of time is one year), because the ruin probability is fixed and the distribution is the same (therefore, the mean value and the standard deviation are constant); the priority increases from 101.8 to 135.1 and the capacity decreases from 562.2 to 529.0, for 15% and 65% safety loading respectively.

Summarizing, the safety loading has a directly impact on the optimization model for the priority and relative capacity. The premium is affected, since it depends on the priority and the capacity amount.

3.2.2. Dispersion of the distribution

The meaning of dispersion of a distribution refers to how the data is spread out; as the dispersion increases the range of values that losses can assume increases. It is interesting to analyse the impact of distributions with different dispersion on the optimized variables of the limited stop-loss reinsurance model. The standard deviation is a measure of dispersion, which is considered the only factor free to vary in the following simulation, in which the expected value of the Gamma distribution is set equal to 360 (Since the generation of random numbers using Matlab for each distribution requires the input of specific parameter, like shape and scale parameters, the mean value is in the interval between 359.9 and 360.1, because the parameters are obtained from the reverse formula of the mean value (3.5), for more details see the Matlab codes in the appendix A.2.), the probability level is 0.5%, safety loading equal to 20%.

In the following table 3.2.2.1. the results of the simulation are reported, it is possible to notice that with increasing standard deviation the skewness and kurtosis increase, meaning that the distribution has a right tail that extends with increasing skewness and the peak increases too (not always these two parameters increase with an increase of the standard deviation, because the distribution is generated randomly). As expected, the value at risk tends to increase with the increase of the dispersion of losses (also here this is not always the case, as for values 228.6 and 249.0 of standard deviation, where the value at risks are 2095.6 and 2088.5, respectively, due to random generation of distributions), meaning that the loss distribution is riskier for a higher value of the standard deviation. The priority does not increase with the dispersion, it decreases, whereas the capacity increases with the value at risk, because for a riskier loss distribution and a fixed ruin probability the insurer cedes more risk and retains less in terms of priority, however the value at risk increases.

St. Dev.	Priority	Capacity	$VaR_{0.5\%}$	Premium	Mean	Skewness	Kurtosis	ROL
83.6	278.9	331.1	610.0	104.0	359.9	0.4	3.2	31.4%
119.2	247.9	482.3	730.2	143.0	360.0	0.7	3.5	29.7%
148.6	219.4	642.7	862.2	177.9	359.9	0.8	4.1	27.7%
166.3	202.2	709.2	911.4	198.6	360.0	0.9	4.2	28.0%
187.9	184.4	839.1	1023.5	220.8	360.1	1.0	4.6	26.3%
208.0	169.5	941.2	1110.7	238.1	360.0	1.1	4.8	25.3%
223.8	155.0	1047.3	1202.3	255.4	360.0	1.2	5.1	24.4%
242.7	143.2	1160.0	1303.2	268.7	360.0	1.5	6.7	23.2%
255.2	131.3	1194.8	1326.1	283.1	360.0	1.4	5.5	23.7%
266.3	122.7	1282.4	1405.1	293.1	360.1	1.5	6.2	22.9%
281.1	112.5	1398.0	1510.5	304.6	359.9	1.6	7.1	21.8%
289.4	106.5	1438.8	1545.3	312.0	360.0	1.6	6.5	21.7%
310.0	95.7	1529.4	1625.1	324.0	360.0	1.8	7.6	21.2%
315.6	89.4	1579.3	1668.7	331.4	360.0	1.7	7.1	21.0%
324.4	83.0	1638.2	1721.2	338.4	359.9	1.8	7.6	20.7%
340.2	75.8	1710.1	1785.9	346.8	360.0	1.8	7.5	20.3%
350.3	73.6	1779.4	1853.0	349.4	360.0	1.9	7.9	19.6%
364.7	64.7	1887.4	1952.1	359.4	360.0	2.0	8.2	19.0%
370.4	59.4	1917.7	1977.0	364.7	360.0	2.0	9.0	19.0%
385.5	56.4	2019.8	2076.1	367.3	359.9	2.4	12.4	18.2%
388.1	53.2	2042.4	2095.6	371.9	359.9	2.1	9.0	18.2%
397.1	49.5	2039.0	2088.5	375.8	360.1	2.1	9.9	18.4%
409.8	43.0	2142.8	2185.8	382.3	359.9	2.2	10.4	17.8%
420.2	40.3	2266.4	2306.8	386.3	360.1	2.3	10.5	17.0%
424.1	39.1	2296.6	2335.7	386.8	360.0	2.3	10.9	16.8%
426.0	36.3	2270.0	2306.3	390.0	360.0	2.3	10.8	17.2%
444.5	31.8	2412.2	2444.0	395.6	360.0	2.3	10.4	16.4%
451.0	28.5	2470.5	2499.0	398.3	360.0	2.4	11.7	16.1%
462.3	26.8	2515.0	2541.8	400.5	360.0	2.5	11.5	15.9%
466.9	25.7	2598.0	2623.7	400.4	360.0	2.7	14.9	15.4%

Table 3.2.2.1.: The impact of the dispersion on the optimization model

If the cedant faces a portfolio with higher standard deviation, it is exposed to a higher potential loss (holding everything else constant) and hence it has a higher risk exposure: since the priority decreases and the capacity increases for an increase in dispersion, the premium must increase to compensate the reinsurer for reinsuring a riskier portfolio and his higher relative potential risk contribution. However, the ROL decreases meaning that the price of reinsurance coverage decreases in relative terms for a higher standard deviation, due to the high capacity increase.

3.2.3. Probability level α and relative $VaR_{\alpha}(X)$

The following analysis is particularly important for the cedant point of view, since for different risk levels the values of the optimal limited stop-loss reinsurance can be examined. After such analysis, the ceding company could choose the probability level of interest, according to his risk aversion and the purpose of its reinsurance coverage.

α	Priority	Capacity	VaR_{α}	Premium	Mean	St. Dev.	ROL
0.5%	202.2	672.0	874.2	156.1	327.0	139.9	23.2%
1%	202.2	578.0	780.1	155.5	327.0	139.9	26.9%
5%	202.2	384.2	586.4	149.4	327.0	139.9	38.9%
9%	202.2	317.7	519.8	144.3	327.0	139.9	45.4%
13%	202.2	276.9	479.1	138.1	327.0	139.9	49.9%
17%	202.2	243.7	445.9	132.5	327.0	139.9	54.4%
21%	202.2	217.2	419.4	126.2	327.0	139.9	58.1%
25%	202.2	195.4	397.5	120.5	327.0	139.9	61.7%
29%	202.2	176.7	378.8	114.6	327.0	139.9	64.9%
33%	202.2	157.2	359.3	107.8	327.0	139.9	68.6%
37%	202.2	140.7	342.9	100.9	327.0	139.9	71.7%
41%	202.2	127.4	329.6	94.0	327.0	139.9	73.8%
45%	202.2	113.4	315.5	87.0	327.0	139.9	76.7%
49%	202.2	99.7	301.9	79.4	327.0	139.9	79.6%
53%	202.2	87.8	290.0	72.0	327.0	139.9	82.0%
57%	202.2	76.0	278.2	64.3	327.0	139.9	84.5%
61%	202.2	65.5	267.6	56.8	327.0	139.9	86.7%
65%	202.2	53.6	255.8	47.7	327.0	139.9	89.0%
69%	202.2	42.9	245.1	39.3	327.0	139.9	91.6%
73%	202.2	31.9	234.1	29.9	327.0	139.9	93.7%
77%	202.2	19.6	221.8	18.8	327.0	139.9	95.9%
81%	202.2	7.0	209.1	6.9	327.0	139.9	98.7%
85%	197.4	0.0	197.4	0.0	327.0	139.9	-
89%	182.4	0.0	182.4	0.0	327.0	139.9	-
93%	166.0	0.0	166.0	0.0	327.0	139.9	-
97%	140.0	0.0	140.0	0.0	327.0	139.9	-
99%	116.8	0.0	116.8	0.0	327.0	139.9	-

Table 3.2.3.1. The impact of the probability level α on the optimization model

For the following simulation, reported in table 3.2.3.1., the lognormal distribution and a safety loading of 20% have been assumed, whereas the probability is the parameter free to vary. The mean value and the standard deviation of the distribution are equal to 327.0 and 139.9, respectively and the histogram and probability density function of the distribution are reported in figure 3.2.3.1..

The distribution is positive skewed with 1.34 skewness and has high kurtosis, equal to 6.04.

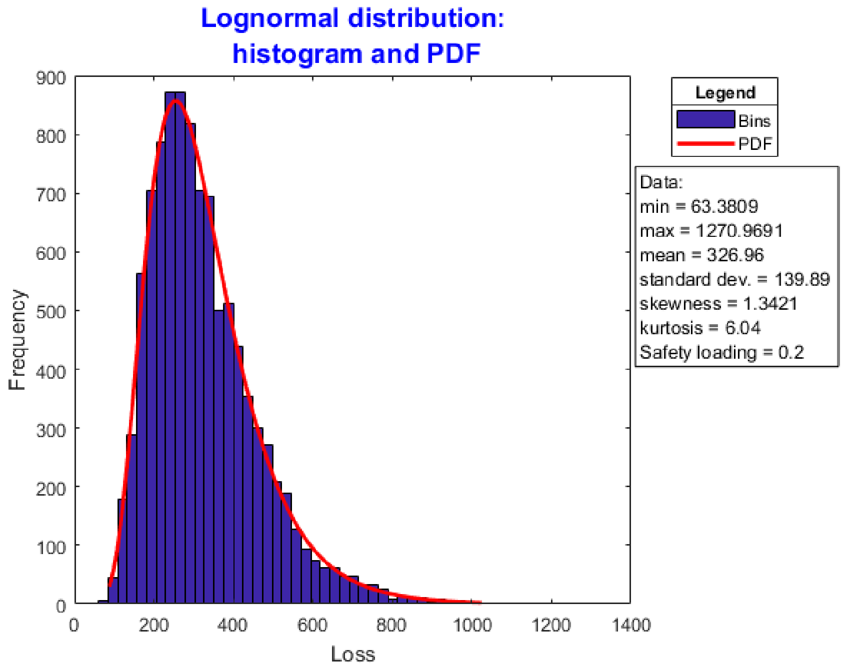

Figure 3.2.3.1. Lognormal distribution $X \sim Lognormal(5.7,0.4)$

As the risk in terms of probability increases, the priority stays constant until a certain point (see figure 3.2.3.2.), which does not mean that the ceding company contributes less to the potential loss, since the related value at risk decreases with increasing probability (see figure 3.2.3.3.). Hence the insurance company is exposed to the higher potential losses for large losses present in the right tail of the distribution.

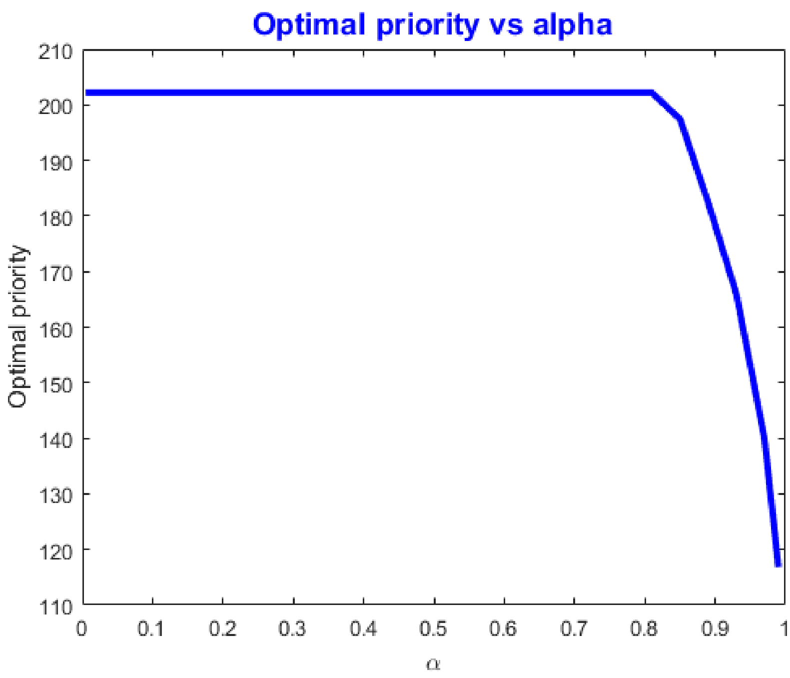

Figure 3.2.3.2. The impact of α on the optimal priority

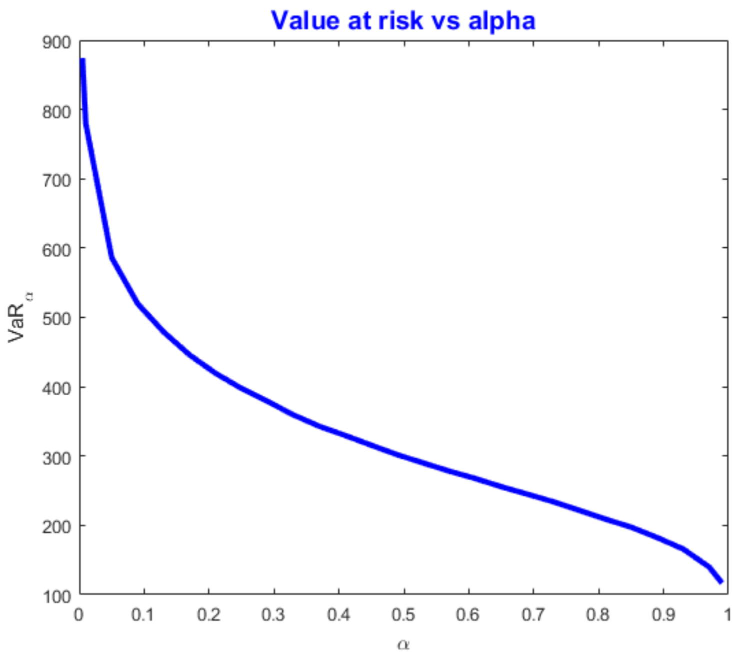

Figure 3.2.3.4. The impact of α on the value at risk

The capacity decreases and the relative premium decreases, reaching a ruin probability equal to 85% where no reinsurance take place since it does not seem to be convenient. The curve is reported in figure 3.2.3.3..

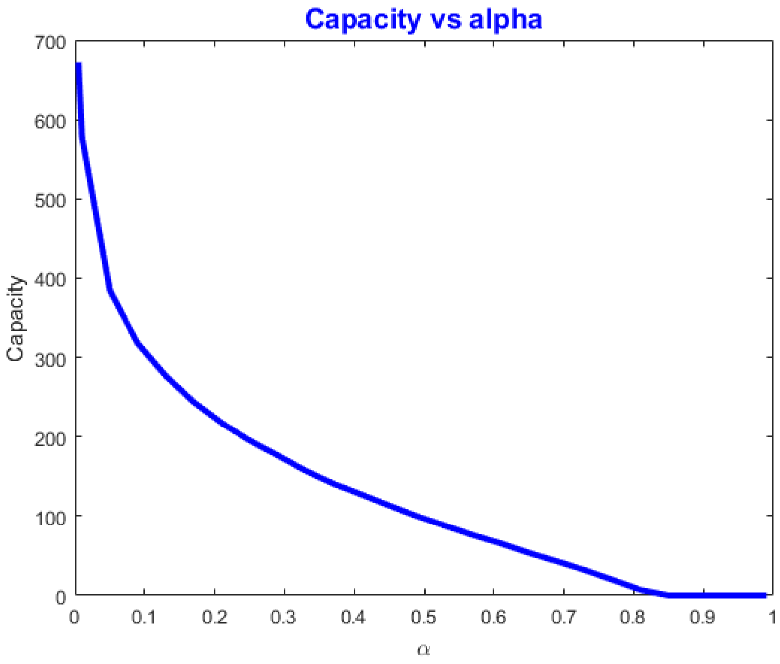

Figure 3.2.3.3. The impact of α on the capacity

Since the capacity decreases, and so the potential contribution of the reinsurer to the loss, the premium owed, decreases with increasing probability. Nevertheless, the ROL increases meaning that the relative price of reinsurance for a lower value at risk increases. The premium payment curve is reported in the following figure:

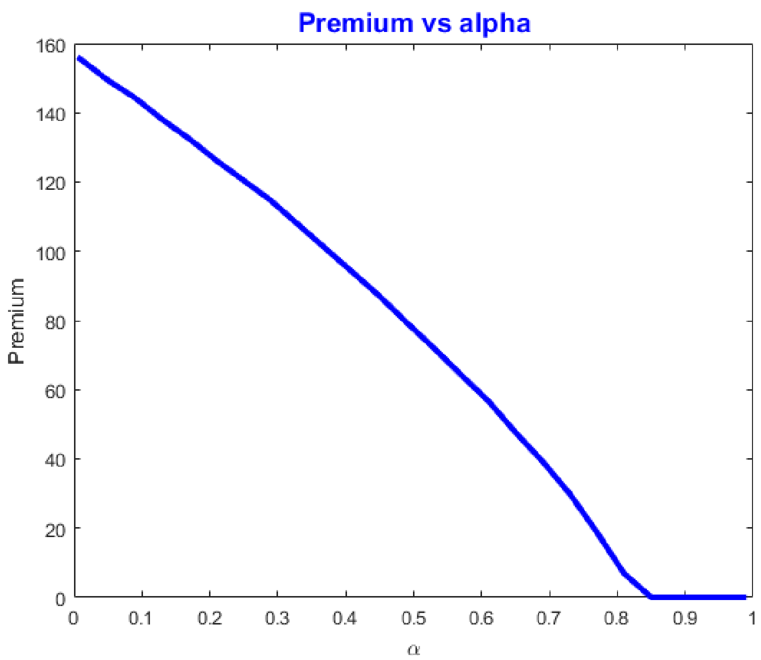

Figure 3.2.3.4. The impact of α on the premium

In this simulation, the whole range of probability levels have been simulated to study the behaviour of the variables. It is worth to mention that some further assumptions should be taken into consideration: for example, if a cedant is seeking coverage for high loss, it should never take a value at risk that is close or below the expected value of the loss. Suppose that the insurer chooses a ruin probability equal to 45% in the above simulation (see table 3.2.3.1.), it means that the loss is expected to be greater than 315.5 forty-five percent of the times in a given period ($VaR_{45\%} = 315.5$), but the expected value of the loss is 327, so any loss value greater than 315.5 must be covered by the cedant, which is below the expected loss value. This is not really the purpose of reinsurance. Therefore, if the ceding company seek reinsurance protection for large losses the ruin probability level should be taken such that the value at risk is greater than the expected value, and the particular level should reflect the risk aversion of the ceding company. Such analysis is important for the insurer to choose the right ruin probability looking at the figure of the expected value and the value at risk.

The figure 3.2.3.5. shows that risk sharing between the cedant and the reinsurer for different probability level in one graph.

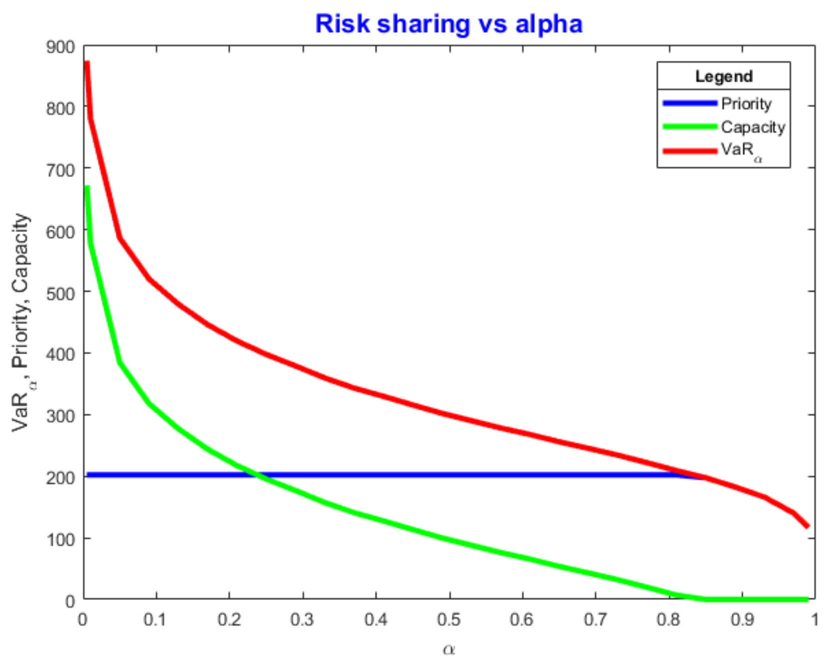

Figure 3.2.3.5.: The impact of α on the value at risk, priority and capacity

The optimal model predicts that for a given safety loading and a certain distribution the optimal priority stays constant until a certain point and then decreases. However, this decreasing point depends on the value of the safety loading. To see this the results of a further simulation are provided in table 3.2.3.2., assuming a Pareto distribution with shape and scale parameter equal to 20 and 1800, respectively, and a safety loading of 20%; whereas in table 3.2.3.3., the results for the same distribution are reported, assuming a safety loading equal to 50%.

The distribution is shown in the following figure:

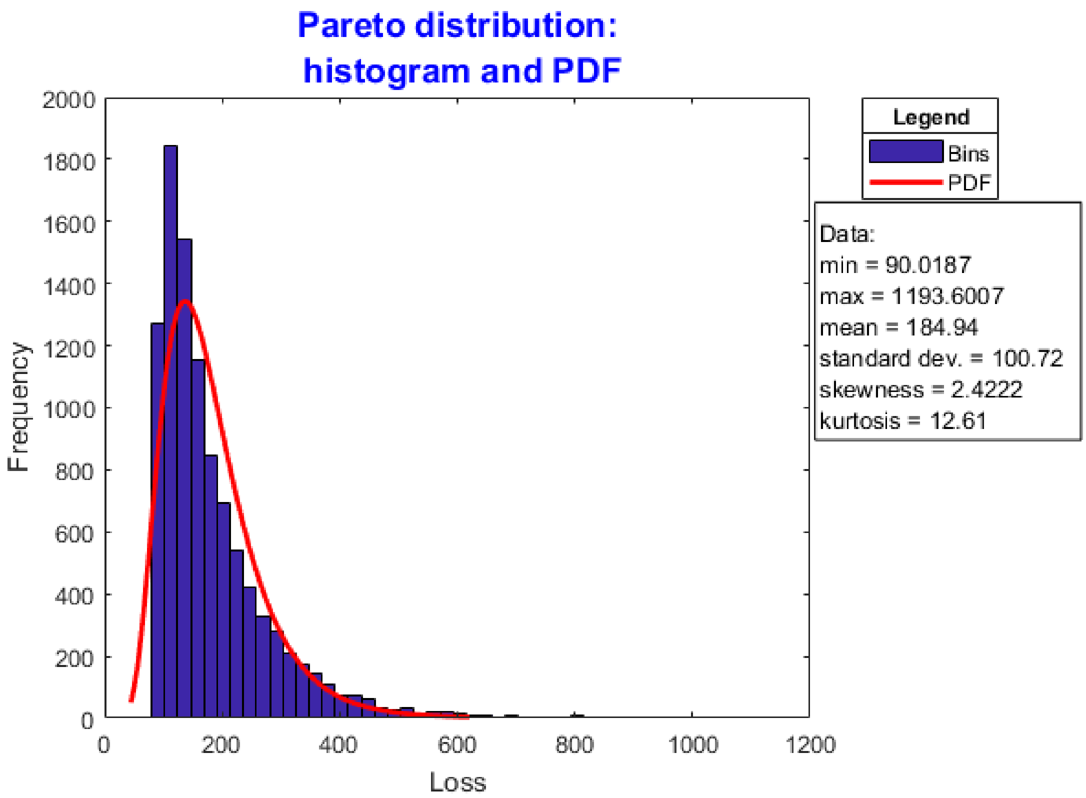

Figure 3.2.3.5.: Pareto distribution $X \sim Pa(20,1800)$

It is possible to notice that for an increase in the safety loading and the increase in the probability level the optimal model results are different. The value of the priority increases with the increase in the safety loading, and the point from which the priorities starts to decrease changes: for the value $\omega = 20\%$ the priority starts to decrease at a probability level equal to 81% (table 3.2.3.2.), whereas for $\omega = 50\%$, the decreasing point is at a ruin probability equal to 65% (table 3.2.3.3.). It is possible to see this relationship on figure 3.2.3.6. and 3.2.3.7., in which the sharing of the risk is reported for both values of the safety loading.

α	Priority	Capacity	VaR_α	Premium	Mean	St. Dev.	ROL
0.5%	106.7	541.3	648.0	94.7	184.9	100.7	18%
1%	106.7	459.3	566.1	94.1	184.9	100.7	20%
5%	106.7	271.0	377.7	88.7	184.9	100.7	33%
9%	106.7	214.1	320.9	84.0	184.9	100.7	39%
13%	106.7	178.0	284.7	79.3	184.9	100.7	45%
17%	106.7	150.6	257.3	74.3	184.9	100.7	49%
21%	106.7	129.8	236.5	69.6	184.9	100.7	54%
25%	106.7	112.6	219.3	64.9	184.9	100.7	58%
29%	106.7	98.3	205.0	60.3	184.9	100.7	61%
33%	106.7	85.9	192.6	55.7	184.9	100.7	65%
37%	106.7	74.1	180.8	50.7	184.9	100.7	68%
41%	106.7	65.0	171.7	46.4	184.9	100.7	71%
45%	106.7	56.4	163.1	42.0	184.9	100.7	75%
49%	106.7	47.8	154.5	37.2	184.9	100.7	78%
53%	106.7	40.9	147.6	33.0	184.9	100.7	81%
57%	106.7	34.4	141.1	28.7	184.9	100.7	83%
61%	106.7	28.6	135.3	24.6	184.9	100.7	86%
65%	106.7	22.9	129.7	20.3	184.9	100.7	89%
69%	106.7	17.7	124.4	16.1	184.9	100.7	91%
73%	106.7	12.1	118.8	11.4	184.9	100.7	94%
77%	106.7	7.3	114.0	7.0	184.9	100.7	96%
81%	106.7	2.7	109.4	2.6	184.9	100.7	99%
85%	104.9	0.0	104.9	0.0	184.9	100.7	-
89%	100.6	0.0	100.6	0.0	184.9	100.7	-
93%	96.7	0.0	96.7	0.0	184.9	100.7	-
97%	93.0	0.0	93.0	0.0	184.9	100.7	-
99%	91.0	0.0	91.0	0.0	184.9	100.7	-

Table 3.2.3.2.: The impact of α on the optimization model for $\omega = 20\%$

Alpha	Priority	Capacity	VAR	Premiums	Mean	St. Dev.	ROL
0.5%	127.5	520.5	648.0	95.1	184.9	100.7	18%
1%	127.5	438.5	566.1	94.3	184.9	100.7	21%
5%	127.5	250.2	377.7	87.5	184.9	100.7	35%
9%	127.5	193.3	320.9	81.7	184.9	100.7	42%
13%	127.5	157.1	284.7	75.8	184.9	100.7	48%
17%	127.5	129.7	257.3	69.6	184.9	100.7	54%
21%	127.5	109.0	236.5	63.7	184.9	100.7	58%
25%	127.5	91.8	219.3	57.8	184.9	100.7	63%
29%	127.5	77.5	205.0	52.0	184.9	100.7	67%
33%	127.5	65.0	192.6	46.3	184.9	100.7	71%
37%	127.5	53.3	180.8	40.1	184.9	100.7	75%
41%	127.5	44.1	171.7	34.7	184.9	100.7	79%
45%	127.5	35.6	163.1	29.3	184.9	100.7	82%
49%	127.5	26.9	154.5	23.1	184.9	100.7	86%
53%	127.5	20.1	147.6	17.9	184.9	100.7	89%
57%	127.5	13.6	141.1	12.6	184.9	100.7	92%
61%	127.5	7.8	135.3	7.4	184.9	100.7	96%
65%	127.5	2.1	129.7	2.1	184.9	100.7	99%
69%	124.4	0.0	124.4	0.0	184.9	100.7	-
73%	118.8	0.0	118.8	0.0	184.9	100.7	-
77%	114.0	0.0	114.0	0.0	184.9	100.7	-
81%	109.4	0.0	109.4	0.0	184.9	100.7	-
85%	104.9	0.0	104.9	0.0	184.9	100.7	-
89%	100.6	0.0	100.6	0.0	184.9	100.7	-
93%	96.7	0.0	96.7	0.0	184.9	100.7	-
97%	93.0	0.0	93.0	0.0	184.9	100.7	-
99%	91.0	0.0	91.0	0.0	184.9	100.7	-

Table 3.2.3.3.: The impact of α on the optimization model for $\omega = 50\%$

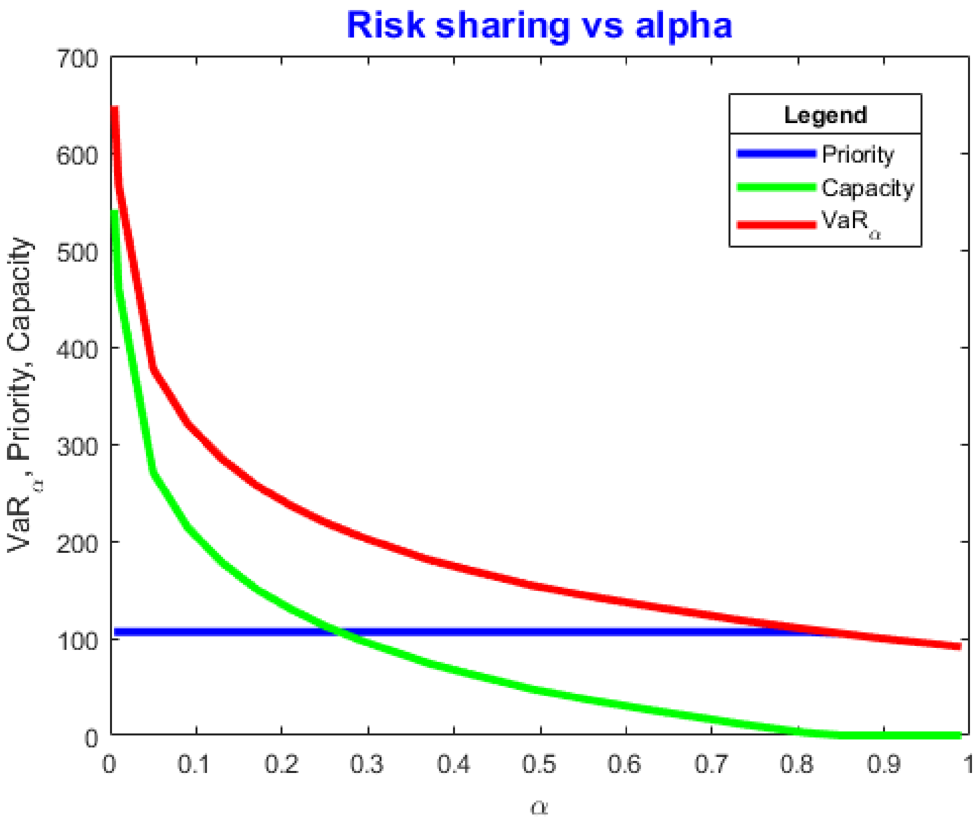

Figure 3.2.3.6.: The impact of α on the value at risk, priority and capacity for $\omega = 20\%$

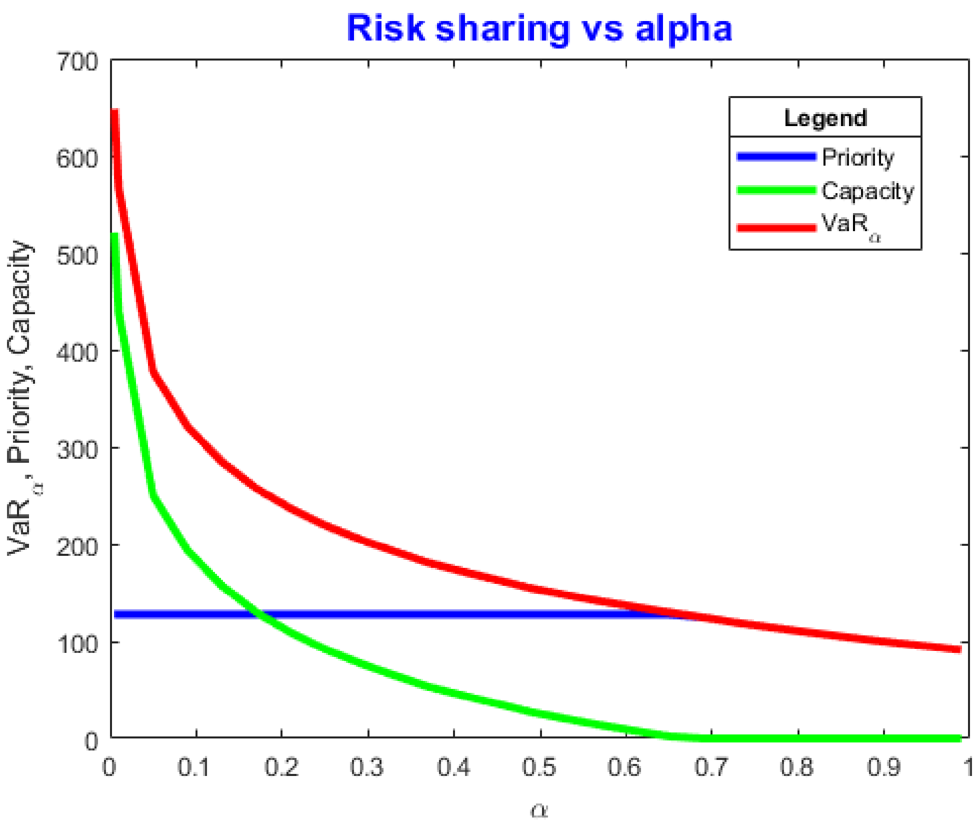

Figure 3.2.3.7.: The impact of α on the value at risk, priority and capacity for $\omega = 50\%$

What is clearly visible from the tables 3.2.3.1., 3.2.3.2. and 3.2.3.3. and from the figures 3.2.3.5., 3.2.3.6. and 3.2.3.7. is that there exists only one optimal priority value, since when it starts to decrease, the capacity is equal to zero and so no reinsurance takes place, meaning that it is not worth to enter into such a reinsurance agreement for that probability level.

3.3. Danish Fire Loss analysis

In this analysis, the optimal value at risk reinsurance model (3.23) is applied to real data on Danish fire loss collected by Copenhagen Reinsurance between 1980 and 1990. The amount of losses is expressed in millions of Danish Krone and has been adjusted for inflation to reflect 1985 values. Each loss amount is the sum of the damage to building, damage to content (for example, furniture and personnel property) and loss of profit (Beirlant et al (2017)). This data has been used by Tan and Weng (2014) to study the optimal reinsurance model under variance risk measure.

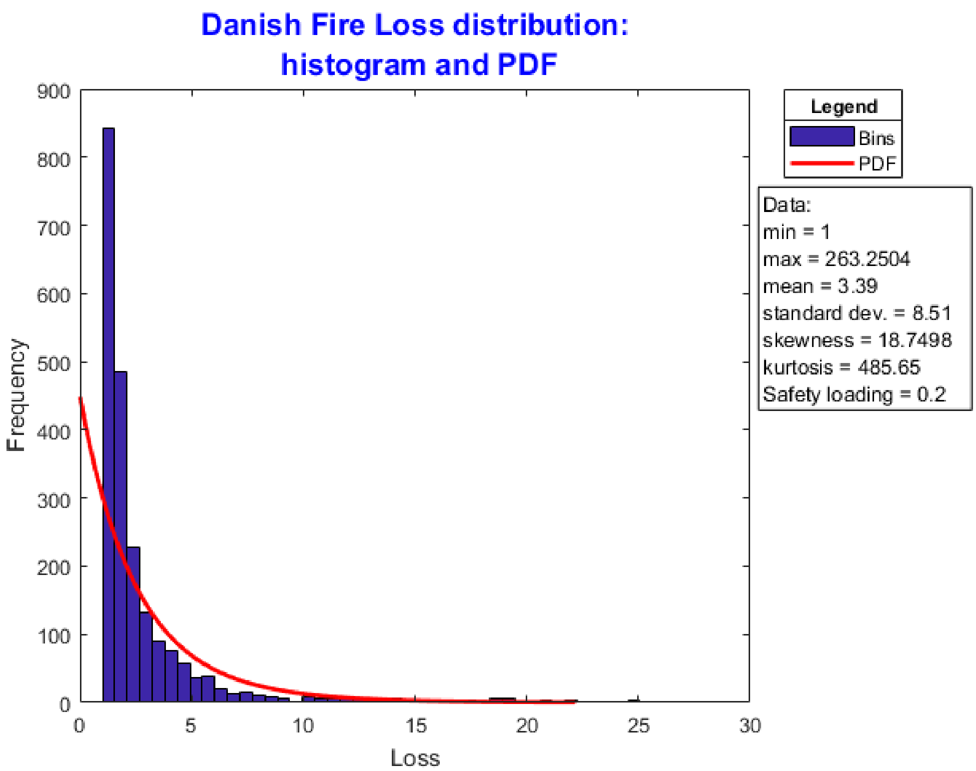

Figure 3.3.1.: Danish fire loss distribution

The distribution is reported in figure 3.3.1. (excluding 17 values from the figure, for a better view of the data, since the las few values are very high with respect to the mean value). The mean of the distribution is equal to 3.39 and the standard deviation is equal to 8.51. The data ranges from 1 million to 363.3 million. Supposing that the insurer has a large fire insurance portfolio to reinsure and that potential losses can be describe by the Danish fire loss distribution, it is possible to conduct a similar analysis as the one conducted in section 3.2.3.. The insurer facing such a loss distribution, has to choose the ruin probability that better represents its risk aversion looking at different possibilities. The simulation of the different possible ruin probabilities is reported in the following table 3.3.1.:

α	Priority	Capacity	VaR_α	Premium	Mean	St. Dev.	ROL
0.5%	1.21	35.60	36.81	2.33	3.39	8.51	6.5%
1%	1.21	24.96	26.17	2.24	3.39	8.51	9.0%
5%	1.21	8.81	10.02	1.79	3.39	8.51	20.3%
9%	1.21	4.61	5.82	1.46	3.39	8.51	31.8%
13%	1.21	3.45	4.65	1.31	3.39	8.51	38.0%
17%	1.21	2.76	3.96	1.19	3.39	8.51	43.0%
21%	1.21	2.16	3.36	1.05	3.39	8.51	48.7%
25%	1.21	1.76	2.97	0.94	3.39	8.51	53.5%
29%	1.21	1.42	2.62	0.83	3.39	8.51	58.6%
33%	1.21	1.19	2.40	0.75	3.39	8.51	62.7%
37%	1.21	0.98	2.19	0.66	3.39	8.51	67.2%
41%	1.21	0.83	2.03	0.59	3.39	8.51	71.0%
45%	1.21	0.70	1.90	0.52	3.39	8.51	74.7%
49%	1.21	0.59	1.80	0.46	3.39	8.51	77.8%
53%	1.21	0.50	1.71	0.41	3.39	8.51	80.8%
57%	1.21	0.43	1.64	0.36	3.39	8.51	83.1%
61%	1.21	0.36	1.57	0.31	3.39	8.51	85.7%
65%	1.21	0.28	1.49	0.25	3.39	8.51	88.7%
69%	1.21	0.22	1.42	0.20	3.39	8.51	91.1%
73%	1.21	0.15	1.36	0.14	3.39	8.51	93.6%
77%	1.21	0.08	1.29	0.08	3.39	8.51	96.5%
81%	1.21	0.04	1.24	0.04	3.39	8.51	98.7%
85%	1.18	0.00	1.18	0.00	3.39	8.51	-
89%	1.12	0.00	1.12	0.00	3.39	8.51	-
93%	1.08	0.00	1.08	0.00	3.39	8.51	-
97%	1.03	0.00	1.03	0.00	3.39	8.51	-
99%	1.01	0.00	1.01	0.00	3.39	8.51	-

Table 3.3.1.: The impact of α on the Danish fire loss data

The results on real data behave as the theoretical ones. It is possible to notice that the priority decreases from the 85% as before. However, the choice of such a ruin probability does not seem reasonable, as has been seen before, because the value at risk is below the expected loss and because for such a ruin probability no reinsurance agreement takes place. The optimal priority is therefore equal to 1.21. The capacity deceases for an increasing probability level, such as the value at risk. The relationship between the three variables for each probability level α can be seen in figure 3.3.2.:

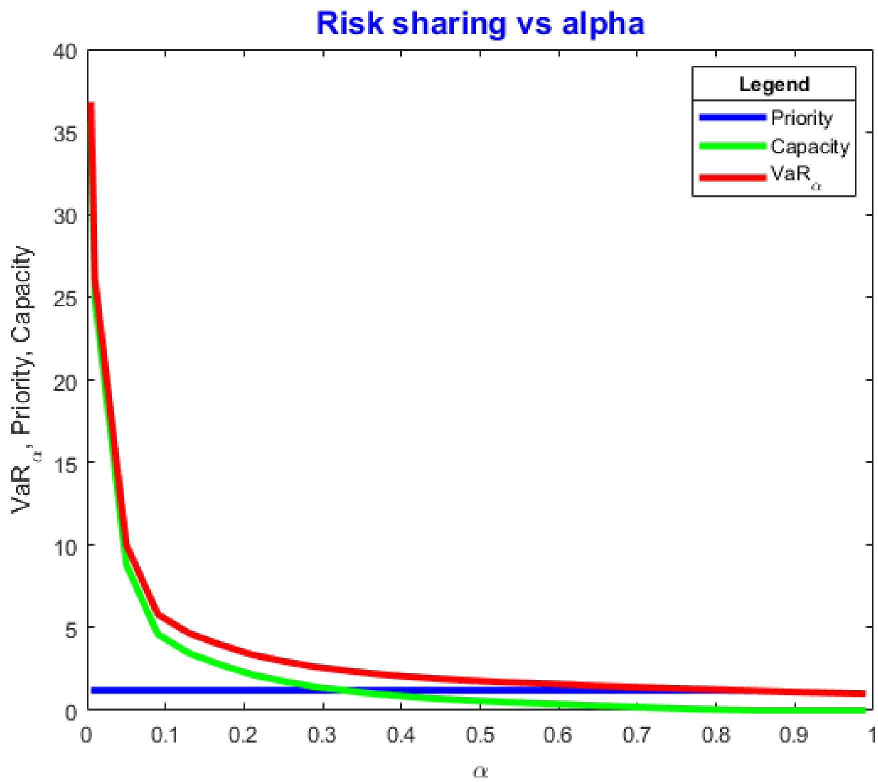

Figure 3.3.2.: The impact of α on the value at risk, priority and capacity for Danish fire loss data

Looking at the ROL in table 3.3.1., the reinsurance coverage payment is lower for a lower ruin probability. This mean that a low ruin probability α is convenient in term of reinsurance contribution, since the relative payment is lower. This parameter suggests that the insurer should purchase a reinsurance contract with very low ruin probability.

Supposing that the reinsurer chooses a 0.5% probability level, the priority is very low (1.21 million), the capacity is equal to 35.6 million and the value at risk is equal to 36.8. The following figure 3.3.3. shows the ceded loss function and the retained loss function:

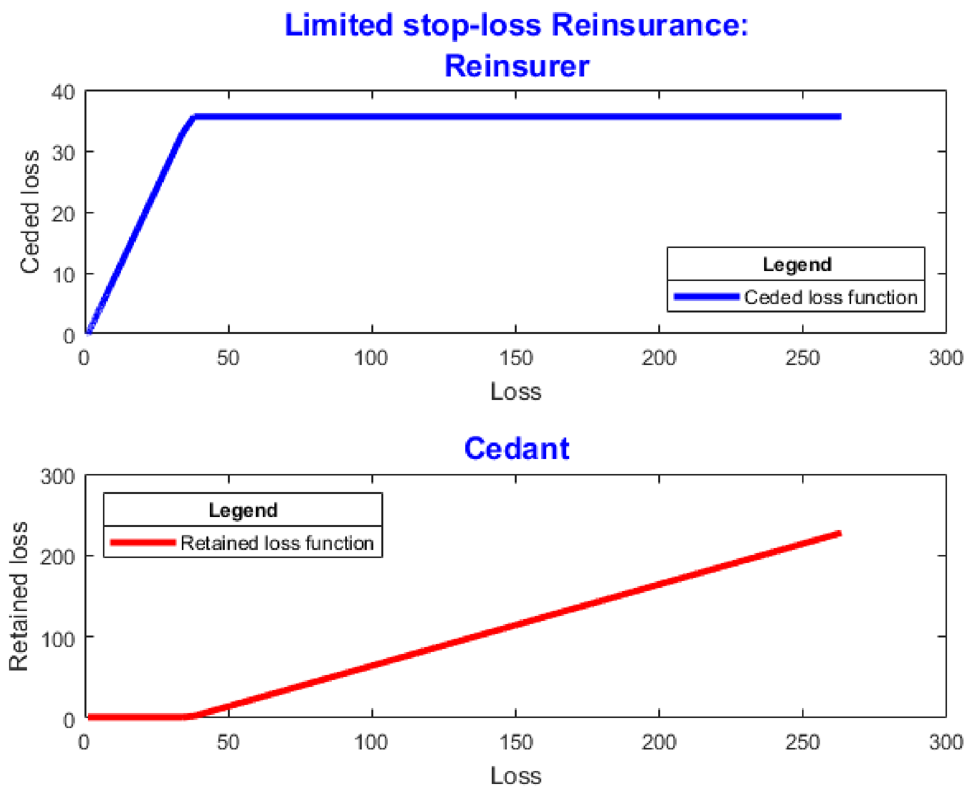

Figure 3.3.3.: Limited stop-loss reinsurance for Danish fire loss data at 0.5% ruin probability

Since the priority is very low, it is not possible to see behaviour of the first parts of the lines of the ceded loss function and of the retained loss function in the figure 3.3.3. (until the priority). The simulation has been performed using the algorithm in appendix A.3., without the generation of a random distribution but inputting directly the loss distribution data.

Conclusion

In the theoretical approach studied in the second chapter, the ceded loss function and the retained loss function have been assumed to be increasing functions, to prevent moral hazard. However, there are other two possibilities according to Chi and Tan (2011), under VaR risk measure:

- The ceded loss function can be assumed to be in the class of convex and increasing functions, leading the stop-loss reinsurance (without an upper limit) to be optimal.
- The retained loss function can be assumed to be left continuous and increasing, leading the truncated stop-loss reinsurance to be optimal.

The first case is not really applicable in reality since no upper limit is considered, meaning that up to a certain amount only the reinsurance company covers the loss in excess of the priority. Such a reinsurance coverage will be very expensive if a reinsurance company accepts to enter into such a contract. A truncated stop-loss reinsurance has the property that if the loss exceeds the upper limit (the sum of priority and capacity), the reinsurance undertaking will have zero obligation to the cedant. Even this form of reinsurance agreement does not seem to find application in practice, and if it does, just in limited cases, since the insurer would be interested in reinsuring only moderate losses and not large ones. The assumption that both the ceded loss function and the retained loss function are increasing, which lead the optimal limited stop-loss reinsurance to be optimal, is very close to reality, since it is a very common form in the reinsurance industry.

The advantage of non-proportional reinsurance is that the priority can limit the liabilities reflecting the risk appetite of the insurance company and its needs. The risk appetite can be reflected in the ruin probability level, and the relative value at risk; whereas the optimal priority is the output of the optimization problem, solving the right balance between the risk retention and the risk transfer. This objective is captured by the (2.43) and (3.23) models studied in this paper.

The empirical analysis of section 3.2.1. showed that the increase in the safety loading rises the optimal priority and decreases the potential reinsurance contribution to the loss for a given level of ruin probability. However, the premium does not necessarily rise in absolute terms because the change in risk sharing, in terms of priority and capacity, could compensate the increase in safety loading, but in terms of ROL the reinsurance coverage becomes more expensive. This could be expected, since if the reinsurer rises the price of reinsurance the insurer retains more and cedes less.

In section 3.2.3. it has been shown that the increase in the dispersion of a loss distribution, and the riskiness of the loss, increases the capacity and the relative premium due to the reinsurer, whereas the insurer has a lower potential loss contribution in terms of priority. This behaviour of the variables seems reasonable, since the riskier the distribution, the more the insurer will be willing to cede; hence, the reinsurer will have a higher potential loss contribution in terms of capacity and he will demand a higher premium. However, the relative premium payment in terms of ROL decreases.

Finally, the empirical analysis of section 3.3., using the Danish fire loss data provided the same conclusions of section 3.2.3., using a random generated loss distribution. The rise in the ruin probability lead the priority unchanged for a given loss distribution and a given safety loading, while the premium and the capacity decreases. However, the relative premium in terms of ROL increases with the rising ruin probability. It is worth to mention that if the capacity increases (holding the safety loading constant), the ROL decreases, and vice versa. This inversely proportional relationship means that if the amount reinsured increases, the relative price for the coverage decreases.

The choice of the premium principle has a determinant effect on the optimization problem (2.43), but in this paper the cedant point of view was taken into consideration and therefore the effect of different premium principles has not been analysed, since this choice is taken by the reinsurance undertaking. The use of other premium principles to calculate the reinsurance premium is left for future research.

Appendix A

A.1. Safety loading ω simulation: Matlab algorithm

The following algorithm can be used to reproduce a similar simulation as the one of section 3.2.1., inputting a vector of values for the safety loading, a desired ruin probability and the scale and shape parameters of a desired Pareto distribution.

```
%% Impact of Safety Loading on Optimal Limited Stop-Loss reinsurance

% This script provides an Excel file reporting the optimal priority, the
% capacity, the value at risk, the premium payment, the mean value, the
% standard deviation and the Rate Online (ROL) for each chosen value of
% the safety loading.
% It also provides a figure with histogram and the probability density
% function of the Pareto distribution.

% OPTIMIZATION PROBLEM (assuming expected value premium principle and
% Pareto distribution):

% min Priority+(1+omega)*E(min(max(X-Priority,0),VaR-Priority))

% subject to 0<=Priority<=VaR
% where X is loss random variable, VaR is the value at risk and E(.) is the
% expected value function.

% INPUT: - omega = safety loading value or vector;
%        - alpha = probability level;
%        - mu = shape parameter of Pareto distribution;
%        - sigma = scale parameter of Pareto distribution;
% OUTPUT: - Excel file 'SafetyLoading_vs_Optimization.xls'
%         - Histogram and PDF figure

clear; % Clean Workspace
clc; % Clean Command window
% INPUTS
omega=input('Enter the value/range of values of the safety loading (row
vector):'); % Reinsurer safety loading range
alpha=input('Enter the ruin probability:'); % Ruin probability
mu=input('Enter the shape parameter of the Pareto distribution:'); % Shape
parameter
sigma=input('Enter the scale parameter of the Pareto distribution:'); % Scale
parameter
N = 10000;
% LOSS DISTRIBUTION GENERATION AND FIGURE
X =random('Generalizedpareto',1/mu,sigma/mu,sigma/mu,1,N); % Generation of
10000 random number
L = sort(X); % Sort data to determine the quantile
figure;
histfit(X,50,'Generalizedpareto'); % histogram with pdf
title({'Pareto distribution:' , ' histogram and PDF'},...
    'Color','b','FontSize',14); % Title of graph
xlabel('Loss'); % x-axis label
ylabel('Frequency'); % y-axis label
lgd=legend('Bins','PDF','Location','northeastoutside'); % Create legend
title(lgd,'Legend');
MIN=min(L); MAX=max(L); MEAN=mean(L); % Computation of descriptive statistics
SD=std(L); kur=kurtosis(L); sk=skewness(L); % Computation of descriptive
statistics
% Create text annotation on the figure with descriptive statistics
str={'Data:'  ['min = ' num2str(MIN)] ['max = ' num2str(MAX)]...
```

```
    ['mean = ' num2str(MEAN,'%.2f')] ['standard dev. = '
num2str(SD,'%.2f')]...
    ['skewness = ' num2str(sk)] ['kurtosis = ' num2str(kur,'%.2f')] };
annot=annotation('textbox',[0.78 0.45 0.1
0.1],'string',str,'FitBoxToText','on',...
    'VerticalAlignment', 'bottom','HorizontalAlignment', 'left' ); % Put
annotation on figure
annot.Margin=2; % Set margin width of annotation

% OPTIMIZATION
Priority = NaN(length(omega),1); % Vector of NaN to store for loop values (for
speed)
Premiums = NaN(length(omega),1);
VaR = quantile(L,1-alpha); % Value at risk (the 1-alpha quantile)

for k=1:length(omega)
        ob  = @(d)(0);
        for i=1:N
        f_i = @(d)((min(max(X(i)-d,0),VaR-d))); % Computation for each X(i) of
expected value object
        ob = @(d)(f_i(d) + ob(d)); % Sum of expected value object
        end
ob = @(d)(d+(1+omega(k))*(ob(d)/N)); % Final optimization object
Priority(k) = fminbnd(ob,0,VaR); % Optimal priority

P=0;
for l=1:N
         % Calculation of the premium for each omega
         pr = (1+omega(k))*(min(max(X(l)-Priority(k),0),VaR-Priority(k)));
         P = P + pr; % Sum of premiums
end
     Premiums(k)=P/N; % Calculation of mean value of premium
     disp(['Iterations left ' num2str(length(omega)-k) ' of '
num2str(length(omega))])
     % Display the number of simulations left
end

% EXCEL FILE
Omega = omega'; % Transpose of omega for table creation
VAR = VaR*ones(length(omega),1); % Value at risk vector for table creation
Capacity = VaR-Priority; % Capacity calculation
Mean = MEAN*ones(length(omega),1); % Mean value vector for table creation
StDev = SD*ones(length(omega),1); % Standard deviation vector for table
creation
ROL = Premiums./Capacity; % ROL calculation
VALUE = table(Omega,Priority,Capacity,VAR,Premiums,Mean,StDev,ROL);
% Create a table with all values of interest
writetable(VALUE,'SafetyLoading_vs_Optimization.xls'); % Save table in Excel
file
```

A.2. Dispersion simulation: Matlab algorithm

The following algorithm can be used to reproduce a similar simulation as the one of section 3.2.2., inputting the safety loading, the ruin probability, the desired expected value of the Gamma distribution and different values of the scale parameter (in an increasing order), which generate different distributions with increasing standard deviation, since there exists a correlation between expected value and variance of the distribution and the parameters of the distribution (see equation (3.5) and (3.6)).

```
%% Impact of Dispersion on Optimal Limited Stop-Loss Reinsurance

% This script provides an Excel file reporting the standard deviation, the
% priority, the capacity, the value at risk, the premium, the mean value,
% the skewness, the kurtosis and the Rate On Line (ROL), of the following
% optimization problem:

% OPTIMIZATION PROBLEM (assuming expected value premium principle and
% Gamma distribution):

% min Priority+(1+omega)*E(min(max(X-Priority,0),VaR-Priority))

% subject to 0<=Priority<=VaR
% where X is loss random variable, VaR is the value at risk and E(.) is the
% expected value function.

% INPUT: - omega = safety loading value (omega>0);
%        - alpha = probability level;
%        - E = expected value of Gamma distribution;
%        - sigma = scale parameter of Gamma distribution;
% OUTPUT: - Excel file 'Dispersion_vs_Optimization.xls'


clear; % Clean Workspace
clc; % Clean Command window
omega = input('Enter the value of the safety loading (for example, 0.3):'); %
Reinsurer safety loading
alpha = input('Enter the ruin probability:'); % Ruin probability
E = input('Enter the desired expected value of the Gamma distribution:'); %
Expected value Gamma
sigma = input('Enter a range of values for the scale parameter of a Gamma
distribution (increasing order):'); % Scale parameter
N = 10000;
Priority = NaN(length(sigma),1); % Vector of NaN to store for loop values (for
speed)
Mean = NaN(length(sigma),1);
StDev = NaN(length(sigma),1);
VAR = NaN(length(sigma),1);
Skewness = NaN(length(sigma),1);
Kurtosis = NaN(length(sigma),1);
Premiums = NaN(length(sigma),1);
STD_1=std(random('Gamma',E/sigma(1),sigma(1),1,N)); % Initial standard
deviation
for k=1:length(sigma)
    range=false;
    while~range % Creation of random distribution with constant mean and
                % increasing standard devaition at each iteration for each
                % sigma
            mu=E/sigma(k); % Calculation of shape parameter (mu=Expected
value/sigma)
            X = random('gamma',mu,sigma(k),[1,N]); % Random distribution
            L=sort(X); % Sort data for VaR calculation
            MEAN=mean(X);
            STD=std(X);
            if MEAN>=(E-E*0.05) && MEAN<=(E+E*0.05) && STD>STD_1
                range=true; % Exit while loop if conditions are met
            else
                range=false; % Restart while loop to generate another random
distribution
            end
    end
    VaR=quantile(L,1-alpha); % VaR, the 1-alpha quantile
    ob=@(d) (0);
    for i=1:N
            f_i = @(d)((min(max(X(i)-d,0),VaR-d))); % Compuation for each X(i)
of expected value object
            ob = @(d)(f_i(d) + ob(d)); % Sum of expected value object
```

```
    end
    ob = @(d)(d + (1+omega)*(ob(d)/N)); % Final optimization object
    Priority(k) = fminbnd(ob,0,VaR); % Optimal priority
    VAR(k) = VaR;
    Mean(k) = MEAN;
    StDev(k) = std(X);
    Prem = NaN(1,N);
    P = 0;
    for l=1:N % Number of premium calculation foe each sigma
         pr = (1+omega)*(min(max(X(l)-Priority(k),0),VaR-Priority(k)));
         P = P + pr; % Sum of premium calculation for each X(i)
    end
    Premiums(k) = P/N; % Calculation of mean value of premium
    STD_1  =std(X); % Calculation of the standard deviation to met next random
distribution conditions
    Skewness(k) = skewness(X); % Skewness
    Kurtosis(k) = kurtosis(X); % Kurtosis
    disp(['Iterations left ' num2str(length(sigma)-k) ' of '
num2str(length(sigma)) '.'])
    % Display remaining interations
end

Capacity = VAR-Priority; % Capacity
ROL = Premiums./Capacity; % ROL calculation
dispersion =
table(StDev,Priority,Capacity,VAR,Premiums,Mean,Skewness,Kurtosis); % Creation
of table
writetable(dispersion,'Dispersion_vs_Optimization.xls'); % Save table in Excel
file
```

A.3. Probability level α simulation: Matlab algorithm

The following algorithm can be used to reproduce a simulation as in section 3.2.3., inputting the safety loading, the parameters of a lognormal distribution (location parameter and scale parameter, see section 3.1.3. for more details) and a vector (or single value) of probability levels α. This script has also been used for the Danish fire loss analysis in section 3.3..

```
%% Impact of the ruin probability on Optimal Limited Stop-Loss reinsurance

% This script provides and excel file reporting the chosen ruin probability,
% the priority, the capacity, the value at risk, the premium, the mean
% value and the standard deviation of the distribution and the Rate On Line
% (ROL), of the following optimization problem:

% OPTIMIZATION PROBLEM (assuming expected value premium principle and
% Gamma distribution):

% min Priority+(1+omega)*E(min(max(X-Priority,0),VaR-Priority))

% subject to 0<=Priority<=VaR
% where X is loss random variable, VaR is the value at risk and E(.) is the
% expected value function.

% The script provides also the following graphs:
% - Histogram and PDF of the distribution.
% - Optimal priority vs ruin probability.
% - Value at risk vs ruin probability.
% - Capacity vs ruin probability.
% - Premium vs ruin probability.
```

```
% - Risk sharing vs ruin probability.
% INPUT: - omega = safety loading value(omega>0);
%        - alpha = probability level;
%        - mu = location parameter of Lognormal distribution;
%        - sigma = scale parameter of Lognormal distribution.
% OUTPUT: - figures;
%          - Excel file 'Alpha_vs_Oprimization.xls'.

clear; % Clean Workspace
clc; % Clean Command window
% INPUTS
omega=input('Enter the value of the safety loading (for example,0.3):'); %
Reinsurer safety loading range
alpha=input('Enter the ruin probability vector (row vector):'); % Ruin
probability
mu=input('Enter the shape parameter of the Lognormal distribution:'); %
Location parameter
sigma=input('Enter the scale parameter of the Lognormal distribution:'); %
Scale parameter
N = 10000; % Number of losses generated
X = random('Lognormal',mu,sigma,[1,N]); % Generation of the loss distribution
L=sort(X); % Sort data to determine the quantile

% HISTOGRAM AND PDF
figure;
histfit(X,50,'Lognormal'); % Histogram with pdf
title({'Lognormal distribution:' , ' histogram and PDF'},...
    'Color','b','FontSize',14);
xlabel('Loss');
ylabel('Frequency');
lgd=legend('Bins','PDF','Location','northeastoutside');
title(lgd,'Legend');
MIN=min(L); MAX=max(L); MEAN=mean(L); standard_dev=std(L); kur=kurtosis(L);
sk=skewness(L);
str={'Data:'  ['min = ' num2str(MIN)] ['max = ' num2str(MAX)]...
    ['mean = ' num2str(MEAN,'%.2f')] ['standard dev. = '
num2str(standard_dev,'%.2f')]...
    ['skewness = ' num2str(sk)] ['kurtosis = ' num2str(kur,'%.2f')] ['Safety
loading = ' num2str(omega)]};
annot=annotation('textbox',[0.78 0.40 0.1
0.1],'string',str,'FitBoxToText','on',...
    'VerticalAlignment', 'bottom','HorizontalAlignment', 'left' );
annot.Margin=2;

% OPTIMIZATION
Priority = NaN(length(alpha),1); % Vector of NaN to store for loop values (for
speed)
Premiums = NaN(length(alpha),1);
VAR = zeros(length(alpha),1);
for k=1:length(alpha)
    VaR = quantile(X,1-alpha(k)); % Value at risk calculation
    ob  = @(d)(0);
    for i=1:N
    f_i = @(d)((min(max(X(i)-d,0),VaR-d))); % Computation for each X(i) of
expected value object
    ob = @(d)(f_i(d) + ob(d)); % Sum of expected value object
    end
    ob = @(d)(d+(1+omega)*(ob(d)/N)); % Final optimization object
    Priority(k) = fminbnd(ob,0,VaR); % Optimal priority

    P=0;
    for l=1:N % Number of premium calculation simulations
             % Calculation of the premium for each omega by simulation
             pr = (1+omega)*(min(max(X(l)-Priority(k),0),VaR-Priority(k)));
             P = P + pr;
    end
    Premiums(k)=P/N; % Calculation of mean value of premium simulation
```

```
    VAR(k)=VaR;
    disp(['Iterations left ' num2str(length(alpha)-k) ' of '
num2str(length(alpha)) '.'])
    % Display remaining iterations
end
% OPTIMAL PRIORITY VS RUIN PROBABILITY GRAPH
figure;
plot(alpha,Priority,'Color','b','LineWidth',3);
xlabel("\alpha");
ylabel("Optimal priority");
title('Optimal priority vs \alpha','Color','b','FontSize', 14);

% PREMIUM VS RUIN PROBABILITY GRAPH
figure;
plot(alpha,Premiums,'Color','b','LineWidth',3); % Reinsurer payment curve
hold on
xlabel("\alpha");
ylabel("Premium");
title('Premium vs \alpha','Color','b','FontSize', 14);
lgd1=legend('Premium','Location', 'southeast');
title(lgd1,'Legend');
hold off

% CAPACITY VS RUIN PROBABILITY
Capacity=VAR-Priority; % Capacity calculation
figure;
plot(alpha,Capacity,'Color','b','LineWidth',3);
hold on
lgd1=legend('Capacity','Location', 'southeast');
title(lgd1,'Legend');
title('Capacity vs \alpha','Color','b','FontSize', 14);
xlabel("\alpha");
ylabel("Capacity");
hold off

% VALUE AT RISK VS RUIN PROBABILITY
figure;
plot(alpha,VAR,'Color','b','LineWidth',3);
hold on
lgd1=legend('VaR','Location', 'southeast');
title(lgd1,'Legend');
title('Value at risk vs \alpha','Color','b','FontSize', 14);
xlabel("\alpha");
ylabel("VaR_\alpha");

% RISK SHARING VS RUIN PROBABILITY
figure;
plot(alpha,Priority,'Color','b','LineWidth',3);
hold on
plot(alpha,Capacity,'Color','g','LineWidth',3);
plot(alpha,VAR,'Color','r','LineWidth',3);
u=legend('Priority','Capacity','VaR_{\alpha}','Location', 'Best');
title(u,'Legend')
xlabel("\alpha");
title('Risk sharing vs \alpha','Color','b','FontSize', 14);
ylabel("VaR_\alpha, Priority, Capacity");
hold off

% EXCEL FILE
Alpha=alpha'; % Transpose of ruin probability for table creation
ROL = Premiums./Capacity; %ROL calculation
Mean=ones(length(alpha),1)*MEAN;
StandardDeviation=ones(length(alpha),1)*std(X);
A=table(Alpha,Priority,Capacity,VAR,Premiums,Mean,StandardDeviation); %
Creation of table
writetable(A,'Alpha_vs_Oprimization.xls'); % Save table in Excel file
```

A.4. Limited stop-loss reinsurance: ceded loss function and retained loss function

The following script provides a graph representing the limited stop-loss reinsurance curves for the reinsurer and the cedant. It indicates the respective potential loss contributions for the two parties. It is necessary to input the priority, the value at risk and the sorted loss distribution. These values can be calculated using one of the previous scripts. An example of the output of this scrips is figure 3.3.3..

```
%% Limited stop-loss reinsurance: ceded loss function and retained loss
function

% This script provides a graph for the ceded loss function and the retained
% loss function of a limited stop-loss reinsurance.

% INPUT: - Priority.
%        - VaR.
%        - Sorted loss distribution.
% OUTPUT: - Figure.

% INPUTS
P = input('Enter the priority:');
V = input('Enter the value at risk:');
Loss = input('Enter the sorted loss distribution:');

% CEDED LOSS FUNCTION
RT = min(max(Loss-P,0),V-P);

% RETAINED LOSS FUNCTION
Cedant = Loss - RT;

% CREATION OF THE FIGURE
figure;
subplot(2,1,1) % Divide figure
plot(Loss,RT,'Color','b','LineWidth',3); % Reinsurer curve
hold on
xlabel("Loss");
ylabel("Ceded loss");
title({'Limited stop-loss Reinsurance:' 'Reinsurer'},'Color','b','FontSize',
14);
lgd1=legend('Ceded loss function','Location', 'southeast');
title(lgd1,'Legend');
hold off

subplot(2,1,2); % Divide figure
plot(Loss,Cedant,'Color','r','LineWidth',3); % Insurer curve
hold on
xlabel("Loss");
ylabel("Retained loss");
title('Cedant','Color','b','FontSize', 14);
lgd2=legend('Retained loss function','Location', 'southeast');
title(lgd2,'Legend');
hold off
```

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Acknowledgments

I would like to express my sincere thanks to my advisor Prof. Paola Ferretti for your immense knowledge, motivation, remarks and support through the learning process of this thesis.

Besides my advisor, I would like to thank Prof. Martina Nardon, who provided me with the fundamental Matlab skills in the Computational Finance course, essential for the empirical analysis. I am also particularly grateful to the smart PhD student Matteo Iacopini for his help, insightful comments and interest.

I want to say thanks to who went along with me the whole university career from Trieste to Venice, my dear friend Gabriele. I will never forget our stimulating discussions, days of study and all the fun at and outside the university. I am especially thankful to my dear friend Mattia for all the moral support during my university career, the insightful conversations we have every day and the fun we have together. I would like to thank my friend Luca for all the aid you give me, the encouragement and the fun we have the working nights at the carnival float. I hope these friendships will last forever and never fade.

Finally, the warmest and dearest gratitude goes to my family. I am indebted to my parents, Clemens and Beate, you have accompanied me through my life. I would not be who and where I am now without your love, warmth, support and confidence.

I am especially thankful to Giulia, the beautiful person I have by my side. I thank you for every small detail and smile you give me, for everyday endless support and love I receive.

Last, but not least, I would like to thank a wonderful person, my big brother Luca, for always being there even if far away. Your generosity and goodness are endless.