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A VaR-based optimal reinsurance model: the perspective of both the insurer and the reinsurer

Supervisor
Ch. Prof. Paola Ferretti

Assistant Supervisor
Ch. Prof. Stefania Funari

Graduand
Federica Maria Cogo
Matriculation number 859273

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INTRODUCTION

Reinsurance is usually defined as «*insurance for insurers*»: under a reinsurance arrangement, the *reinsurer* agrees on being transferred a portion of the risks underwritten by an insurance company, the so-called *cedant*. In exchange, the reinsurer is paid a *premium*.

In recent decades, reinsurance has often been addressed in economic research: the main objective is to determine the optimal form and level of reinsurance, under either the perspective of the cedant, of the reinsurer or both. Specifically, in light of the major role played by risk measures in the insurance regulatory system, latest research has focused on risk-measure-based reinsurance models.

In particular, Cai and Tan (2007) were among the first to propose an optimal reinsurance model, which explicitly employed the *Value-at-Risk* (VaR) risk measure in its objective function. The authors determined the level of retention, in a stop-loss reinsurance, that minimizes the VaR of the total exposure of the cedant: in this sense, it is optimal (see [2]). In 2011 and 2013, Chi and Tan further investigated the model: by relaxing some of the original assumptions, they managed to provide it with significant robustness (see [4, 5]).

The aim of this thesis is to study the model in Chi and Tan (2013), under the different perspectives of the two parties involved in a reinsurance contract. After an analysis of the VaR-based optimal reinsurance model from the cedant's point of view, the same is evaluated under the reinsurer's one: our interest is to determine whether and how the optimal reinsurance contract for the cedant diverges from the one optimal for the reinsurer. In conclusion, the two agents' antithetical perspectives are considered simultaneously: to the purpose, a field of mathematics called *Multi-Objective Optimization* is introduced.

The structure of the thesis is as follows. In Chapter 1, an overview to reinsurance is presented: in particular, we explain what it is and how and why it should be used by insurance companies. In Chapter 2, we propose the model solved in Chi and Tan (2013): after an exhaustive analysis of the set of preliminary conditions, the main theorem is formalized and proved step-by-step. In Chapter 3, we focus on the perspective of the reinsurer: under the same assumptions, we study whether the reinsurance contract, proved optimal in Chapter 2, might be optimal also for the reinsurer; since it is not, we look for a different reinsurance policy, which minimizes the VaR of the reinsurer's total risk exposure. In Chapter 4, Multi-objective Optimization is introduced: first, we explain why it might be used to formalize an optimal reinsurance model, capable of taking into account both agents' conflicting interests; then, we show how Cai, Lemieux and Liu (2015) and Lo (2017) resorted to two different Multi-objective Optimization methods to derive mutually acceptable optimal solutions.

Chapter 1

Reinsurance: an overview

1.1 Introduction

Non-life insurers usually face fortuitous claims experiences, which threaten the results of their portfolios and may consequently impinge on their overall performances and capitals (see [1]). In example, an unforeseen catastrophe may affect multiple risks in the same portfolio and lead to a major total loss; or frequent small losses might result in an overall burden far greater than expected. As a consequence, the only way to ensure stable results is through diversification. By the nature of their business, though, insurers do not have the means, nor the possibility, to maximally diversify the risks they hold. A valid solution is offered by *reinsurance*.

Reinsurance is a form of risk sharing, usable as a risk mitigating tool: buying reinsurance cover for a given premium, the insurer (cedant) can transfer part of its risk to the reinsurer. The latter, operating worldwide and across different lines of business, can exploit diversification to achieve a more efficient use of its capital, which is translated into capital relief for the cedant. Also, resorting to reinsurance, the cedant gains the possibility to underwrite a larger number of risks, improving its position in the market and spreading, and potentially better diversifying, its portfolios.

1.2 Types of reinsurance

The offerings of reinsurance covers are as varied as the different and evolving needs of insurers. There exists both *Facultative Reinsurance*, which can be bought to ensure protection against major single risks, and *Treaty reinsurance*, which covers entire portfolios. According to how much time claims might take to be settled, it is also possible to choose between *Short-Tail Reinsurance* or *Long-Tail Reinsurance*. In some business segments claims are usually settled within a short period (i.e. property lines): in these cases, Short-Tail Reinsurance is suggested. In others, years or decades may pass before claims are paid off (i.e. liabilities lines): for these, Long-Tail Reinsurance is suitable. Also, the cedant can choose whether to buy a *Direct Reinsurance*, without an intermediary, or a *Brokered Reinsurance* (see [11]). The main difference, though, is between *Proportional* and *Non-Proportional Reinsurance*.

1.2.1 Proportional Reinsurance

With a proportional reinsurance treaty, also known as Pro Rata Reinsurance, the cedant proportionally shares one or more of its policies with the reinsurer, paying a percentage of the received premiums: in exchange, the reinsurer complies with underwriting the same proportion of the risks and paying the relative claims.

It follows a brief description of the main characteristics of the two more common kinds of Proportional Reinsurance (see [6, 1]).

Quota Share Reinsurance

- A Quota Share reinsurance allows the cedant to cede to the reinsurer a fixed proportion of all the policies, within the scope of the Treaty.
- The net amount of risk the insurer agrees to keep for its own account is defined as *retention*. In this case, the cedant retains a fixed percentage and the reinsurer bears the exceeding quota share.

- Underwriting the risks, the reinsurer accepts all the conditions originally agreed between the policyholder and the direct insurer.

- It is usual to agree on an absolute quota share limit, to within the potential loss the reinsurer may be burdened with. In case the quota share of the risk exceeds this limit, a new proportion is computed as the ratio between the *quota share limit* and the *original risk*. Consequently, also the premiums are rectified.

- If none of the risks exceeds the quota share limit, it is possible to directly cede the agreed proportion of the whole portfolio: in this sense, this kind of reinsurance is easy to administer.

- The only purpose of a similar structure is to improve the insurer's solvency, by reducing its potential losses: it does not affect, in fact, their distribution and possible peaks.

Surplus Reinsurance

- With a Surplus reinsurance, the retention is fixed at a certain amount: if the risk is lower than this, the insurer takes it on in full.

- If the risk outbreaks the retention, it is reinsured proportionally: the proportion is established on the basis of the size of the overall liability. Smaller risks are likely to be reinsured for the whole percentage exceeding the retention; greater risks, on the other hand, may be reinsured only in part, leaving to the insurer not only the burden of the retention, but also an additional quota.

- This kind of reinsurance requires the retained and reinsured part of each risk to be defined individually. The surplus premium can then be computed on the overall portfolio.

- Since Surplus reinsurance cannot be applied directly to the entire portfolio, it is harder to administer.

- On the other hand, it allows to improve the homogeneity of the portfolio, by eliminating the peaks in it.

1.2.2 Non Proportional Reinsurance

With a non proportional coverage, the insurer only bears a part of the loss, up to a fixed limit called *deductible*. The exceeding part is covered by reinsurance, according to the terms specified in the treaty. It is common practice, when entering into this kind of contract, to fix a ceiling, up to which the loss is recoverable, and to divide the ceded liability into different *layers*.

The conditions of the coverage are agreed directly between the insurer and the reinsurer: these do not depend on the original terms set between the insurer and the policy holder.

It is possible to identify two categories of Non Proportional Reinsurance. The first one is called *Excess of Loss*, in short *XL*; the second *Stop Loss*.

Excess of Loss Reinsurance

It is mainly built upon the specific definition of loss term. As a matter of fact, each loss event may differ significantly, both in terms of occurrence and amount: an insurer needs to be able to buy protection from a single major loss, as well as from many small losses (see [7]).

According to this concept, there exist three types of Excess of Loss coverage: Per Risk XL, Per Event XL and Catastrophe XL (see [1]).

With a *Working excess of loss cover per risk*, *WXL/R*:

- Every risk is considered singularly and requires a cover designed *ad hoc*.

- Each risk's coverage is characterized by properly specified retention, layers and, if necessary, uncovered top.
- In case an event triggers simultaneously multiple *per risk covers*, it will result in an equal number of losses, each to be considered individually.

Both the insurer and the reinsurer need to take into account the total combined risks that may be affected by a single loss event: in insurance, this is referred to as *accumulation*. To this purpose, it is advisable to consider a *Working excess of loss cover per event*, *WXL/E*:

- Its peculiarity is that, independently on how many risks, only one cumulative loss is considered by the coverage and recovered according to the specific terms of the policy.
- It is in charge of the insurer to correctly estimate whether a per event structure better fits its particular needs: despite common beliefs, a similar coverage does not always guarantee higher contributions.
- It is common idea, to consider a reinsurance on a per event basis when the risks are difficult or impossible to define.

A third option is given by the *Catastrophe excess of loss cover*, or *Cat XL*:

- Just like the per event cover, it provides special protection from accumulation losses.
- Instead of being suggested as a substitute for the WXL/R, though, it is intentionally tailored to complement it. In fact, it is designed so as to not be triggered by a loss affecting only one individual risk.
- It only comes into play if the accumulation is a true catastrophe, that is when the loss event involves several risks. Otherwise, the cover is limited to the per risk one.

Stop Loss Reinsurance

It is sometimes referred to as *Aggregate Excess Reinsurance* and it provides the most comprehensive protection for the insurer, since it can be specifically designed to cover any loss event taking place during the year of occurrence. It can not be used as a protection from the general entrepreneurial risk distinctive of the insurance business, but it is a valid means to lower the net retention resulting from the combination of other reinsurance covers. It represents a good solution for those insurers, who want an additional protection against the possibility of several losses impinging on their businesses during the same year of occurrence.

1.3 Costs and benefits of reinsurance

An insurance company has to deal with a severe entrepreneurial risk, exacerbated by the high-risk nature of its business. It does not only have to guarantee its competitiveness, such as any other company, but it also has to manage higher solvency and liquidity risks. The reverse product cycle, which endows the company with advance liquidity inflows, is obviously not enough to guarantee its ability to tackle all its potential future outflows: in case many events happened in a short period of time, the insurer could have to face a loss so severe as to threaten its own existence.

Reinsurance offers a solution to this problem, since it allows the insurer to recover part of the losses, specifically the one which exceeds the agreed retention: this way, the cedant does not risk to incur in major unforeseen and potentially unpayable losses. It helps the cedant stabilize its losses, easing the task of correctly estimating them and reducing the risk of setting aside less capital than necessary. It also works on the side of the profits: in fact, it offers the possibility to balance the results of not maximally diversified portfolios, which would show significant fluctuations otherwise.

Furthermore, the cedant can rely on a specifically designed Catastrophe XL treaty to ensure protection against any catastrophic event: the probability of their occurrence is low, but if they do they can cause an incredibly high number of claims, which may result in the impossibility of underwriting new policies in the best scenario, and in the company going under in the worst.

An efficient risk management strategy, though, should not be limited to contain the company's solvency and liquidity risks. It should also take into account what is known as *underwriting capacity*. By definition (see [16]), this is «the maximum liability that an insurance company is willing to take on from its underwriting activities» and it «[...] represents an insurer's capacity to retain risk».[1] It causes significant restrictions: by stopping the company from writing a larger number of risks, it impinges not only on the results, but also on the degree of diversification of the portfolios. Fewer risks, in fact, make diversification harder. Non proportional reinsurance allows for an increased underwriting capacity, which is likely to be reflected on better diversified portfolios and less foregone profits. Moreover, considering that the transferred risks are most of all tail risks, the portfolios will have a lower volatility and they will be, overall, less risky.

At this point, it is unavoidable to mention the biggest disadvantage: reinsurance is expensive. It is up to the insurer to analyze the cost opportunity of a reinsurance treaty and choose whether to give up on a part of the profits in exchange for reliance on reinsurance, or to take the risk of having to bear all of the claims. To the purpose, it needs to be noted that basing the decision on a simple comparison, between the results of the reinsurered and not-reinsured portfolio, may be unproductive. To ensure a decision taken on the basis of all the relevant criteria, it may be better to compare the *Return On Risk Adjusted Capital* (*RORAC*)

[1] Read more: Underwriting Capacity Definition `http://www.investopedia.com/terms/u/underwriting-capacity.asp`

values of the portfolio, calculated both with reinsurance cover and without (see [11]). The RORAC measures how much capital is at risk because of a certain investment and it indicates how much to reserve so to not become insolvent; for each investment, it is computed as:

$$RORAC = \frac{Expected\,Profits}{Risk - Weighted\,Assets} \tag{1.1}$$

Despite the lower results in the presence of reinsurance, a comparison between these two values may highlight a reduction of the risk much greater than the one of the results. This would imply a smaller requirement of capital by the reinsured portfolio, making it preferable.

Speaking of disadvantages, it is important to point out that any reinsurance purchase entails a credit and liquidity risk, which, however small, must not be neglected. Nevertheless, by paying attention to the rating of the reinsurer and through diversification and collateralization, the insurer can easily minimize it.

In conclusion of this analysis, it follows an outline of the effects of reinsurance under the regulatory framework Solvency II.

1.3.1 Solvency II

Starting from 2016, the new Solvency II regime has been in force in the European Economic Area (EAA). As a consequence of its introduction, today's insurance industry is required to operate under a more demanding regulatory system: built upon the promises of a better match to the true risks faced by the insurance companies, it now imposes stricter standards in terms of risk, value and capital management of their portfolios (see [12]).

The main aim of this new regime is to minimize the possibility of bankruptcy of an undertaking: to the purpose, it is expected to cover all the risks within an interval of probability of 99,5%.

Inspired by the banking regulation *Basel III*, it develops on three pillars: *Pillar 1* concerns the quantitative requirements, *Pillar 2* the qualitative ones and *Pillar 3* regulates the transparency and supervision obligations (see [17]). The first pillar entails the biggest change of this new framework: the shift from a volume-based to a risk-based capital regime.

To satisfy the capital requirements defined by the first pillar, every insurance company is required to meet a solvency ratio of at least 100%. The Solvency Ratio is defined as follows:

$$Solvency\ Ratio = \frac{Own\ Funds}{Solvency\ Capital\ Requirement} \tag{1.2}$$

The Own Funds of the company are calculated as the market value of the assets, after the subtraction of the best estimate of liabilities and a risk margin. The Solvency Capital Requirement (SCR) indicates how much capital the undertaking is required to set aside, so to be able to tackle all the potential outflows over the next 12 months with a probability of 99.5%. According to the new regulation, this should imply its ability to survive 199 out of 200 years. Its calculation is based on different risk modules: this way, the regulator managed to include in the computation of the capital requirements all the actual risks of the insurance business.

Reinsurance's effects reflect on three of these modules: underwriting risk, market risk and counterparty default risk.

The insurer can lower the underwriting risk thanks to reinsurance: as already explained, the undertaking can transfer part of the written risks to one or more reinsurance companies, through the purchase of proper reinsurance covers. According to its needs, it can then decide whether to fulfill its increased capacity, underwriting new risks and enlarging its portfolios, or whether to benefit in terms of capital relief. A smaller underwriting risk, in fact, allows for a reduction in the SCR and, consequently, for an increase in the Solvency Ratio.

Reinsurance indirectly affects also the market risk borne by the insurer. Any reinsurance cover requires the payment of a premium, which constitutes a heavy cost for the buyer and a significant reduction in the resources at its disposal. As a consequence, fewer market assets are available and the market risk is, therefore, reduced. It follows a further decrease in the SCR.

On the other hand, counterparty default risk is negatively affected by reinsurance. This implies an increase in the SCR and a smaller Solvency Ratio. Anyway, as mentioned in paragraph 1.3, the probability of default of a reinsurer is very low and many solutions are available to minimize any potential implication.

Besides the effects on the denominator, reinsurance impacts also on the numerator of the ratio. Under Solvency II, indeed, the risk mitigation effect proper of reinsurance can be reflected on the own funds of the company. As long as both the risk transfer and the deriving default risk are transparent, it can be considered in the computation of a lower risk margin (see [13]). Doing so, the resulting value of own funds is increased, likewise the Solvency Ratio.

Overall, the effects on the ratio can be expected to be positive, proving that a wise use of tailored reinsurance solutions is a valid means to obtain capital relief.

CHAPTER 2

THE PERSPECTIVE OF THE INSURER

2.1 Outline of the model and relevant literature

In the last decades, reinsurance has been addressed in significant researches: the purpose was to determine a model capable of estimating the optimal level and form of reinsurance. Different optimality criterion have been chosen: some authors were interested in maximizing the insurer's profit, some in minimizing its risk. Others were more concerned with the perspective of the reinsurer. In this chapter, we provide an analysis of an optimal non-proportional reinsurance model under the perspective of the insurer.

As explained in the previous chapter, an insurer buys reinsurance to transfer part of its underwritten risks to a third agent, so to reduce its liquidity and insolvency risk. In other words, its aim is to lower its total risk exposure. This model is indeed set as to determine the reinsurance treaty that minimizes the Value-at-Risk of an insurer's total risk exposure.

This model was first introduced by Cai and Tan (2007), whose aim was to determine the optimal retention for a stop-loss reinsurance, under the perspective of the insurer (see [2]). In 2011, Chi and Tan (see [4]) tested the robustness of the model under different set of constraints: in particular, assuming a premium calculated according to the Expected Value principle, they studied how the optimal solution changed with different assumptions on the set of feasible ceded and retained loss functions. Their results can be summarized as follows:

- when the ceded loss function is assumed to be increasing convex, the stop-loss reinsurance is optimal;
- when the ceded and retained loss functions are assumed to be increasing, the stop-loss reinsurance with an upper limit is optimal;
- when the retained loss function is assumed to be increasing and left-continuous, the truncated stop-loss reinsurance is optimal.

Both these works, though, were criticized for being too narrow with concern to the class of admissible premium principles. In response, Chi and Tan (2013) relaxed the assumption of the Expected Value principle and provided a solution to the model under a wider set of premium principles (see [5]).

After defining the set of preliminary conditions, the model proposed by Chi and Tan (2013) is presented and demonstrated step-by-step.

2.2 Preliminary conditions

We denote with X the amount of loss assumed by an insurer, before the purchase of any reinsurance cover. In accordance with the definition proposed by Denuit et al (2005, see [8]), we assume X is a non-negative real-valued random variable on a probability space $(\Omega, \mathcal{F}, P)$.

Definition 1 *A random variable (rv) X is a measurable function mapping Ω to the real numbers, that is, $X : \Omega \to \mathbb{R}$ is such that $X^{-1}((-\infty, x]) \in \mathcal{F}$ for any $x \in \mathbb{R}$, where $X^{-1}((-\infty, x]) = \{\omega \in \Omega | X(\omega) \leq x\}$.*

Random variables are often used by the actuaries, since they allow to measure and compare the outcomes of different events, by mapping them into real numbers.

Each random variable is endowed with a probability distribution, which specifies how likely it is, that its value falls within a given interval. The probability distribution of X is described by a cumulative density function (c.d.f.)

$$F_X(x) = P(X \leq x), \tag{2.1}$$

a survival function (or complementary c.d.f.)

$$S_X(x) = 1 - F_X(x) = P(X > x), \tag{2.2}$$

and a mean

$$0 < \mathbb{E}[X] < \infty.$$

Given X, an insurer purchases a reinsurance cover to the purpose of reducing its borne loss, transferring part of it to the reinsurer. In presence of reinsurance, then:

$$X = f(X) + R_f(X), \tag{2.3}$$

where $0 \leq f(X) \leq X$ represents the amount of loss ceded to the reinsurer, while $R_f(X)$ is the part retained by the insurer. We denote $f(X)$ the *ceded loss function* and $R_f(X)$ the *retained loss function*. Under this notation, the insurer's optimal problem can be rearranged in terms of the optimal partitioning of X: the insurer needs to determine the optimal ceded loss function.

In this model, the ceded loss function is constrained to the so-defined set

$$\mathcal{C} \triangleq \{0 \leq f(x) \leq x : \text{both } R_f(x) \text{ and } f(x) \text{ are increasing functions}\}, \tag{2.4}$$

which assumes the ceded and retained loss functions to be non-decreasing[1] functions. This assumption is important for both its mathematical and practical implications. In fact, it is substantial for demonstrating our statement and providing a solution to the model. Also, it implies that the insurer and the reinsurer have to pay more when losses are larger: this prevents the risk for the reinsurer of having

[1] In this thesis, the term *increasing* means *non-decreasing* and *decreasing* means *non-increasing*.

to bear a loss larger than necessary, due to a dishonest behavior of the insurer, who takes advantage of the fact that it has to pay only up to the deductible. In this sense, this increasing assumption is said to prevent *moral hazard*. Moreover, Chi and Tan (2011), who also studied this set of feasible ceded loss functions, managed to draw two additional remarkable properties.

First, $f(X) \in \mathcal{C}$ and $R_f(X) \in \mathcal{C}$ are Lipschitz continuous:

Definition 2 *A real-valued function $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous if there exists a Lipschitz constant L_f, such that*

$$|f(x_{1)} - f(x_2)| \leq L_f|x_1 - x_2| \qquad \forall\, x_1, x_2 \in \mathbb{R} \textit{ and } L_f > 0.$$

The L_f-constant here measures how fast the function grows: the larger the constant, the faster the growth. The increasing and Lipschitz-continuity properties of the ceded and retained loss functions imply:[2]

$$0 \leq f(x_2) - f(x_1) \leq x_2 - x_1, \quad \forall\, x_1,\, x_2 \text{ s.t. } 0 \leq x_1 \leq x_2 \tag{2.5}$$

Second, the layer reinsurance treaty belongs to $\mathcal{C}$. The importance of this statement is straightforward, since we expect this layer reinsurance to be optimal for the insurer. Given $(x)_+ \triangleq \max\{x, 0\}$, its form is as follows:

$$min\{(x-a)_+, b\} = (x-a)_+ - (x-(a+b))_+ \qquad a, b \geq 0 \tag{2.6}$$

It represents a stop-loss reinsurance, with *deductible* $a > 0$ and *upper limit* $b > 0$. If $b = \infty$ the stop-loss reinsurance is unbounded, meaning that no matter the value taken on by X, the insurer will have to bear an amount of loss at most equal to a. Otherwise, if $b < \infty$ the stop-loss is limited: if X takes on a value $x > a$, the reinsurer will only cover the loss up to b; the exceeding part will be borne by

[2] The derivation is straightforward, since for all x_1 and x_2 such that $0 \leq x_1 \leq x_2$

$$0 \leq R_f(x_1) \leq R_f(x_2) \quad 0 \leq x_1 - f(x_1) \leq x_2 - f(x_2) \quad 0 \leq f(x_2) - f(x_1) \leq x_2 - x_1$$

the insurer. According to the value taken on by X, then, the ceded loss function behaves as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \leq a \\ x - a & \text{if } a < x < (a+b) \\ b & \text{if } x \geq (a+b) \end{cases}$$

As far as the retained loss function is concerned, the three cases are the following:

$$R_f(x) = x - f(x) = \begin{cases} x & \text{if } x \leq a \\ a & \text{if } a < x < (a+b) \\ x - b & \text{if } x \geq (a+b) \end{cases}$$

With the necessary notation and a basic knowledge been provided, we can now define the two characterizing assumptions of this model: the *class of premium principles* and the *VaR risk measure*.

2.2.1 Class of admissible Premium Principles

As we have already mentioned, in exchange for the burden of the underwritten risks, the reinsurer charges an expense to the insurer. This expense can be determined through the use of specific rules, the so-called *Premium Principles*.

Definition 3 *A premium calculation principle is a functional π assigning to a non-negative random variable $X \in \chi$ a non-negative real number $N \in \mathbb{R}_+$*

Since the premium principle depends on the portion of loss ceded to the resinsurer, from now on it will be denoted as $\pi(f(X)$.

Every optimal reinsurance model is required to specify which premium principles guarantee its effectiveness: usually, this choice is made on the basis of some desired properties, the principles need to be endowed with. Moreover, the assumptions on the admissible premium principles constitute one of the measures of a model's robustness: the higher the number of admissible premium principles, the more robust the model. This is what led Chi and Tan to relax the assumption of a premium calculated according to the Expected Value principle (see [4]). In Chi and Tan (2013), in fact, the model is not constrained to a specific premium principle, but to a wider set of them. In particular, this set is composed of every premium principle satisfying these three properties (see [5]):

1. Distribution invariance: For any $X \in \chi$, $\pi(X)$ depends only on the c.d.f. $F_X(x)$.

   Also known as *independence*, the meaning of this property is that the premium does not depend on the cause of a loss, but only on its monetary value and on the probability that it occurs.

2. Risk-loading: $\pi(X) \geq \mathbb{E}[X]$ for all $X \in \chi$.

   The reinsurer needs to charge not only the expected value of the risk X, but also the uncertainty that it entails: otherwise, on average, it will lose money;

3. Stop-loss ordering preserving: For $X, Y \in \chi$, we have $\pi(X) \leq \pi(Y)$, if X is smaller than Y in the stop-loss order (denoted as $X \leq_{sl} Y$).

   Hürlimann (2002) defines the *degree n stop-loss transform* of a non-negative random variable X, for each $n = 0, 1, 2, ...$, as «the collection of partial moments of order n» given by $\Pi_X^n(x) = \mathbb{E}[(X - x)_+^n]$, with $x \in \mathbb{R}$ (see **[Hurlimann2002]**). An exception is made for the *0th stop-loss transform*, which is conventionally determined as $\Pi_X^0(x) = S_X(x) = 1 - F_X(x)$.

   The following definitions result:

Definition 4 *The random variable X precedes Y in the 0th stop-loss order, written $X \leq_{st} Y$, if the moments of order 0 are finite and*

$$\Pi_X^0(x) = S_X(x) \leq \Pi_Y^0(x) = S_Y(x), \text{ uniformly for all } x \in \mathbb{R}.$$

Definition 5 *The random variable X precedes Y in the 1st stop-loss order, written $X \leq_{sl} Y$, if the moments of order 1 are finite and*

$$\Pi_X(x) = \mathbb{E}[(X - x)_+] \leq \Pi_Y(x) = \mathbb{E}[(Y - x)_+], \text{ uniformly for all } x \in \mathbb{R}.$$

It is possible to identify at least 8 premium principles, among the most common ones, that satisfy the above axioms: this corroborates the model, proving its robustness. For the purpose of this thesis, we can confine the analysis of the individual premium principles to their statements (see [25]).

1. Net Premium Principle: $\pi(X) = \mathbb{E}[X]$;

2. Expected Value Premium Principle: $\pi(X) = (1 + \theta)\mathbb{E}[X]$, for some $\theta > 0$;

3. Exponential Premium Principle: $\pi(X) = (1/\alpha) \cdot \ln \mathbb{E}[e^{\alpha X}]$, for some $\alpha > 0$;

4. Proportional Hazards Premium Principle: $\pi(X) = \int_0^\infty [S_X(t)]^c \, dt$, for some $0 < c < 1$;

5. Principle of Equivalent Utility: $\pi(X)$ solves the equation

   $$u(w) = \mathbb{E}[u(w - X + \pi(X))],$$

   where u is an increasing, concave utility of wealth and w in the initial wealth;

6. Wang's Premium Principle: $\pi(X) = \int_0^c g[S_X(t)] \, dt$, where g is an increasing, concave function that maps [0,1] onto [0,1];

7. Swiss Premium Principle: $\pi(X)$ solves the equation

   $$\mathbb{E}[u(X - pH)] = u((1 - p)H),$$

   for some $p \in [0, 1]$ and some increasing, convex function u;

8. Dutch Premium Principle: $\pi(X) = \mathbb{E}[X] + \theta\mathbb{E}[(X - \alpha\mathbb{E}[x])_+]$, with $\alpha \geq 1$ and $0 < \theta \leq 1$.

All the principles above can be used by insurers, as well as reinsurers: in the first case, the premium is computed on X; in the second, on $f(X)$.

Besides the premium principle, the other characterizing assumption of an optimal reinsurance model concerns the risk measure. This model was studied under both the VaR and the CVaR risk measure, but we have chosen to focus only on the Value-at-Risk.

2.2.2 VaR Risk Measure

The *Value-at-Risk*, from now on denoted as VaR, is a common risk measure among both the banking and insurance sectors and it plays a major role in their regulations, as far as the capital requirements are concerned.

Before entering into a deeper analysis of its meaning and properties, it is useful to introduce its definition. Chi and Tan (2013) propose the following:

Definition 6 *The VaR of a non-negative random variable X at a confidence level $1-\alpha$ where $0 < \alpha < 1$ is defined as*

$$VaR_\alpha(X) \triangleq \inf\{x \geq 0 : P(X > x) \leq \alpha\}$$

In simple words, it is the *(1-α)-quantile of the random variable* X and it indicates how much, at most, it is possible to lose at the confidence level $1-\alpha$. Being expressed in units of lost money, its interpretation is immediate.

Denuit et al (2004) finds worth noticing that $VaR_\alpha(X) = F_X^{-1}(1-\alpha)$, where $F_X^{-1}(p)$ is the inverse of the cumulative density function of the random variable X, with $0 < p < 1$ (see [8]). Considering definition in (2.1), the above states that the value (a real number) taken on by X with probability $p = 1-\alpha$ is exactly $VaR_\alpha(X)$; also, it implies that the probability of X taking on a value smaller than $VaR_\alpha(X)$ is $p = 1-\alpha$. Equivalently, $VaR_\alpha(X) = S_X^{-1}(\alpha)$, where $S_X^{-1}(p)$ with

$0 < p < 1$ is the inverse of the survival function. According to the definition in (2.2), this statement emphasizes that the probability of X taking on a value greater than $VaR_\alpha(X)$ is $p = \alpha$. It is clear that $VaR_\alpha(X) = 0$ when $\alpha \geq S_X(0)$: therefore, we assume $0 < \alpha < S_X(0)$.

In light of the above, it emerges that α is to be intended as the level of risk accepted by the insurer. Moreover, we can deduce that the VaR is endowed with all the intrinsic properties of a *quantile function*: in particular, it is an increasing, left-continuous function. As a consequence, the property demonstrated by Dhaene et al (2002) holds (see Theorem 1, [9]), and

$$VaR_\alpha(g(X)) = g(VaR_\alpha(X)) \tag{2.7}$$

is true for any increasing and left-continuous function g. Moreover:

1. For any constant c,

$$VaR_\alpha(X + c) = VaR_\alpha(X) + c;$$

2. For any *comonotonic* random variables X and Y,

$$VaR_\alpha(X + Y) = VaR_\alpha(X) + VaR_\alpha(Y),$$

where the concept of comonotonicity can be explained through the sequence of definitions that follows (see Dhaene et al (2002), [9]).

Definition 7 *A bivariate random vector (X, Y) is said to be comonotonic if it has a comonotonic support.*

Definition 8 *Any subset $A \subseteq \mathbb{R} \times \mathbb{R}$ is called support of (X, Y), if $P((X, Y) \in A) = 1$ holds true.*

Definition 9 *The set $A \subseteq \mathbb{R} \times \mathbb{R}$ is said to be comonotonic, if each two bivariate random vectors in A are ordered componentwise.*

3. For any random variables $X \leq Y$,

$$VaR_\alpha(X) \leq VaR_\alpha(Y).$$

It is important to underline that the VaR does not satisfy the property of *subadditivity*, meaning that $VaR_\alpha(X+Y) \leq VaR_\alpha(X) + VaR_\alpha(Y)$ is not always true. This entails the most argued downside of this risk measure: it is not a *coherent risk measure*. A coherent risk measure is usually eligible, since it is translative, positive homogeneous, subadditive and monotone: in our case, though, the lack of coherence does not constitute a severe limitation.

With all the assumptions been set and all the fundamental tools been introduced, we can finally proceed with the statement and the solution of the model.

2.3 A VaR minimization model

Initially, the *total risk exposure* of the insurer is represented by X. Buying reinsurance, the insurer lowers its exposure by transferring part of its underwritten risks to the reinsurer; in exchange, it pays a premium. Therefore, considering the partitioning of X in (2.3) and the premium paid for reinsurance, the total risk exposure $T_f(X)$ of the cedant is given by:

$$T_f(X) = R_f(X) + \pi(f(X))$$

Consequently, $VaR_\alpha(T_f(X))$ measures the maximum loss reasonably predictable by the cedant at a confidence level $1-\alpha$.

The model is set so as to determine the ceded loss function that minimizes the VaR of the total exposure of the cedant: in this sense, it is said to be optimal. In short, the model can be written as

$$Var_\alpha(T_{f^*}(X)) = \min_{f \in \mathcal{C}} VaR_\alpha(T_f(X)), \tag{2.8}$$

where $T_{f*}(X)$ is the total exposure of the cedant, when the ceded loss function is optimal (denoted as f^*). Obviously, since f is assumed to belong to $\mathcal{C}$ (defined in (2.4)), also $f^* \in \mathcal{C}$.

The solution of the model provided by Chi and Tan (2013) builds on specific key steps. We think it is useful to first introduce them theoretically and then proceed with their mathematical demonstration. These are:

1. Firstly, for any ceded loss function $f \in \mathcal{C}$,

$$h_f(x) \triangleq \min\Big\{\big(x - (VaR_\alpha(X) - f(Var_\alpha(X)))\big)_+, \, f(VaR_\alpha(X))\Big\}, \quad x \geq 0 \tag{2.9}$$

   is defined. We can see that $h_f(x)$ is a layer reinsurance policy of the form (2.6), having deductible equal to $VaR_\alpha(X) - f(Var_\alpha(X))$ and upper limit equal to $f(VaR_\alpha(X))$. Its peculiarities are:

   a) The ceded loss function $h_f(x)$ is increasing, and so is the relative retained function $R_{h_f}(x)$. Then:

$$h_f(x) \in \mathcal{C}.$$

   b) The VaR of the ceded loss function $h_f(x)$ is equal to the VaR of any ceded loss function $f(x)$. Obviously, the same is true for the relative retained functions. Moreover, exploiting VaR's property in (2.7):

$$h_f(VaR_\alpha(X)) = f(VaR_\alpha(X)). \tag{2.10}$$

   c) The total exposure of the insurer, when the ceded loss function is equal to $h_f(x)$, is

$$T_{h_f}(X) = R_{h_f}(X) + \pi(h_f(X))$$

2. Secondly, this layer reinsurance treaty is proved to be optimal, by showing that $VaR_\alpha(T_{h_f}(X))$ is smaller or equal than $VaR_\alpha(T_f(X))$. The proof relies

on the fact that the VaRs of the two retained functions are equal, while the premium paid for (2.9) is smaller.

3. Lastly, it is shown that the layer reinsurance (2.6), with $b < \infty$, is always optimal. In addition, it is possible to rewrite (2.9) as

$$\begin{aligned} h_f(x) &\triangleq \min\Big\{(x-a)_+,\,(VaR_\alpha(X)-a)\Big\}, \\ &= (x-a)_+ - (x - VaR_\alpha(X))_+ \end{aligned} \tag{2.11}$$

and the problem of determining the optimal ceded loss function can be rearranged in terms of the optimal deductible a of a limited stop-loss reinsurance.

Finally, the solution of the model is the following (see [5].

Theorem 1 *For the VaR-based optimal reinsurance model (2.8), the layer reinsurance of the form (2.9) is optimal in the sense that*

$$VaR_\alpha(T_{h_f}(X)) \le VaR_\alpha(T_f(X)), \qquad \forall\, f \in \mathcal{C} \tag{2.12}$$

Moreover, we have

$$\begin{aligned} &\min_{f\in\mathcal{C}} VaR_\alpha(T_f(X)) = \min_{f\in\mathcal{C}_v} VaR_\alpha(T_f(X)) \\ &= \min_{0\le\alpha\le VaR_\alpha(X)} \Big\{a + \pi\big(min\{(X-a)_+,\,VaR_\alpha(x)-a\}\big)\Big\}, \end{aligned} \tag{2.13}$$

where

$$\mathcal{C}_v \triangleq \Big\{min\{(x-a)_+,\,VaR_\alpha(X)-a\} : 0 \le a \le VaR_\alpha(X)\Big\} \tag{2.14}$$

Proof. First of all, we need to demonstrate that for any $f \in \mathcal{C}$

$$h_f(x) \le f(x), \qquad \forall\, x \ge 0 \tag{2.15}$$

with $h_f(x)$ defined in (2.9). We identify two cases.

If $0 < x < VaR_\alpha(x)$, we can use the Lipschitz-continuity property of f in (2.5), so that

$$0 \le f(Var_\alpha(x)) - f(x) \le VaR_\alpha(X) - x, \qquad \forall\, 0 \le x \le VaR_\alpha(X)$$

Rearranging, we obtain

$$0 \leq \Big(x + f(VaR_\alpha(X)) - VaR_\alpha(x)\Big)_+ \leq f(x), \qquad \forall\, 0 \leq x \leq VaR_\alpha(X) \tag{2.16}$$

Since $f \in \mathcal{C}$ is increasing, $x < VaR_\alpha(x)$ implies $f(x) < f(VaR_\alpha(x))$. Then

$$0 \leq \Big(x + f(VaR_\alpha(X)) - VaR_\alpha(x)\Big)_+ \leq f(x) \leq f(VaR_\alpha(X)),$$

from which it can be deduced that

$$h_f(x) = \Big(x + f(Var_\alpha(X)) - Var_\alpha(X)\Big)_+$$

Finally, from (2.16) it is straightforward that

$$f(x) \geq h_f(x), \qquad \forall\, 0 \leq x \leq VaR_\alpha(X)$$

proving that in this case (2.15) holds.

If $x \geq VaR_\alpha(x)$, we can rely on the increasing property of f to derive:

$$f(x) \geq f(VaR_\alpha(X))$$

It is immediate to deduce that $f(VaR_\alpha(X)) \leq \Big(x + f(VaR_\alpha(X)) - VaR_\alpha(X)\Big)$, and then

$$h_f(x) = f(VaR_\alpha(X))$$

$$f(x) \geq h_f(x), \qquad \forall\, x \geq VaR_\alpha(X)$$

In both cases, and then for any $x \geq 0$, equation (2.15) holds and the evaluation of $f(x)$ on x is greater or equal to the evaluation of $h_f(x)$. Moreover, from definitions 4 and 5, we can draw that $h_f(X)$ precedes $f(X)$ in both the $0th$ and $1st$ stop-loss orders. This result can be written as:

$$h_f(X) \leq_{st} f(X)$$

$$h_f(X) \leq_{sl} f(X)$$

Let us now recall one of the properties satisfied by the assumed set of admissible premium principles: *stop-loss ordering preserving*. Given the above findings, this property allows us to state that

$$\pi(h_f(X)) \leq \pi(f(X)), \tag{2.17}$$

meaning that the premium computed on $h_f(X)$ is lower than any other premium computed on $f \in \mathcal{C}$. To determine whether the layer reinsurance treaty $h_f(x)$ is optimal, one last step is required. We know that

$$VaR_\alpha(T_f(X)) = VaR_\alpha\Big(R_f(X) + \pi(f(X))\Big).$$

Thanks to the translation invariance property of VaR, we can rewrite it as

$$VaR_\alpha(T_f(X)) = VaR_\alpha(R_f(X)) + \pi(f(X))$$

Since $R_f(x)$ is assumed to be increasing and Lipschitz-continuos, we can use the property of VaR defined in (2.7) to write

$$VaR_\alpha(T_f(X)) = R_f(VaR_\alpha(X)) + \pi(f(X))$$

and then use the partitioning of X, in (2.3), to rearrange it as

$$VaR_\alpha(T_f(X)) = VaR_\alpha(X) + \pi(f(X)) - f(VaR_\alpha(X))$$

But we know that $h_f(VaR_\alpha(X)) = f(VaR_\alpha(X))$, so that

$$VaR_\alpha(T_f(X)) = VaR_\alpha(X) + \pi(f(X)) - h_f(VaR_\alpha(X))$$

At this point, it is obvious that the only difference between $VaR_\alpha(T_{h_f}(X))$ and $VaR_\alpha(T_f(X))$ consists in the premium: thanks to (2.17), we can deduce that

$$VaR_\alpha(T_{h_f}(X)) \leq VaR_\alpha(T_f(X)), \qquad \forall\, f \in \mathcal{C}$$

proving that the constructed layer reinsurance treaty $h_f(X)$ is optimal. Now, we can set $a = VaR_\alpha(X) - f(VaR_\alpha(X))$ ans substitute it into the definition of $h_f(x)$. Clearly, $0 \leq a \leq VaR_\alpha(X)$. The set described in (2.14),

$$\mathcal{C}_v \triangleq \Big\{min\{(x-a)_+,\, VaR_\alpha(X) - a\} : 0 \leq a \leq VaR_\alpha(X)\Big\},$$

is then composed of all the layer reinsurance policies of the form $h_f(x)$. In particular, every layer reinsurance treaty in $\mathcal{C}_v$ is a stop-loss reinsurance treaty with deductible a and upper limit $(VaR_\alpha(X) - a)$. Since $\mathcal{C}_v \subseteq \mathcal{C}$, the equation

$$\min_{f \in \mathcal{C}} VaR_\alpha(T_f(X)) = \min_{f \in \mathcal{C}_v} VaR_\alpha(T_f(X))$$

does not need further explanations. Finally, substituting $VaR_\alpha(T_f(X))$ with its definition in terms of a, we obtain (2.13). Its meaning is crucial: the minimum VaR obtainable through the choice of the optimal ceded loss function is equal to the minimum VaR obtainable through the choice of the optimal deductible and upper limit of a layer reinsurance. The optimization problem faced by the insurer results significantly simplified, since it only depends on three parameters: a, $VaR_\alpha(X)$ and $\pi(f(X)$.

2.4 Conclusions

In this chapter, we studied the solution proposed in Chi and Tan (2013), to a VaR-based optimal reinsurance model from the perspective of the insurer (see [5]). The hypothesis were confined to a set of premium principles satisfying three basic axioms (distribution invariance, risk loading and stop-loss ordering preserving) and to the assumption that both the insurer and the reinsurer need to pay more for larger losses.

The authors built an *ad hoc* layer reinsurance treaty, which was characterized by a VaR equal to the one of any other admissible ceded loss function, but which entailed a lower premium: it was proved to be optimal. Introducing a new parameter a, it was possible to generalize the optimal layer reinsurance policy to the standard form of a limited stop-loss reinsurance: under the specified set of assumptions, a limited stop-loss reinsurance with deductible a and upper limit $(VaR_\alpha(X) - a)$ is always optimal. In light of these results, it was possible to simplify the optimal reinsurance problem of the insurer. In fact, to minimize the

VaR of the total exposure of the cedant, it is enough to determine the optimal deductible and upper limit of a layer reinsurance.

The robustness of the model is upheld by the wider set of admissible reinsurance premium principles. By choosing a speific premium principle, explicit solutions of the optimal parameters are derivable: going beyond the purpose of this thesis, though, this topic will not be discussed.

CHAPTER 3

THE PERSPECTIVE OF THE REINSURER

3.1 Topic of research and existing literature

In reinsurance research, the perspective of the reinsurer, considered singularly, has not been extensively studied. In our opinion, though, meaningful results might derive from its unconstrained optimization problem.

Among the studies concerning the reinsurer, it is worth citing Huang and Yu (2017) (see [14]). The authors assume the insurer to behave rationally, meaning that it chooses the form and level of reinsurance proved optimal, and the premium to be computed according to the Expected Value principle $\pi(X) = (1+\theta)\mathbb{E}[X]$, for some $\theta > 0$. The optimal safety loading θ is then studied, with respect to three different optimality criteria: maximizing the expectation and the utility of the reinsurer's profit and minimizing the VaR of its total loss.

In this chapter, we contribute to the existing literature by analyzing the VaR-based problem of Chi and Tan (2013), from the perspective of the reinsurer: our aim is to determine the ceded loss function that minimizes the VaR of its total risk exposure. The assumptions underlying the model are the same. This is novel in the literature, since a general solution to the unconstrained optimization problem of the reinsurer, under the VaR criterion, is yet to be determined. Moreover, our wide class of admissible premium principle would endow the model with remarkable robustness.

3.2 The optimization problem of the reinsurer

We denote the total risk exposure of the reinsurer as $T_f^R(X)$ and we define it as

$$T_f^R(X) = f(X) - \pi(f(X)), \tag{3.1}$$

where $f(X)$ is the portion of loss ceded to the reinsurer and $\pi(f(X))$ is the premium paid by the insurer. It is immediate how the reinsurer's exposure is reversed, with respect to the insurer's one. In this case, the total exposure increases with the ceded loss function: it is obvious, since the greater the portion of loss ceded by the insurer, the higher the risk borne by the reinsurer. The premium, on the other hand, constitutes a cash inflow: this is why it lowers the total exposure of the undertaker.

In our model, we are interested in assessing whether it is possible to determine a general form of reinsurance, which minimizes the VaR of the total risk exposure of the reinsurer. The model can then be summarized as

$$VaR_\beta(T_{f^*}^R(X)) = \min_{f \in \mathcal{C}} VaR_\beta(T_f^R(X)), \tag{3.2}$$

where β is the level of risk accepted by the reinsurer and $\mathcal{C}$ is the set defined in (2.4). It is reasonable to assume $\beta \neq \alpha$, since reinsurance companies can afford better risk management strategies and can be expected to tolerate a higher level of risk. Furthermore, we assume $0 < \beta < S_x(0)$ and all the assumptions discussed in section (2.2) continue to apply.

As first question of research, we wonder whether the optimal solution derived for the insurer could be a solution also to the reinsurer's problem. The question is answered in the following subsection.

3.2.1 Different solutions for the two agents

In section (2.3), we proved that the optimal ceded loss function for the insurer, under our set of assumptions, has the form of a limited stop-loss reinsurance. In particular, a key step in the proof was the construction of $h_f(X)$: thanks to some of its peculiar properties, its optimality was first proven and then generalized. Below, we recall its definition (first proposed in (2.9)):

$$h_f(X) \triangleq \min\Big\{\big(x - (VaR_\alpha(X) - f(VaR_\alpha(X)))\big)_+,\ f(VaR_\alpha(X))\Big\}, \qquad x \geq 0$$

When the point of view of the reinsurer is considered, an adjustment of the above is required. In the derivation of the optimal form of reinsurance for the insurer, we considered its level of risk α. The level of risk of the reinsurer, though, is different: in particular, it is assumed to be equal to β. When trying to determine whether this form of reinsurance may be optimal for the reinsurer, too, it makes sense to take into account its own level of risk. Thus, we define $h_{\beta,f}(x)$ as

$$h_{\beta,f}(x) \triangleq \min\Big\{\big(x - (VaR_\beta(X) - f(VaR_\beta(X)))\big)_+,\ f(VaR_\beta(X))\Big\}, \quad x \geq 0 \tag{3.3}$$

Ideally, it is simply the optimal solution we would have derived in section (2.3), if we had assumed the level of risk of the insurer to be defined by β. Since the provided proof only required $0 < \alpha < S_X(0)$, as long as β is included in the same interval, our results hold. Of course, all the properties of $h_f(x)$ are valid also for $h_{\beta,f}(x)$.[1]

By exploiting some of these properties, it is possible to prove that, as far as the reinsurer is concerned, $h_{\beta,f}(x)$ is not optimal: under the VaR criterion, the optimal form of ceded loss function for the cedent is not optimal for the cessionary. So, our statement is

$$VaR_\beta(T^R_{h_{\beta,f}}(X)) \neq \min_{f \in \mathcal{C}} VaR_\beta(T^R_f(X)), \tag{3.4}$$

which can be proven by showing that

$$VaR_\beta(T^R_f(X)) < VaR_\beta(T^R_{h_{\beta,f}}(X)) \tag{3.5}$$

[1] For the details, see page 21.

The proof follows. Given the above definition of total exposure of the reinsurer,

$$\begin{aligned} VaR_\beta(T_f^R(X)) &= VaR_\beta\big(f(X) + (-\pi(f(X)))\big) \\ &= VaR_\beta(f(X)) + (-\pi(f(X))) \\ &= f\big(VaR_\beta(X)\big) - \pi(f(X)), \end{aligned}$$

where the second inequality relies on the translation invariance of the VaR, whereas the third uses the increasing and Lipschitz-continuity property of $f \in \mathcal{C}$. Further, knowing that $f(VaR_\beta(X)) = h_{\beta,f}(VaR_\beta(X))$,

$$VaR_\beta(T_f^R(X)) = h_{\beta,f}\big(VaR_\beta(X)\big) - \pi(f(X))$$

Finally, we can use $\pi(f(X)) \geq \pi(h_{\beta,f}(X))$, to determine:

$$\begin{aligned} VaR_\beta(T_f^R(X)) &\leq h_{\beta,f}\big(VaR_\beta(X)\big) - \pi(h_{\beta,f}(X)) \\ &\leq VaR_\beta(T_{h_{\beta,f}}^R(X)) \end{aligned}$$

At this point, to prove that $h_{\beta,f}(X)$ is not optimal, it is necessary to show that the strict inequality in (3.5) holds true, at least in one case. Mathematically, a similar proof would require more stringent assumptions. Conceptually, though, a similar conclusion is reasonable: in fact, when the strict inequality

$$\pi(h_{\beta,f}(X)) < \pi(f(X)) \tag{3.6}$$

is true, also (3.5) is valid.

This result is trivial. If both the cedent and the cessionary' decisions are taken upon the same criterion, i.e. minimizing a chosen risk measure on their total exposures, a conflict of interests arises. The cedent would rather transfer a greater portion of loss, paying the lowest premium possible. The cessionary, on the other hand, would rather bear a lower risk, being paid as high as possible. Because of this, $h_{\beta,f}(X)$ could not have been optimal for the reinsurer. As explained earlier, the peculiarities of this ceded loss function perfectly fit the interests of the insurer: it guarantees a Value-at-Risk of the retained loss function equal to the one of any other admissible, but it allows for a lower premium to be paid. Obviously, this same peculiarities make it not eligible for the reinsurer.

3.2.2 An optimal solution for the reinsurer

In this section, we try to solve the Var-based optimal reinsurance model in (3.4). Our purpose is to derive a general form of optimal ceded loss function, which minimizes the total exposure of the reinsurer under our set of assumptions. To do so, we resort to a simple intuitive reasoning, starting from the results obtained for the insurer.

In theorem 1, a solution to the VaR-based optimal reinsurance model defined in (2.8) was obtained. Our findings can be summarized as[2]

$$\min_{f\in\mathcal{C}} VaR_\alpha(T_f^I(X)) = VaR_\alpha(T_{h_f}^I(X))$$

As explained before, a change in the assumed level of risk does not compromise our results. Hence, considering our aim of deriving further insights from the point of view of the reinsurer, we substitute the level of risk of the insurer, α, with the reinsurer's one, β.

$$\min_{f\in\mathcal{C}} VaR_\beta(T_f^I(X)) = VaR_\beta(T_{h_{\beta,f}}^I(X)) \tag{3.7}$$

The meaning is simple: when the ceded loss function takes the form of the layer reinsurance $h_{\beta,f}(x)$, the Value-at-Risk of the total exposure of the cedent is minimized. Using the definition of $T_{h_{\beta,f}}^I(X)$, we can write:

$$VaR_\beta^I(T_{h_{\beta,f}}^I(X)) = VaR_\beta^I\big(X - h_{\beta,f}(X) + \pi(h_{\beta,f}(X))\big) \tag{3.8}$$

Let us recall that $X - h_{\beta,f}(X)$ represents the retained function of the insurer, while the premium $\pi(h_{\beta,f}(X)$ depends on the ceded loss function $h_{\beta,f}(X)$. Further, since $h_{\beta,f}(X)$ is a layer reinsurance, it can be rewritten according to (2.6) as

$$h_{\beta,f}(X) = \big(x - VaR_\beta(X) + f(VaR_\beta(X))_+ - (x - VaR_\beta(X))_+\big)$$

[2]To avoid misunderstandings, from now on the superscript I will be used to identify the perspective of the insurer: i.e. $T_f^I(X)$ will denote the total exposure of the cedent.

Then, for any value $x \geq 0$ taken on by the random variable X, equation (3.8) is

$$\begin{aligned} VaR_\beta(T^I_{h_{\beta,f}}(x)) =& VaR_\beta\Big(x - \big(x - VaR_\beta(X) + f(VaR_\beta(X))\big)_+ + \\ &+ \big(x - VaR_\beta(X)\big)_+ + \pi(h_{\beta,f}(X))\Big) \end{aligned} \tag{3.9}$$

Now, let us assume full transparency between the insurer and the reinsurer: in particular, the latter is informed not only of $f(X)$, but also of the original X assumed by the insurer. This allows us to introduce a function $l_f(x)$, such that

$$\begin{aligned} l_f(x) &\triangleq x - \big(x - VaR_\beta(X) + f(VaR_\beta(X))\big)_+ + \big(x - VaR_\beta(X)\big)_+ \\ &= x - min\Big\{\big(x - (VaR_\beta(X) - f(VaR_\beta(X)))\big)_+, \, f(VaR_\beta(X))\Big\} \\ &= x - h_{\beta,f}(x) \end{aligned} \tag{3.10}$$

Since $h_{\beta,f}(x) \in \mathcal{C}$, it is straightforward that also $l_f(x) \in \mathcal{C}$, making it an admissible ceded loss function under our set of assumptions. When the ceded loss function is $l_f(x)$, the VaR of the total exposure of the reinsurer, for any $x \geq 0$, is

$$\begin{aligned} VaR_\beta(T^R_{l_f}(x)) =& VaR_\beta\Big(x - \big(x - VaR_\beta(X) + f(VaR_\beta(X))\big)_+ + \\ &+ \big(x - VaR_\beta(X)\big)_+ - \pi(l_f(X))\Big) \end{aligned} \tag{3.11}$$

Let us compare equations (3.9) and (3.11). In equation (3.9), the first part of the argument,

$$\Big(x - \big(x - VaR_\beta(X) + f(VaR_\beta(X))\big)_+ + \big(x - VaR_\beta(X)\big)_+\Big),$$

represents the retained loss function $R_{h_{\beta,f}}(x)$, or rather the portion of loss borne by the cedent; the same, in equation (3.11), represents the ceded loss function $l_f(x)$, or rather the burden of risk of the cessionary. Despite the conceptual difference among the two, they are mathematically identical. What changes, on the other hand, is the premium. Being dependent on the ceded loss function, in the total exposure of the insurer it is computed according to $h_{\beta,f}(x)$; in the total exposure of the reinsurer, instead, it is computed according to $l_f(x) = x - h_{\beta,f}(x)$. Furthermore, since the premium has opposite impacts on the exposures of the agents, in equation (3.11) it is of opposite sign.

In light of these hints, it is noticeable that $l_f(x) = R_{h_{\beta,f}}(x)$, for any $x \geq 0$. It follows that

$$l_f(VaR_\beta(X)) = R_{h_{\beta,f}}(VaR_\beta(X)) \tag{3.12}$$

We can now use one of the properties $h_{\beta,f}$ was endowed with, by construction: $VaR_\beta(h_{\beta,f}(X)) = VaR_\beta(f(X))$. Since $h_{\beta,f}(x),\ f(x) \in \mathcal{C}$, both the functions are increasing and continuous. We are then allowed to use a property of the VaR, defined in (2.7), to derive $f(VaR_\beta(X)) = h_{\beta,f}(VaR_\beta(X))$. Straightforwardly, also

$$VaR_\beta(X) - f(VaR_\beta(X)) = VaR_\beta(X) - h_{\beta,f}(VaR_\beta(X))$$

is true. Recalling the definition of the retained loss function, this can be rewritten as

$$R_{h_{\beta,f}}(VaR_\beta(X)) = R_f(VaR_\beta(X)), \tag{3.13}$$

which leads to

$$l_f(VaR_\beta(X)) = R_f(VaR_\beta(X)), \tag{3.14}$$

thanks to the transitive property of equality. Now, under the same assumptions used to introduce $l_f(x)$, let us introduce a ceded loss function $g_f(x)$, such that

$$g_f(x) \triangleq x - f(x)$$

Since $f(x) \in \mathcal{C}$, then also $g_f(x)$ is included in our set of admissible ceded loss functions. Moreover, basing on the reasoning leading to $l_f(x) = R_{h_{\beta,f}}(x)$, we can draw that $g_f(x) = R_f(x)$, for any $x \geq 0$. Then, $g_f(VaR_\beta(X)) = R_f(VaR_\beta(X))$ and equation (3.14) becomes:

$$l_f(VaR_\beta(X)) = g_f(VaR_\beta(X)) \tag{3.15}$$

In the previous chapter, in the proof of theorem 1, we demonstrated that $h_f(x) \leq f(x)$, for any $x \geq 0$. The result holds up for any $0 < \alpha < S_X(0)$, so

$$h_{\beta,f}(x) \leq f(x)$$

is also true. A direct consequence is that, for any $x \geq 0$,

$$\begin{aligned} l_f(x) &= x - h_{\beta,f}(x) \\ &\geq x - f(x) \\ &\geq g(x) \end{aligned}$$

Basing on the same proof, we use definition 5 to infer that $g_f(x)$ precedes $l_f(x)$ in the 1st stop-loss order, written

$$g_f(X) \leq_{sl} h_f(X)$$

Following through, since all the admissible premium principles are assumed to satisfy the *stop-loss ordering preserving* property, we can conclude that

$$\pi(g_f(X)) \leq \pi(l_f(X)) \tag{3.16}$$

What we found, is that a ceded loss function of the form of $l_f(X)$ consents the reinsurer to collect a higher premium. This result is of particular significance. The premium is in inverse proportion to the total exposure of the reinsurer: a ceded loss function allowing for a higher premium, and all else being equal, can then be expected to lead to a lower total exposure of the reinsurer.

Given the results summarized in equations (3.14) and (3.16), we can write

$$l_f(VaR_\beta(X)) - \pi(l_f(X)) \leq g_f(VaR_\beta(X)) - \pi(g_f(X)),$$

which is equal to

$$VaR_\beta(l_f(X)) - \pi(l_f(X)) \leq VaR_\beta(g_f(X)) - \pi(g_f(X)),$$

provided that both $l_f(x)$ and $g_f(x)$ are increasing and continuous functions. Then, thanks to the translation invariance of the VaR, we can rewrite the above as

$$VaR_\beta\big(l_f(X) - \pi(l_f(X))\big) \leq VaR_\beta\big(g_f(X) - \pi(g_f(X))\big)$$

Therefore, according to the definition of $T_f^R(X)$,

$$VaR_\beta(T_{l_f}^R(X)) \leq VaR_\beta(T_{g_f}^R(X)), \tag{3.17}$$

which proves that a ceded loss function of the form of $l_f(x)$ allows the reinsurer for a total exposure always equal or lower than the one obtainable with a generic $g_f(x) = x - f(x)$. Then

$$\min_{g \in \mathcal{C}} VaR_\beta(T^R_{g_f}(X)) = VaR_\beta(T^R_{l_f}(X)) \tag{3.18}$$

But can this result be generalized to any function $f \in \mathcal{C}$?

Let us consider our set of admissible ceded loss function $\mathcal{C}$: it is composed of all the increasing $f(x)$, such that the retained loss function $R_f(x) = x - f(x)$ exists and it is increasing. Noticing that $I(x) = x$ is the *identity function*, which assigns every real number x to the same real number x, we can write $R_f(x)$ as

$$R_f(x) = I(x) - f(x)$$

and $f(x)$ as

$$f(x) = I(x) - R_f(x)$$

Within $\mathcal{C}$, then, $f(x)$ can always be expressed as the identity function minus a generic increasing function $R_f(x)$. Moreover, $R_f(x)$ belongs to $\mathcal{C}$. It follows that, under the perspective of the insurer, if

$$\min_{f \in \mathcal{C}} VaR_\beta(T^I_f(X)) = VaR_\beta(T^I_{h_{\beta,f}}(X)),$$

then

$$VaR_\beta(T^I_{h_{\beta,f}}(X)) \leq VaR_\beta(T^I_{I(x)-f(x)}(X))$$

is also true. But, by construction,

$$\begin{aligned} I(x) - g_f(x) &= I(x) - I(x) + f(x) \\ &= f(x) \end{aligned}$$

and so, from (3.18), it is possible to deduce that

$$VaR_\beta(T^R_{l_f}(X)) \leq VaR_\beta(T^R_f(X)) \tag{3.19}$$

Thus, we have proved that

$$\min_{f \in \mathcal{C}} VaR_\beta(T_f^R(X)) = VaR_\beta(T_{l_f}^R(X)) :$$

the optimal ceded loss function for the reinsurer, among all the functions included in $\mathcal{C}$, has the form of $l_f(x)$, defined as

$$l_f(x) \triangleq x - \big(x - VaR_\beta(X) + f(VaR_\beta(X))\big)_+ + \big(x - VaR_\beta(X)\big)_+$$

To conclude, we set $a_\beta = VaR_\beta(X) - f(VaR_\beta(X))$ and we substitute it in $l_f(x)$. Doing so, we can rewrite the optimal form of ceded loss function, as

$$\begin{aligned} l_f(x) &= x - min\{\big(x - (VaR_\beta(X) - f(VaR_\beta(X)))\big)_+, \, f(VaR_\beta(X))\} \\ &= x - min\{(x - a_\beta)_+, \, VaR_\beta(X) - a_\beta\} \\ &= x - (x - a_\beta)_+ + (x - VaR_\beta(X))_+ \end{aligned} \tag{3.20}$$

3.3 A comparative analysis of the results

In sections (2.3) and (3.2.2), we proposed two optimal-reinsurance models from the perspectives of the insurer and of the reinsurer, respectively. The point of view of the insurer boasts a wide dedicated literature: the vast diversity of optimality criteria have allowed researchers to develop numerous models. Among the many, we chose a VaR-minimization model, as proposed by Chi and Tan (2013, see [5]): after an introduction of all the necessary assumptions and through every step of the mathematical demonstration, the main findings were derived.

Under the perspective of the insurer, the ceded loss function minimizing the Value-at-Risk of its total risk exposure has the form of a layer reinsurance treaty, with deductible $0 \leq a \leq VaR_\alpha(X)$ and upper bound $VaR_\alpha(X) - a$:

$$\begin{aligned} h_f(x) &= min\Big\{(x - a)_+, \, VaR_\alpha(X) - a\Big\} \\ &= (x - a)_+ - (x - VaR_\alpha(X))_+ \end{aligned}$$

This layer reinsurance is known as the *limited stop loss* reinsurance. Under this contract, the behaviors of the ceded and retained loss functions depend on the value x, taken on by X. In particular, it is possible to identify three ranges, on the basis of the agreed deductible and upper limit.

$$h_f(x) = \begin{cases} 0 & \text{if } x \leq a \\ x - a & \text{if } a < x < VaR_\alpha(X) \\ VaR_\alpha(X) - a & \text{if } x \geq VaR_\alpha(X) \end{cases}$$

$$R_{h_f}(x) = \begin{cases} x & \text{if } x \leq a \\ a & \text{if } a < x < VaR_\alpha(X) \\ x - (VaR_\alpha(X) - a) & \text{if } x \geq VaR_\alpha(X) \end{cases}$$

It follows a textual explanation of the behavior of both the ceded and retained loss function, in each of the three cases above.

1. When $x \leq a$, the loss is smaller than the deductible a: the insurer bears the whole loss and nothing is covered by the reinsurer;

2. When $a < x < VaR_\alpha(X)$, the loss is greater than the deductible. The insurer bears an amount of loss equal to a and it cedes to the reinsurer the exceeding part, given by $x - a$. The presence of an upper bound implies that the reinsurer agrees on covering a portion of the loss, only as long as this portion is smaller than a fixed cap, $VaR_\alpha(X) - a$: this limit is reached when
$$x - a = VaR_\alpha(X) - a$$
$$x = VaR_\alpha(X)$$

3. When $x \geq VaR_\alpha(X)$, the reinsurer limits its coverage to a maximum amount, equal to $VaR_\alpha(X) - a$. The insurer is required to pay for what exceeds it, that is $x - (VaR_\alpha(X) - a)$.

The two functions are graphed in figure (3.1a) and (3.1b).

Then, we shifted to the agent on the other side of the contract: the reinsurer. Keeping our set of assumptions constant, we introduced an optimal-reinsurance model aimed at determining the ceded loss function, which minimizes the Value-at-Risk of the reinsurer's total risk exposure.

First, we proved that the form of reinsurance, which is optimal for one party, is not optimal for the other: this was expected. When both of the agents want to minimize their maximum potential loss, their antithetical positions cannot avoid leading to a conflict of interests: each party would rather the other to bear the most of the loss. This conflict is only partially mitigated by the premium: a reinsurer might be willing to accept the transfer of a greater portion of the loss, if this allows it to gain in terms of the receivable premium. Obviously, the opposite is true for the insurer.

Secondly, we derived a general form of the ceded loss function, which is proved optimal for the reinsurer:

$$\begin{aligned} l_f(x) &= x - min\{(x - a_\beta)_+, VaR_\beta(X) - a_\beta\} \\ &= x - (x - a_\beta)_+ + (x - VaR_\beta(X))_+ \end{aligned}$$

The most important implication of this result is that

$$l_f(x) = x - h_{\beta,f}(x),$$

where $h_{\beta,f}$ is the optimal ceded loss function of an insurer, which accepts a level of risk equal to the reinsurer. Also in this case, it is possible to study the behavior of $l_f(x)$, on the basis of the value x.

$$l_f(x) = \begin{cases} x & \text{if } x \leq a \\ a & \text{if } a < x < VaR_\beta(X) \\ x - (VaR_\beta(X) - a) & \text{if } x \geq VaR_\beta(X) \end{cases}$$

It is straightforward that when $\alpha = \beta$, meaning that the levels of risk of the insurer and of the reinsurer are equal,

$$l_f(x) = R_{h_f}(x) \qquad \forall\, x > 0$$

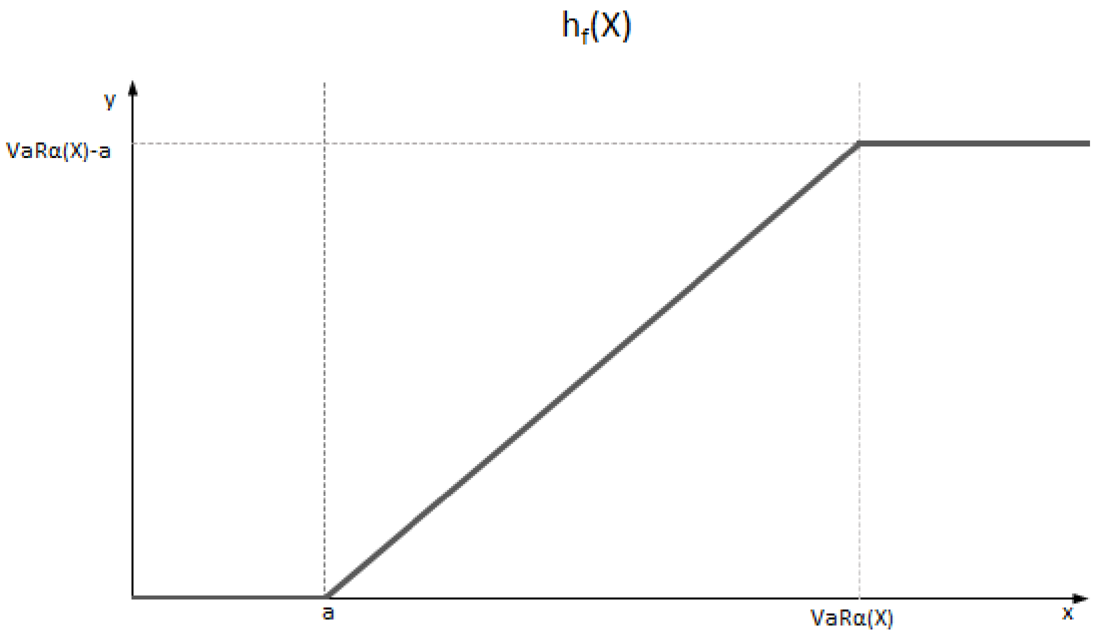
In simple words, $l_f(x)$ describes the portion of loss that, if borne by either of the agents, allows it to minimize the VaR of its total risk exposure. Therefore, the graphical representation of $l_f(x)$ is the one in figure (3.1b): it shows that the reinsurer is willing to cover the whole loss, as long as this is smaller than an agreed level a; when the loss is bigger than a, moved by the aim of protecting itself, it is disposed to cover only a fixed amount, that is a; when the loss exceeds $VaR_\alpha(X)$, it agrees on covering $x - (VaR_\alpha(X) - a)$. Practically, he would rather be on the other side of the contract.

When $\beta \neq \alpha$, the difference between the levels of risk of the two agents leads to an asymmetry: the optimal solutions of the agents are not exactly reversed. If the reinsurer is willing to tolerate a higher level of risk, $\beta \geq \alpha$, it is more likely to be disposed of covering a greater portion of loss, lowering the threshold, up to which it only pays for a fixed amount a. Anyway, though, a difference between α and β as big as to allow for an agreement, is hardly imaginable in practice.

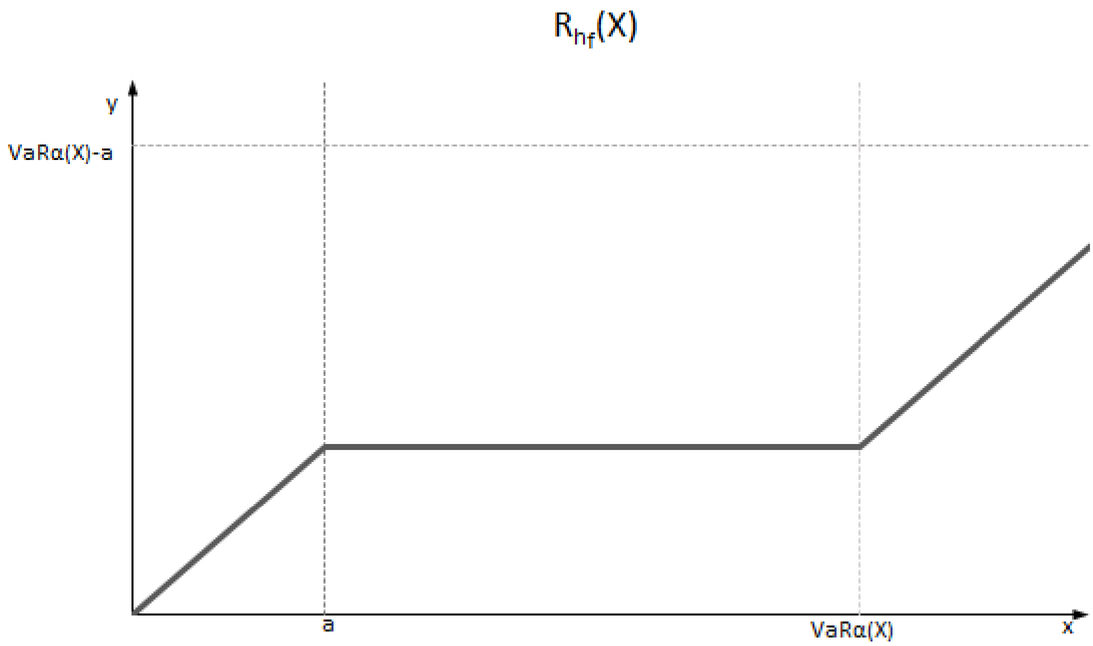
3.4 Conclusions

In this chapter, we proved that when both the insurer and the reinsurer aim at minimizing the VaR of their total exposures, they are unlikely to conclude a contract. In fact, the optimal solution of one party would lead to an unacceptable high value of the other's VaR. Moreover, in the extreme scenario, in which the level of risk is the same for both agents, the solutions to their unconstrained optimization problems are exactly $h_f(x)$ and $x - h_f(x)$: in simple words, the reinsurer would need to be transferred the exact portion of loss, that the insurer would need to retain.

From these results, it arises the necessity of a third reinsurance model, capable of taking into account the opposite interests of the parties, in the attempt of determining an optimal halfway solution. In the literature, two main models of this kind have been proposed: these are studied in the next chapter.



(a) Optimal *ceded loss function* of the insurer



(b) *Retained loss function* of the insurer,
when the ceded loss function is optimal

Figure 3.1: Graphical representation of $h_f(X)$ and $R_{h_f}(X)$

CHAPTER 4

THE PERSPECTIVE OF BOTH

4.1 Multi-objective Optimization

So far, we have studied the individual perspectives of the two agents involved in a reinsurance contract. For each, a VaR-based optimal reinsurance model has been analyzed and solved: what emerged, is that the solutions of the respective unconstrained optimization problems are incompatible.

As a consequence, the necessity of a third optimization problem arises: its peculiarity is the presence of two objective functions, which need to be optimized simultaneously. In economics, this problem is endowed with a clear interpretation: indeed, it theorizes the attempt of the reinsurer to design a contract which satisfies its optimality requirements, being at the same time attractive for the insurer. In mathematics, it is a *Two-objective Optimization Problem*.

Multi-objective Optimization is an area of research, which allows to formalize and solve optimization problems with two or more optimality criteria. Miettinen (1998, see [22]) describes the general form of a multicriteria optimization problem as

$$\min_{x \in S}(f_1(x),\ f_2(x),\ ...,\ f_k(x))^T \tag{4.1}$$

Given each objective function $f_i(x) : \mathbb{R}^n \to \mathbb{R}$, for $i = 1, 2, ..., k$ and $k > 1$,

- $f(x) = (f_1(x),\ f_2(x),\ ...,\ f_k(x))^T$ is the *vector of objective functions*,

- $x = (x_1, x_2, ..., x_n)^T$ is the *decision variable vector*,

- $z = (z_1, z_2, ..., z_k)^T$ is the *objective vector*.

S is the set of constraints defining the *feasible region*, to which the variable vectors belong: obviously $S \subseteq \mathbb{R}^n$, with the latter being the *decision variables space*. $Z = f(S)$ is the *feasible objective region*, computed as the image of the feasible region: it is defined as

$$Z = \{z \in \mathbb{R}^k : z = f(x), x \in S\}$$

and it is a subset of the *objective space* $\mathbb{R}^k$. Every objective vector $z \in Z$ is admissible.

The need for alternative solving methods arises when the *ideal vector* z^{id}, with components

$$z_i^{id} = \min_{x \in S} f_i(x), \qquad \text{with } 1 \leq i \leq k \tag{4.2}$$

does not belong to the feasible objective region. Otherwise, (4.2) would be a banal solution to problem (4.1), easily determinable through the resolution of k single-objective optimization problems, one for each objective function. Assuming $z^{id} \notin Z$ is the same as saying that the objective functions are at least partly conflicting: in this sense, the resulting Multi-objective Optimization Problem (MOP) is nontrivial. In this case, a single solution that optimizes all the objective functions simultaneously does not exist: there exist, though, a number of Pareto optimal solutions. Thus, we can say that the main aim of Multi-objective Optimization is to seek Pareto optimal solutions.

Definition 10 *A decision variables vector $x^* \in S$ is* Pareto optimal, *or* efficient, *if $\nexists\, x \in S$, s.t:*

$$f_i(x) \leq f_i(x^*) \qquad \forall i,\ i = 1, 2, ...k$$

$$f_j(x) < f_j(x^*) \qquad \textit{for at least one index } j$$

According to the above, a decision variable vector is said Pareto optimal, when it is impossible to further minimize at least one of the objective functions, without degrading any of the other objective values. On the same conditions is founded the definition of Pareto optimal objective vector.

Definition 11 *An objective vector $z^* \in Z$ is* Pareto optimal, *or* efficient, *if $\nexists\, z \in Z$, s.t:*

$$z_i \leq z_i^* \qquad \forall i,\ i = 1, 2, ...k$$
$$z_j < z_j^* \qquad \textit{for at least one index } j$$

or if the decision variables vector corresponding to z^* is Pareto optimal.

Usually, the Pareto optimal solution of a MOP is not unique: S_E denotes the *efficient set*, composed of all efficient solutions $x^* \in S$; Z_N is called *nondominated set* or *efficient frontier* and it includes all the *equilibria points*, each representing a nondominated vector $z^* = f(x^*) \in Z$ (see [26, 19]).

Considering the existence of multiple optimal solutions, the goal of a MOP is not always straightforward. Sometimes, solving a MOP means to derive the whole efficient set and corresponding efficient frontier; others, it implies the choice of a single optimal solution, among the many. The latter requires the presence of a better informed *Decision Maker* (DM), who can make the selection according to its subjective preferences. On the basis of the role played by the DM, it is possible to distinguish four main categories of Multi-objective Optimization methods:

1. *No-preference methods*: in the absence of a DM, a compromise solution is determined;

2. *A priori methods*: in light of the DM's preferences, the solution which best satisfies them is chosen;

3. *A posteriori methods*: in light of the set of Pareto optimal solutions, the DM chooses the one which better satisfies its preferences.

4. *Interactive methods*: the DM chooses the best solution through an iterative procedure, by describing how the Pareto optimal solution derived at each round can be improved.

The above is just one of the many classification criteria used in the literature: i.e. Kaucic and Daris (2015), referring to Schukla and Deb (2017), identify two major categories (see [18, 23]). They define as *classical* the methods resorting to direct or gradient-based procedures, on the basis of specific mathematical principles; *non-classical* are, instead, the ones following natural or physical principles.

In view of what just stated about Multi-objective Optimization, it is clear why it well-fits our purpose. The objective functions of the insurer and of the reinsurer have been proven to be conflicting: a unique optimal solution, which simultaneously optimizes both of them, does not exist. Through the formalization and resolution of a MOP, though, we can investigate the existence of a Pareto optimal solution.

Our Two-objective Optimization problem is formalized as

$$\min_{f \in \mathcal{C}} \big(VaR_\alpha(T_f^I(X)),\, VaR_\beta(T_f^R(X))\big)^T \tag{4.3}$$

The two objective functions are the Values-at-Risk of the total risk exposures of the insurer and of the reinsurer. Their levels of risk are $0 < \alpha < S_X(0)$ and $0 < \beta < S_X(0)$, respectively. The set $\mathcal{C}$ defines our feasible region: $0 \leq f(X) \leq X$ and both $f(x)$ and $R_f(x)$ are increasing.

As already mentioned, several are the approaches usable to determine a solution: the most commonly used are the *weighted sum method* and the *ϵ-constraint methods*. Both of them are classical, a posteriori methods. In the following sections, we explain how these were used in the optimal reinsurance models proposed in Cai, Lemieuz and Liu (2015) and Lo (2017).

4.1.1 Cai, Lemieux and Liu (2015): the weighted-sum method

In Cai, Lemieux and Liu (2015), the optimal reinsurance contract under the perspectives of an insurer and a reinsurer is studied (see [3]). Under the assumption that both of the agents aim at minimizing the VaR of their losses and under the same set of feasible ceded loss functions $\mathcal{C}$, the solution to each agent's individual problem is first derived. The optimal reinsurance contract for the insurer is

$$f_i^*(x) = (x - VaR_{\frac{1}{1+\theta}}(X))_+ - (x - VaR_\alpha(X))_+,$$

whereas for the reinsurer it is

$$f_r^*(x) = x - (x - VaR_{\frac{1}{1+\theta}}(X))_+ - (x - VaR_\alpha(X))_+$$

Easily, it is possible to trace back the above to the general optimal forms of ceded loss functions, derived in the previous chapters and defined in (2.11) and (3.20), respectively. In addition, the choice of a premium computed according to the Expected Value premium principle,

$$\pi(f(x)) = (1+\theta)\mathbb{E}[f(X)],$$

makes it possible for the authors to specify the optimal levels of deductible a and upper limit b. Specifically,

$$a = a_\beta = VaR_{\frac{1}{1+\theta}}(X)$$

is the optimal deductible for both the insurer and the reinsurer, while the optimal upper limits are

$$b = \begin{cases} VaR_\alpha(X) - VaR_{\frac{1}{1+\theta}}(X), & \text{for the insurer} \\ VaR_\beta(X) - VaR_{\frac{1}{1+\theta}}(X), & \text{for the reinsurer} \end{cases}$$

It can be deduced that the different levels of risk, α and β, only affect the upper limits. The deductible, instead, is a function of the *safety loading* θ, only.

Given the contradiction of the optimal individual solutions, the authors address the problem of determining a reinsurance contract, which simultaneously takes into account the insurer's objectives and the reinsurer's goals. To do so, they design a Single-objective Optimization Problem, with objective function equal to the convex combination of the VaR risk measures of the insurer's loss and the reinsurer's loss. Thus,

$$V(f) = \lambda VaR_{\alpha}\big(X - f(X) + \pi(f(X))\big) + (1-\lambda)VaR_{\beta}\big(f(X) - \pi(f(X)\big)$$

is the objective function, with $\lambda \in [0,1]$. Despite not explicitly stated, it is possible to notice that this is the first step of resolution of a Two-objective Optimization problem, when resorting to the *weighted sum method*. This method allows to reformulate a Multi-objective to a Single-objective Optimization Problem thanks to a *scalarization* technique, which is applied by associating each objective function with a weighting coefficient. The resulting weighted sum is the single-objective function of the Single-objective Optimization Problem, also called *scalar* optimization problem or *weighted sum scalarization* (see [10]). Given a general MOP, as the one described in (4.1), the scalar optimization problem is formalized as

$$\min_{x \in S} \sum_{i=1}^{k} \lambda_i f_i(x),$$

with $i = 1, 2, ..., k$ and $k > 1$. Each weighting coefficient λ_i is a real number, such that $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$: the latter means that the weights are normalized. Then,

$$\min_{f \in \mathcal{F}} V(f), \tag{4.4}$$

is exactly the weighted sum scalarization of the Two-objective Optimization Problem

$$\min_{f \in \mathcal{F}} \Big(VaR_{\alpha}\big(X - f(X) + \pi(f(X))\big),\, VaR_{\beta}\big(f(X) - \pi(f(X))\big)\Big)^T$$

This is the model studied in Cai, Lemieux and Liu (2015). The feasible region $\mathcal{F}$ is defined as

$$\mathcal{F} \triangleq \{f \in \mathcal{C} : f(VaR_\beta(X)) - L_2 \leq \pi(f(X)) \leq f(Var_\alpha(X)) - VaR_\alpha(X) + L_1\}, \quad (4.5)$$

where L_1 and L_2 are the upper bounds of the VaR risk measures of the insurer ans reinsurer' losses, respectively. In fact, the authors assume

$$VaR_\alpha\big(X - f(X) + \pi(f(X))\big) \leq L_1 \quad (4.6)$$

and

$$VaR_\beta\big(f(X) - \pi(f(X))\big) \leq L_2, \quad (4.7)$$

meaning that both agents are willing to reduce the VaR of their total exposure to a tolerated value, such that

$$L_1 \leq VaR_\alpha(X), \qquad \text{with } L_1 > 0$$

and

$$L_2 \leq VaR_\beta(X), \qquad \text{with } L_2 > 0$$

Given these constraints, the set $\mathcal{F}$ is easily derived in a few steps. Considering that X and $f(X)$ are comonotonic random variables and thanks to the the translation invariance of the VaR, as well as to its property defined in (2.7), inequalities (4.6) and (4.7) can be rewritten as

$$VaR_\alpha(X) - f(VaR_\alpha(X)) + \pi(f(X)) \leq L_1$$

$$f(VaR_\beta(X)) - \pi(f(X)) \leq L_2$$

Then,

$$\pi(f(X)) \leq f(VaR_\alpha(X)) - VaR_\alpha(X) + L_1 \quad (4.8)$$

$$f(VaR_\beta(X)) - L_2 \leq \pi(f(X)) \quad (4.9)$$

are also true. Since $\pi(f(X))$ is equal in both equations, constraints (4.8) and (4.9) define the set of reinsurance contracts acceptable by both agents, namely $\mathcal{F}$. Furthermore, a premium computed according to the Expected Value premium principle continues to be assumed.

At this point, the Pareto optimal solutions are sought by the authors. The weighted sum method uses a *one-at-a-time* strategy to generate the entire non-dominated set. This means that an optimal solution to the problem needs to be derived, for each value taken on by the weight λ. The authors carry out this analysis through the study of the following cases:

1. If $\lambda = 0$, or if $\alpha = \beta$ and $0 \leq \lambda < \frac{1}{2}$, problem (4.4) is reduced to

$$\min_{f \in \mathcal{F}} VaR_\beta\big(f(X) - \pi(f(X))\big)$$

2. If $\lambda = 1$, or if $\alpha = \beta$ and $\frac{1}{2} < \lambda \leq 1$, problem (4.4) is reduced to

$$\min_{f \in \mathcal{F}} VaR_\beta\big(X - f(X) + \pi(f(X))\big)$$

3. If $\alpha = \beta$ and $\lambda = \frac{1}{2}$

4. If $\alpha < \beta$ and $0 < \lambda \leq \frac{1}{2}$

5. If $\alpha < \beta$ and $\frac{1}{2} \leq \lambda < 1$

6. If $\alpha > \beta$ and $0 < \lambda \leq \frac{1}{2}$

7. If $\alpha > \beta$ and $\frac{1}{2} \leq \lambda < 1$

Besides the trivial solutions of the first two cases, the authors manage to find the optimal reinsurance contract for each of the cases from 4 to 7: for the details, we refer to [3, pp. 825-826]. Moreover, each of these solutions is optimal also under the assumptions in case 3.

The structure of the analysis described above allows for a deeper significance of the obtained results: it enables a full understanding of how the optimal reinsurance contract changes, according to how much importance is put by the reinsurer on having a competitive contract. In cases 5 and 7, in fact, a greater weight is assigned to the insurer's optimality criterion; in cases 4 and 6, on the other hand, the reinsurer's one prevails. Furthermore, it is worth noticing that the convexity of the objective function $V(f)$ endows this method with greater efficiency, since when the problem is convex, all the Pareto optimal solutions can be found.

To conclude, the weighted sum method proves efficient in the attempt of solving the Two-objective Optimization problem, which formalizes the authors' aim of deriving the reinsurance contracts optimal for both the insurer and the reinsurer. In the next section, we investigate the use of a different Multi-objective Optimization method: the ϵ-constraint method.

4.1.2 Lo (2017): the ϵ-constraint method

The *ϵ-constraints method* differs from the weighted sum method, since it does not involve any aggregation of the objective functions. Indeed, only one objective function is chosen to be optimized, while all the others are transformed into constraints. The *ϵ-constraint problem* obtained by the MOP in 4.1 is formalized as:

$$\begin{aligned} &\min_{x \in S} f_j(x) \\ &\text{s.t. } f_l(x) \leq \epsilon_l, \qquad \forall\, l,\ l = 1, 2, ..., k \text{ and } j \neq l \end{aligned} \tag{4.10}$$

with ϵ_l being a real number, which represents the upper bound (or tolerated value) set to $f_l(x)$. The main limit of this method is that its solution is weakly Pareto optimal (see [22]).

Definition 12 *A decision variables vector $x^* \in S$ is* weakly Pareto optimal, *or* weakly efficient, *if $\nexists\, x \in S$, s.t:*

$$f_i(x) < f_i(x^*), \qquad \forall i,\ i = 1, 2, ...k$$

As a consequence, to guarantee that the solution produced by the ϵ-constraint method is Pareto optimal, it is necessary to solve k different problems or to obtain a unique solution. Moreover, also this approach exploits a one-at-a time strategy: this implies that to ensure its convergence to a Pareto optimal solution, it is necessary to solve several constrained optimization problems through a systematic variation of the values taken on by each ϵ_l.

In light of these considerations, it is possible to explain how this approach was used in Lo (2017) (see [20]). In his work, the author implements three different constrained optimization model: one of them, in particular, drew our interest. Coherently with our notation, this can be formalized as:

$$\begin{aligned} &\inf_{f \in \mathcal{C}} TVaR_\alpha(T_f^I(X)) \\ &s.t.\, VaR_\beta\big(f(X) - \pi(f(X))\big) \leq \epsilon \end{aligned} \tag{4.11}$$

It can be noticed that the problem is formalized so as to look for an infimum, instead than for a minimum: this is beacuse the existence of a minimum is not guaranteed. As said above, in fact, the use of this method does not ensure the possibility of finding a Pareto optimal solution.

The feasible region assumed by Lo is our set $\mathcal{C}$. Though, he puts a restriction on the admissible premium principles, since he specifies the premium to be computed as

$$\pi(f(X)) = \int_0^\infty r(S_{f(X)}(t))dt,$$

where $r : [0, 1] \to \mathbb{R}^+$ is an increasing function, such that $r(0) = 0$. Moreover, the risk measure chosen for the insurer is not the usual VaR, but the *Tail-VaR*, also known as *Conditional VaR*. Here we introduce this risk measure by proposing the definition used in Chi and Tan (2013, see [5]).

Definition 13 *Based upon the definition of VaR in 6, the Tail-VaR of X at a confidence level $1 - \alpha$ is defined as*

$$TVaR_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha VaR_s(X)ds \tag{4.12}$$

It is worth mentioning that the TVaR, unlike the VaR, is a coherent risk measure. Nevertheless, we believe the author chose the TVaR over the VaR not because of its subadditivity property, but mainly for a conceptual reason: under the TVaR, in fact, it is more visible how the optimal solution is affected by the addition of a constraint to the optimal reinsurance model.

Looking at the model in (4.11), the similarities with the ϵ-constraint problem in (4.10) are easy to notice. The author chooses as objective function to optimize the TVaR of the insurer's loss; the VaR of the reinsurer's total exposure, instead, is converted into a constraint by putting an upper bound ϵ to it. Doing so, a Single-objective Optimization problem is obtained. Usually, at this point, the resolution of the constrained optimization problem is pursued by drawing on classical mathematical tools, i.e. through the derivation of the Lagrangian function and the study of the Khun-Tucker optimality conditions. The novelty of this work is that the author managed to avoid the use of these sophisticated mathematical techniques by transcribing some heuristic considerations into mathematical statements. Doing so, he was able to study all the possible values of ϵ, as required by the ϵ-constraint method, by dividing them into four sets: he then found the optimal reinsurance treaty for each of them.

What emerges is that, in presence of a reinsurer's risk constraint, the insurer tends to search protection from small losses. This diverges from its usual optimal solution: in Chi an Tan (2011), in fact, it was proven that, under the TVaR criterion, the stop-loss reinsurance is always optimal (see [4]). This means that, despite its willing of fully hedging its right tail, this optimal solution would lead to an unacceptable value of the reinsurer's VaR. As a consequence, it accepts to cede small-sizes losses in order to allow for a mutually acceptable contract. Here, we do not dwell on the detailed forms of optimal reinsurance contracts, for which we refer to [20].

In conclusion, it was possible to show how the ϵ-constraint method has already been used in the literature dedicated to optimal reinsurance. Moreover, the same proved effective in the derivation of new optimal solutions, admissible by both agents and therefore more significant in practice.

4.2 Conclusions and future research suggestions

In this chapter, we introduced a field of mathematics called Multi-objective Optimization, which is aimed at the resolution of multicriteria optimization problems. Its peculiarity is that it provides a vast diversity of methods, which proves effective in the derivation of Pareto optimal solutions in presence of conflicting objective functions. Against this background, we studied whether this discipline might fit our purpose of deriving an optimal, and at the same time mutually acceptable, reinsurance contract.

We demonstrated that, despite the resort to this field of mathematics is not made explicit by the authors, there already exist examples of its use in the literature: in fact, through a critical analysis of two optimal reinsurance models, we managed to explain how these made use of two specific Multi-objective Optimization methods.

In conclusion, our suggestion is to further draw on this discipline to the purpose of finding a new method, yet to be used in the search for a reinsurance contract, optimal for both the parties it concerns. An example might be given by the *value function method*: through this method, the opposite interests of the parties might be taken into account in the formulation of a unique utility function. The same would then become the objective function of a Single-objective Optimization problem, whose solutions also solve the original MOP. Moreover, two main advantages follow from the use of this method. Being an a Priori method, it is not necessary to derive the entire efficient set: the utility function can be designed on the basis of the specific interests of the DM, making the resolution of the problem easier. Furthermore, the utility function does not need to be linear.

CONCLUSIONS

In this thesis, we studied the optimal reinsurance arrangement under the perspectives of both the concerned parties.

We chose the VaR-based optimal reinsurance model in Chi and Tan (2013) because of its proven robustness. In fact, the authors proved that a limited stop-loss reinsurance is always optimal, as long as only two assumptions are true. The first is that both the ceded loss function and the retained loss functions are increasing. The second is that the premium principle satisfies three properties: distribution invariance, risk loading and stop-loss ordering preserving.

Then, we analyzed this result under the reinsurer's point of view: in particular, we studied whether stop-loss reinsurance might be optimal also for the reinsurer. Keeping the same assumptions, we proved that this treaty does not allow to minimize the VaR of the reinsurer's total risk exposure: in other words, it is not optimal. Furthermore, starting from an intuitive reasoning and through the use of some properties of the VaR and of the set of admissible ceded loss functions, we derived the form of reinsurance optimal for the reinsurer.

What emerged, is that the solutions to these two optimal reinsurance models are conflicting: if both the agents aim at concluding a contract, which is optimal under the VaR criterion, they are not reaching any agreement. Moreover, if the cedant and the reinsurer accept the same level of risk, the optimal solutions are exactly opposite. As a matter of fact, the ceded loss function, which was proved optimal for the reinsurer, describes the portion of loss that, if borne by either of the agents, allows it to minimize its total risk exposure. Also under the assumptions of different levels of risk, it was possible to conclude that a unique optimal solution to these problems does not exist.

In light of these results, we formalized a Multi-objective Optimization Problem. It is characterized by the simultaneous presence of two objective functions: the VaR of the insurer's total risk exposure and the VaR of the reinsurer's total risk exposure. Since these objective functions were proven to be conflicted, the resort to Multi-objective Optimization is logical: this field of mathematics, in fact, provides a vast diversity of Multi-objective Optimization methods, which allow to investigate the existence of Pareto optimal solutions.

Moreover, our reasoning is supported by the literature: two Multi-objective Optimization methods have already been used, in the attempt of deriving an optimal and mutually acceptable reinsurance contract. In fact, despite the resort to this discipline was not made explicit by the authors, we managed to prove that in Chi, Lemieux and Liu (2015) the weighted sum method was used; Lo (2017), instead, used the ϵ-constraint method.

To conclude, we believe that Multi-objective Optimization might represent a way to overcome one of the main critiques moved to insurance research. As long as the analysis of optimal reinsurance will be confined to one of the two perspectives, the theoretical results will be endowed with low practical significance: the derivation of the optimal form of reinsurance, for the insurer, will serve little purpose, if the same cannot be accepted by the reinsurer. Through the use of a Multi-objective Optimization method, instead, it would be possible to take into account both the perspectives contemporaneously: the so-derived optimal reinsurance contract, then, will also be actually enforceable.

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