Master's Degree programme
in Economics and Finance
Second Cycle (D.M. 270/2004)

Final Thesis

From model-based to model-free implied volatilities:
the VIX index and the new volatility derivatives

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Academic Year
2015/2016
Abstract

The aim of this work is to study volatility and volatility derivatives. We will consider the development of new models and methodologies for the measurement, forecasting and hedging the volatility risk. In particular, we will provide an extensive overview of the main models used to measure volatility in financial markets, starting from traditional model-based approaches, and then considering more recent model-free approaches. The focus of the work will be on the CBOE VIX index, which provides an effective model-free estimation of the S&P500 implied volatility. As a further aim, we intend to study the risk diversification potential of investment strategies involving volatility derivatives during bullish markets, given the negative correlation between the markets and the VIX. In order to do this, we built a rebalancing strategy which systematically invests, besides of a set of base portfolios composed by traditional and alternative assets, in VIX call options with different degrees of moneyness. The obtained results suggest that, during periods in which markets rise discontinuously, the VIX calls diversification effect is unsatisfactory and the protection from downturns is limited to a few circumstances. What we have found is that purchasing VIX calls reduces the volatility of the base portfolios by remarkably shrinking the returns.
To my mum.
Acknowledgments

First and foremost, I would like to express my deepest gratitude to my supervisor Professor Nardon for the insights and expertise that guided me throughout the progress of this research.

I would like to thank my lifetime friends Matteo, Nicola, Davide, Gae, Giorgia, Anna, Max, and Francesco, for being always there for me every time I come back home, their presence is fundamental to me; Marta, for the long chats and the appreciated advices; Riccardo and Luca, for the times spent together, wherever they go there will always be a cozy second home for me; Simone and Elia, the busiest guys on earth, for the laughs and the hints; Matteo and Davide, for the great company and the beers together; my neighbors Doina, Anna, Mirella, Luigi and Diego, for the delicious dinners and their noisy renovations. Thanks also to all the fantastic people I have encountered during my venetian years, in particular to Giacomo, Andrea, Elisabetta, Marco, Chiara, Eda and the whole C.Veronica team, without them no spritz will be the same.

A huge thank goes to Sara, for being my safe harbor during stormy weather, her constant presence is irreplaceable to me.

Finally, I am profoundly grateful to my parents and my sister, for standing by me during this journey, for their silent help and unconditional love. A special mention goes to my mum, whose strength inspired me. Thanks also to the whole family, in particular my grandpas, grandmas and my beloved cousins.
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Introduction

In financial markets, volatility is a measure of uncertainty and represents the degree of variability of the returns of a given asset. Usually, market risk is used as a synonym for volatility but, conceptually speaking, they are extremely different. While risk is relative to the probability of a negative outcome of a certain event, volatility reflects the variability of the possible outcomes, including both the upside and the downside risk. Clearly, in order for anyone to profit from financial markets, there must be necessarily a price movement. The advantage is that as volatility increases, the opportunity to gain increases too. Unfortunately, the drawback is that also the risk increases, and consequently the likelihood of a loss.

Estimating and forecasting the asset return volatility is a challenging task. This constitutes a relevant issue since volatility is the heart of most theoretical and practical applications in financial markets. A common practice is to derive the volatility historically from past prices of a given asset. An alternative approach is to find the implied volatility of an option contract, by reverting the option pricing model used to find the price of those options.

Black and Scholes (1973) implied volatilities are often used to forecast volatility, however many authors witnessed misgivings with this practice, noting the inconsistency of forecasting changes in volatility from a model based on constant volatility. Empirical evidence on volatility surfaces demonstrated that the implied volatility strongly depend both on the strike price and the time to maturity of the options. Furthermore, the assumption of a log-normal distribution of the underlying asset price is not supported by real data and the asset’s returns are characterized by heavy tails, skewness, jumps and negative correlation between volatility and stock price level. A plethora of models have been proposed to bypass the criticisms of the BSM model. Possible solutions have been to incorporate in the option pricing model a deterministic time-varying volatility, as Dupire (1994) proposed, or
to consider a stochastic volatility, as in Heston (1993). Nonetheless, the comparison between the estimated and the realized volatility is a joint test of the assumptions of those models and the efficiency of the markets.

One of the most innovative methods in volatility forecasting literature relates to the estimation of a model-free implied volatility (MFIV) measure which, at least on principle, can be derived directly from a comprehensive set of European option prices, with strikes spanning the full range of possible values of the underlying asset at the option’s expiration. The intuition was introduced and extended by Britten-Jones and Neuberger (2000): starting from the pioneering work of Breeden and Litzberger (1978), they defined the MFIV as the square root of the risk-neutral expectation of the variance. Moreover, Demeterfi et al. (1999) demonstrated that the concept of MFIV can be associated to the price of a variance swap, replicated by conveniently assuming a position in options of all strikes.

Conceptually, the MFIV is deemed to be a superior estimation technique since it does not depend on any option pricing formula and, instead, relies only on the aggregate previsions made by the investors and translated in quoted option prices. Nevertheless, the theoretical requirements to extract the MFIV are not met by existing data thus some approximations are inevitable. In particular, there are practical limitations in terms of the availability of liquid options with strike prices covering the entire support of the risk-neutral distribution.

In recent years, the popularity of volatility indexes has grown sharply. The development of this indexes has been helped by the introduction of the MFIV, which represents the backbone of their calculation. Published by the CBOE, the VIX is the brightest example of volatility index such as it has gained the moniker of “fear index”. The VIX replaced a multitude of volatility measures with a unique value, obtained as a weighted average of S&P500 call and put options prices with a one-month maturity. Given its model-free theoretical foundation, the VIX is considered to be a transparent index: the implied volatility is in fact evaluated under the so-called risk-neutral probability and it remains valid under very general assumptions regarding the returns dynamics.

Both academics and practitioners identified not only its correlation with future realized volatility, but also with global risk factors. Given the tangible implications, the necessity by market participants to invest in volatility has brought the CBOE to enable the trading of VIX derivatives, and in particular of VIX futures and options. Volatility derivatives have been found to
be particularly useful during financial crises as portfolio diversifiers, when alternative assets fail to provide the needed diversification.

During the period 2015-2016, the U.S. stock markets have been characterized by a stable growth interrupted by unexpected drops in the equity indexes, immediately followed by corresponding hikes in the VIX. The opposite relationship between the VIX and the S&P500 deem feasible that it may be possible to exploit such a feature to reduce the risk of traditional portfolios by correctly assuming a long position in VIX call options. However, detecting a straightforward relation between the VIX options prices and the other assets is a complicated issue, and may give rise to even harmful consequences for the portfolio.

The final aim of this study is thus to investigate the benefits and the drawbacks of investing in VIX derivatives during bullish markets, from the risk diversification point of view. An appropriately rebalanced strategy is built by adding a percentage of VIX call options with different degrees of moneyness to a set of traditional and alternative portfolios.

The thesis is organized as follows. In Chapter 1 we present a review of the literature regarding the cornerstones of volatility modeling, in order to understand the basis of more advanced volatility estimation techniques. In Chapter 2 we explain in detail the breakthrough induced by the model-free implied volatility. In Chapter 3 we introduce the VIX index, focusing on its calculation methodology and its interaction with the stock markets. In Chapter 4 we give an overview of the principal features of VIX derivatives. Finally, in Chapter 5 we explain in detail the construction of the strategy investing in VIX call options, along with the analysis of the results we obtained.
Chapter 1

Model-based volatility

Volatility plays a central role in many theoretical and practical applications in the financial markets. These include portfolio theory, derivatives pricing and risk management. Basically, every institution involved in a market transaction is subject to the volatility risk, which potentially leads to large losses in the value of the assets. Losses can also be due to errors in the estimation and the forecast of the volatility. Given its importance, it is crucial to measure the volatility with the greatest accuracy possible.

The first requirement to price an option is the estimation of the volatility. Option dealers make use of several option pricing models in order to derive the quoted prices in the markets. By inverting the pricing procedure it is possible to extract a measure of the estimated volatility, the so-called implied volatility. As a consequence, the estimation of the implied volatility is strictly connected with the assumptions of the pricing model.

In contrast with historical volatility, which is derived from the past prices, implied volatility is forward-looking. As a matter of fact, it is derived by the price of options expiring in the future. Hence, it is used not only as a measure per se but also as a forecasting tool.

In this Chapter we will present the cornerstones of option pricing and volatility modeling, from the pioneering work of Black and Scholes (1973) to the Heston stochastic volatility model (1993). These are called model-based volatility measures, in contrast to the model-free measures, which do not refer to an underlying model.

In the Black-Scholed framework, volatility is treated as a constant. There have been many attempts to extend the Black-Scholes work to account for a more accurate definition of the volatility. One possible approach relates to
the possibility that the volatility is time-dependent, another aims to incorporate a stochastic volatility factor, or allows for discontinuous jumps in the stock price process. However, these extensions may generate several practical difficulties.

The different approaches, together with their criticisms, are analyzed in this Chapter. But first, we need to introduce the most basic concept in volatility estimation, historical volatility.

1.1 Historical volatility

The simplest measure of the risk of holding a certain asset can be recovered from its past price behavior. Since we are interested in giving an estimation of the volatility of the asset, we define its price at time $t$ as $S_t$. The price must be positive, $S_t > 0$, and its logarithmic return $r_t$ is then given by:

$$r_t = \ln\left(\frac{S_{t+\Delta t}}{S_t}\right).$$

The return is a random variable characterized by the expected value $\mu \Delta t$ and volatility $\sigma \sqrt{\Delta t}$. The volatility is a measure that quantifies the amount of dispersion around the average price of a financial asset for a given period of time. Financially speaking, when the volatility is higher, the asset is riskier and therefore the probability of larger gains (or losses) is greater. However, the concept of risk is different: it relates only to the possibility of a negative outcome (a loss), while the volatility refers to the likelihood of a movement of the asset price in both directions.

The most basic measure of volatility is the sample standard deviation. Over a given time horizon $t = 1, ..., T$, its measure is calculated through by the following equation:

$$\hat{\sigma} = \sqrt{\frac{1}{T-1} \sum_{i=1}^{T} (r_i^2 - \bar{r})^2}.$$  \hspace{1cm} (1.2)

where $\bar{r}$ is the mean return. Sometimes the variance, $\sigma^2$, is also considered as a synonym of the concept of volatility. Anyway, the variance does not provide an intuitive interpretation as the standard deviation does. The latter, in fact, has the same unit of measure of the mean and thus it is directly comparable.
When dealing with a more practical approach, however, the historical volatility is not always a sensible measure. The calculation of the volatility can, indeed, be corrupted by using the sample log return as the expected future return. This is due to the fact that the sample mean is not a precise measure of the true mean, unless the considered period is sufficiently long. Inaccurate mean leads to noisy volatility estimates since, as we already have discussed, the volatility is calculated in terms of deviation from the mean. On the other hand, a longer sample period may also be dangerous, even though it can be seen as an improvement. Since volatility is not constant over time, including more data would mean including in the estimation of the volatility irrelevant information.

In order to remedy to the sample mean distortion, often volatility is measured considering a mean return equal to zero. Over the long term very high or very low returns are not realistic, thus assuming a null drift will normally provide a more reliable calculation of the volatility.

As a rule of thumb, the number of observations that has to be taken into consideration depends on the number of days to which the volatility estimation has to be applied to. For example, when evaluating an option expiring in one month, daily data of the last 30 days must be included in the calculations.

In general, the volatility we refer to is the annualized standard deviation, which is the spread the return can have over one year horizon. To obtain the annualized volatility it is sufficient to divide it by the square root of the time period expressed in years $\Delta t$, as:

$$\sigma = \frac{\hat{\sigma}}{\sqrt{\Delta t}}.$$  \hfill (1.3)

In a continuous setting, the theoretical definition of realized volatility for a given price history is the continuous integral:

$$\hat{\sigma} = \frac{1}{T} \int_0^T \sigma(t, Y_t) dt,$$  \hfill (1.4)

where $\sigma(t, Y_t)$ is the volatility process of the asset, which depends on time $t$ and a generic process $Y_t$ of the volatility itself.

1This framework is very general. In fact, the volatility process can even be constant.
1.2 Black-Scholes-Merton model

The most important breakthrough in option pricing theory was introduced by Fischer Black, Myron Scholes and Robert Merton (1973). In their pioneering work, Black and Scholes derived the famous equation, the "Black-Scholes partial differential equation", which describes the relation between the price of a derivative and its underlying asset over time. The mathematical understandings were then developed in the same year by Merton, who first coined the term “Black-Scholes option pricing model” and provided a valuation formula for European options.

The importance of the model was recognized in 1997 when Merton and Scholes, Black unfortunately died a few years before, were awarded with the Nobel prize for Economics. These findings led to a boom in options trading and legitimized scientifically the activities of many options markets around the world.

The basic idea of the Black-Scholes (shortly, BS) model is condensed in these few lines:

“If options are correctly priced in the market, it should not be possible to make sure profits by creating portfolios of long and short positions in options and their underlying stocks”

In fact, in absence of arbitrage opportunities\(^2\), a portfolio composed by an opportune quantity \(n\) of stocks and a short position in one call option must earn the risk-free rate in a way that is completely independent from the stock price movements.

The stock price and the derivative are both affected by the same source of uncertainty: stock price movements. For a very short period of time\(^3\), the riskless portfolio is able to offset the gains or losses in the stock with the gains or the losses of the derivative position. Hence, the portfolio is hedged correctly.

\(^2\)Arbitrage opportunities arise when it is possible to set up a portfolio which guarantees a profit without exposure to negative cash flows. The absence of arbitrage is solely a sufficient requirement to build realistic models in the financial markets, since modeling the chaos of real markets is quite problematic. If it is possible to replicate the payoff of a derivative using a portfolio of underlying securities, the prices of the two must be equal. If not, an arbitrage opportunity exists. Option pricing and risk-neutral valuation are based on the no-arbitrage argument.

\(^3\)Theoretically, the portfolio is riskless only for an instantaneous short period of time. In order to remain riskless, it must be rebalanced very frequently.
The model relies on very strong assumptions to derive the BS differential equation:

- the stock price $S_t$ follows the stochastic process $dS_t = r S_t \, dt + \sigma S_t \, dW_t$, and the risk-free interest rate and the volatility are known and constant;
- the continuously compounded returns follow a log-normal distribution;
- the option is European, meaning that it can be only exercised at maturity;
- there are no dividends or other distributions during the life of the derivative;
- the short selling of securities is permitted without penalties;
- markets are frictionless: there are no transaction costs or taxes;
- it is possible to trade any fraction of a security;
- money can be borrowed at the risk-free interest rate $r$;
- there are no arbitrage opportunities;
- security trading and option trading are synchronous and continuous.

In the BS model the stock price follows the process:

$$dS_t = r S_t \, dt + \sigma S_t \, dW_t,$$  \hspace{1cm} (1.5)

and the respective behavior of the stock return is then:

$$\frac{dS_t}{S_t} = r \, dt + \sigma \, dW_t,$$  \hspace{1cm} (1.6)

where $(W_t)_{t \geq 0}$ is a Wiener process\footnote{The Wiener process $W$ is a a particular type of Markov process. Its increments are independent distributed normally with mean zero and variance of $\Delta t$. A Markov process is a continuous stochastic process in which the past history of the variable does not count, but only the current value of the variable is relevant for predicting the future. It implies that the probability distribution of the price at any particular future time is not dependent on the particular path followed by the price in the past. This property is consistent with the weak form of market efficiency since the present price of a stock contains all the information contained in a record of past prices.}, with $r$ and $\sigma$ constant.
Since the expected rate of return\(^5\) required by the investor is independent from the value of the stock, in the BS model the rate of return is constant. Over a short period of time, \(\Delta t\), the expected increase in the stock price \(S\) should be \(\mu S\Delta t\). In a riskless world, the second term disappears and the stock must grow at the continuously compounded rate of \(\mu\) per unit of time. Since this is not the reality, Black and Scholes made the assumption that the same uncertainty governs the stock price and the returns. The uncertainty, measured by the standard deviation, has to be proportional to the stock price. When the risk-neutral valuation\(^6\) holds, the rate of return \(\mu\) is substituted by the risk-free rate of interest \(r\).

By applying Itô’s formula to (1.5), we find the explicit solution, over a general time period \(T - t\) with \(T \geq t\), as

\[
S_T = S_t e^{(r - \frac{\sigma^2}{2})(T - t) + \sigma(W_T - W_t)}.
\]

(1.7)

The continuously compounded return in (1.1) is then not deterministic but the realization of a random process. It is in fact, the exponential of the realization of a Brownian motion normally distributed with mean \((r - \frac{\sigma^2}{2})(T - t)\) and variance \(\sigma^2(T - t)\). Hence, the stock prices are log-normally distributed because the log-prices are distributed as

\[
\ln S_T \sim N \left( \ln S_t + (r - \frac{\sigma^2}{2})(T - t), \sigma^2(T - t) \right).
\]

(1.8)

**1.2.1 Black-Scholes price of a European option**

A European option is a contract which, traded at some point in time \(t\), gives the right to buy (in the case of call option) or to sell (in the case of a put option) a fixed amount of the underlying asset of price \(S_t\), and at the fixed price \(K\), called strike price, at the fixed time \(T \geq t\), called expiration.

\(^5\)Expected drift divided by stock price.

\(^6\)In a risk-neutral vision, all the variables are not affected by the risk preferences of the investors. If the expected return of the stock \(\mu\) is involved in calculations, this would not be the case. In this framework, investors do not require a premium for taking additional risks and the return on investments in every asset is the risk-free interest rate. In the case of BS, solutions are valid for all the situations, not only in risk neutral but also in risk-averse world. Objective measures and risk-neutral measures differ since the first is based on the view of the investor while the second does not incorporate the investor’s risk preferences, which means that an option can be priced by discounting its expected cash flows using the risk-free rate.
The payoffs of European-style options at expiration are:

\[ f_{\text{Call}}(T, S_T) = (S_T - K)^+, \quad (1.9) \]

\[ f_{\text{Put}}(T, S_T) = (K - S_T)^+. \quad (1.10) \]

For the non-arbitrage principle, the value of an option at expiration must correspond to its payoff, so that

\[ V(T, S_T) = f(S_T). \quad (1.11) \]

In a risk-neutral world, the BS equation describes in time the value of the European option \( V(t, S_t) \) with respect to the underlying asset price \( S \) in this way:

\[ \frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0. \quad (1.12) \]

Considering first the case of a call option, its price process will follow:

\[ C_{BS}(t, S_t) = E[e^{-r(T-t)} f(S_T)|\mathcal{F}_t], \quad (1.13) \]

where \( f(S_T) \) is the payoff of the option at expiration and \( \mathcal{F}_t \) is the \( \sigma \)-algebra.

Following the Wiener process properties along with (1.5) and (1.9), we obtain the price as:

\[ C_{BS}(t, S_t) = S_t N(d_1) - Ke^{-r(T-t)} N(d_2), \quad (1.14) \]

where

\[ d_1 = \frac{\ln(S_t/K) + (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}, \quad (1.15) \]

\[ d_2 = d_1 - \sigma \sqrt{T-t}. \quad (1.16) \]

The function \( N(x) \) is the cumulative probability distribution function for a standardized normal distribution and \( \sigma \) is the volatility.

In order to value the put price we build a portfolio with a long position in a call option and a short position in a put, which has the following payoff:

\[ f_{\text{Call-Put}}(T, S_T) = (S_T - K)^+ - (K - S_T)^+ = S_T - K. \quad (1.17) \]
Applying the non-arbitrage principle, we find the put-call parity as:

\[ P_{BS}(t, S_t) = C_{BS}(t, S_t) - S_t + Ke^{-r(T-t)}. \]  

(1.18)

Substituting inside (1.14) and rearranging we obtain the price of a put option:

\[ P_{BS} = Ke^{-r(T-t)}N(-d_2) - S_tN(-d_1). \]  

(1.19)

### 1.2.2 Implied volatility

When pricing an option, there is one fundamental parameter that must be taken into account: the stock price volatility. Unfortunately, this is not directly observable in the markets. Option dealers use to estimate the future volatility in stock markets and utilize it to quote options. In an inverse manner, knowing the prices in the option market make it possible to find the value of the volatility that has been used to calculate those prices: this measure is called implied volatility (hereinafter, IV).

The matter generates a bit of “chicken or egg” question. In reality, many factors are determinant for the calculation of option prices. Beside the mathematical point of view, there is also a supply-demand mechanism. Prices always adjust. For example, when the market participants “fear” that there will be turmoil in the future, they will ask for call options for protection, increasing the demand and pushing their price higher, like in a vicious circle.

Since the implied volatility changes between options with different strike prices, it represent a crucial quantity that highlights the difference between the BS model and how the pricing of options, and the related estimation of the volatility along the strike prices, actually works in real markets.

Mathematically, the value of an option \(C_{BS}\) depends, through a generic function \(g\), on the volatility \(\sigma\) along with other parameters, such as:

\[ C = g(\sigma, ...). \]

The function \(g\) is monotonically increasing in \(\sigma\). There exist a particular value of \(\sigma\) that leads to a particular value of \(C\). In other words, there exist a function \(l = g^{-1}\) such that

\[ \sigma_{C^*} = l(C^*, ...), \]

where \(C^*\) are the market prices and \(\sigma_{C^*}\) is the IV.
The IV’s original definition is related to the initial BS pricing method. Anyway, whatever method option dealers used to find those prices, it is possible to extract the IV in terms of the BS method. Besides this, other forms of IV have been proposed over the years.

The inverse \( l \) of the pricing model \( g \) has no closed-form solution. Instead, a root finding technique is widely used to solve the equation:

\[
g(\sigma, \ldots) - C = 0,
\]

Among all the possible techniques, Newton-Raphson method and bisection method can be used to derive the IV. While the first method requires the knowledge of partial derivative of the options theoretical value with respect to the volatility, also called \textit{Vega}, the latter is useful also when the derivative is not known. Indeed, it is sufficient to estimate upper and bottom bounds for the implied volatility.

1.2.3 Black-Scholes model inconsistency and volatility smiles

Even though all of its assumptions can be shown to be wrong to a greater or lesser extent, the BS model is profoundly important both in theory and in practice.

The assumptions made about the market functioning are clearly not realistic. Stocks often pay dividends, they must be acquired in units or, sometimes, in batches, trading is asynchronous and discontinuous, transaction fees are required and arbitrage opportunities arise frequently. However, the two main drawbacks are that the assumptions regarding the probability distribution of the stock prices and the constant volatility are found to be wrong.

First, many empirical studies have determined that the asset’s log-return distribution is not Gaussian: the tails are fatter and the peak is higher (leptokurtic distribution). In order to have a normal distribution of the stock returns two conditions must be satisfied: the volatility of the asset has to be constant and the price must change smoothly with no jumps. Both conditions are denied by the empirical evidence because volatility varies over time and the markets often exhibit jumps that cause extreme outcomes more likely.

Second, if we leave the IV as the unknown value while we let vary the other features of the option (time to maturity and strike price), we contradict
the BS assumption of constant volatility. As a matter of fact, the IV is not a linear function of the strike price and the expiration date. The IV function with respect to the strike price has positive concavity. This phenomenon is the well-known “volatility smile”. The effect suggest that the risk premium associated to options far out-of-the-money is greater than what is given by near at-the-money options.

A special case of the smile happens when the IV is a decreasing function of the strike price. In such a situation, the left tail of the price distribution, called the implied distribution, is fatter than the normal distribution and we observe what it is called “volatility skew”. As we can see in Figure 1.1, changing expiration, the same option’s IV function can present a variety of forms.

![S&P500 implied volatility smiles](image)

**Figure 1.1:** Implied volatility smiles. Options written on the SPX for different maturities (in months). Index level 2168.27.
There are also empirical evidence and economic arguments that support a non-constant volatility. The reasons behind the smile curvature are the "leverage effect" and the "crashophobia". The first relates to the possibility that as company’s equity declines in value, the company’s leverage increases. In this way, the equity becomes more risky and its volatility increases. The opposite holds true when the value of the company reduces. The second, instead, as proposed by Rubinstein (1994) and refers to the fact that volatility smiles started to appear after the stock market crash of October 1987. Ever since, traders have been concerned about the possibility that similar events will happen again, hence they started to price options accordingly.

![S&P500 implied volatility surface](image)

**Figure 1.2:** Implied volatility surface. Options written on the SPX. Index level 2168.27

Leaving all the other factors unchanged, the IV function with respect to the time to maturity may generate a downward or an upward sloping curve. Usually, the term structure is downward sloping because in the long period
bigger movements tend to cancel out. This means that the future volatility is expected to drop. The reason is that the volatility tends to stabilize around its long-term mean, and thus it is considered as a mean-reverting process. Anyway, also an upward sloping curve can be found when the market future estimated volatility is assumed to rise.

The two considerations made are strictly correlated. The evidence can be retraced in financial markets. By empirically varying the BS volatility with strike price, traders are implicitly attributing a non-log-normal distribution for the stock prices.

The overall structure of the IV is the volatility “surface”, which merges together the information coming from the volatility smile and the term structure. An example of the surface is given in Figure 1.2.

Given these considerations, models which treat the volatility as a time-dependent and strike-dependent variable came to light.

1.3 Local volatility

By applying strictly the Black-Scholes model to option markets, the volatility would be the same for all the option prices. Reality shows that this is not the case: BS implied volatilities strongly depend both on the strike price and the time to maturity of the considered European options.

The first approach made in order to make the BS implied distribution to account for the leverage effect was made by Cox (1975) and Cox and Ross (1976). He proposed a constant elasticity model (CEV) in which the price process is a function of a parameter that changes in the case of positive or negative leverage effect, that is, if the volatility increases, or decreases, when the stock price grows. However, the main breakthrough came with the intuition of Dupire (1994). He introduced a model that allows the pricing method to be consistent with volatility smiles and prices observed in the market.

The idea behind Dupire’s intuition is that it is possible to have enough freedom in the function \( \sigma_L(S_t, t) \) to make theoretical option values consistent with all option values present in the market, or equivalently, with the implied volatilities. Normally, starting with the specification of the actual volatility, the option prices are derived consequently and also the IVs: it is a forward problem. In this form of calibration, instead, the work is made backwards from the results, the implied volatility, to the inputs, the actual volatility.
Around the same time other methods have been developed. Derman and Kani (1994) made use of the idea of "implied tree". The tree is constructed based on the original Cox-Ross-Rubinstein (1979) method, which implemented a binomial tree to represent the process in (1.5) in the BS environment. The stock evolves along a risk-neutral binomial tree with constant logarithmic stock price spacing and constant volatility. In this case, the tree is the same for all options of a particular stock, irrespective of their strike level or expiration.

Derman-Kani, instead, added a correction term to the implied tree, adjusting the volatility in different points to allow the prices obtained to be consistent with volatility smile. However, the method is not clear-cut and several adjustments have to be made in order to make it work in practice. Therefore, in this work we prefer to present the Dupire’s approach to volatility modeling.

### 1.3.1 Dupire model

Let the collection of undiscounted option prices for different strike prices $K$ in the interval $(0, \infty)$, for a given expiration $t$ and price of the underlying asset $S_0$, be $C(S_0, K, T)$. We define it as:

$$C(S_0, K, T) = \int_K^\infty dS_t \varphi(S_T; T; S_0)(S_T - K), \quad (1.20)$$

where the term $\varphi(S_T; T; S_0)$ is the risk-neutral density function of the final spot price $S_T$.

Differentiating twice with respect to the strike price we obtain:

$$\varphi(K, T; S_0) = \frac{\partial^2 C}{\partial K^2}, \quad (1.21)$$

that are the Arrow-Debreu\(^7\) prices for each expiration.

Dupire shows that there is a unique risk-neutral diffusion process for the price $S_t$ which generates these distributions, that is:

$$dS_t = r_tS_t dt + \sigma_t(S_t, t)S_t dW_t, \quad (1.22)$$

\(^7\)Arrow-Debreu prices are the prices of state-contingent claims, which deliver $\$1$ if a specific state realizes at a specific future date. In finite-state models, Arrow-Debreu securities delivering $\$1$ can be viewed as building blocks for all other state contingent financial claims. Their prices determine a unique arbitrage-free price system.
where \( r_t \) is the risk-free rate and \( \sigma_L(S_t, t) \) is the local volatility function of the stock price \( S_t \) and time \( t \).

Since the volatility term is a deterministic function of price \( S_t \) and time \( t \), it extends the BS model without increasing the dimension of the uncertainty. The function is determined numerically requiring that the prices of the options calculated in this way fit the smile.

Under the risk-neutral assumption, Dupire derived a partial differential equation that is a generalization of the Black-Scholes equation, called Dupire equation. It describes the option price \( C(S_0, K, T) \) with respect to the strike price \( K \) as:

\[
\frac{\partial C}{\partial T} = \frac{1}{2} \sigma_L^2(K, T) K^2 \frac{\partial^2 C}{\partial K^2} + r_t \left[ C(K, T) - K \frac{\partial C}{\partial K} \right].
\] (1.23)

Expressing the option price as a function of the forward price

\[
F_T = S_0 e^{\int_0^T r_t \, dt},
\] (1.24)

we obtain the Dupire Equation (1.23) minus the drift, now as a function of \( C(F_T, T, S_0) \)

\[
\frac{\partial C}{\partial T} = \frac{1}{2} \sigma_L^2(K, T) K^2 \frac{\partial^2 C}{\partial K^2}.
\] (1.25)

Inverting this and obtain the definition for the local volatility function regardless of what kind of process actually governs the evolution of the volatility, as:

\[
\sigma_L^2(K, T) = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}}.
\] (1.26)

The right hand side of this equation can be computed from a known set of European options prices for all strikes and expirations.

### 1.3.2 Strengths and weaknesses of the Dupire model

Local volatility models possess several interesting features. First of all, they improve the BS constant volatility by finding option prices consistent with the smile effect.

Dumas et al. (1998) consider the local volatility implied tree approach consistent with the apparent deficiencies of the Black-Scholes model. For SPX options, the price distribution tends to exhibit negative skewness and
excess kurtosis. An inverse relation exists between the volatility and the index level: as the index falls, the volatility increases. Moreover, the relation is asymmetric because the volatility increases more when a downward movement happens in the index, than when an upward movement occurs.

Second, since the only source of uncertainty remains the stock price, they do not lose the completeness of the BS model as hedge is based only on the underlying asset. Completeness is important, because it guarantees unique prices. Given their simplicity, they are also easy to calibrate.

Despite these attractive features, the general local volatility approach is somewhat tricky to put in practice. The main drawback is that it requires an arbitrary pre-interpolation of the input, the volatility surface. The reason is that the availability of a continuum of prices for all the strike prices and maturities is impossible to obtain.

 Gatheral (2004) argued that, in reality, the local volatility approach is none other than a simplifying assumption of stochastic volatility models, since it is obtained like a sort of average over all possible volatilities in a stochastic volatility world. As a proof, Dumas et al. (1998), found that a deterministic local volatility function is not a better measure than a procedure that merely smooths Black and Scholes implied volatilities through exercise prices and time to expiration.

 For Brigo and Mercurio (2002) the drawbacks of the model lie in its construction. First, the inverse problem is somewhat problematic to be put in practice. The technique to work out implied volatilities is a smoothing technique. The inverse is not so precise due to the sensitivity of the final local volatility to a small change in the IVs. In fact, analytical solutions of the equation are usually not available. Finite difference methods can be used to approximate the derivatives. However, the second partial derivative with respect to the strike price in the denominator of the equation can cause difficulties for extreme strike values and for small times to expiration. Since we divide in (1.26) by the second partial derivative, small errors in its numerical approximation may lead to substantial errors in the overall model.

1.4 Stochastic volatility

From the investor perspective, it is not so difficult to believe that, in financial markets, volatility is exposed to some sort of randomness. Volatility clustering is a very common behavior for stock prices and the Black-Scholes
implied volatilities exhibit smiles that are not compatible with the underlying process. In addition, the distribution of the returns is skewed and fat-tailed.

These characteristics motivate the use of stochastic volatility (SV, in short) models which treat the volatility itself as a random variable.

Merton (1973) proposed a jump-diffusion model which tries to fit better the distribution of returns, including jumps in the stock price process.

Bates (1991) generalized the previous approach by allowing the jumps to be asymmetric. His idea considers the asymmetry often observed in volatility smiles.

Despite their great diffusion, these models failed to consider that, after a jump occurs, volatility does not remain the same.

In a SV model, the only assumption that has to be made for the volatility in a process for the underlying asset price \((S_t)_{t \geq 0}\) of the type

\[
dS_t = \alpha S_t \, dt + v_t S_t \, dW_t,
\]

where \(\alpha\) is the instantaneous drift of the stock returns and \(W_t\) is a Wiener process, is that the volatility is positive. The volatility is itself a random variable described by some function \(f\) as:

\[
v_t = f(Y_t).
\]

where \((v_t)_{t \geq 0} > 0\).

The volatility process will then follow the stochastic equation

\[
dY_t = \alpha(m - Y_t) \, dt + g(t, Y_t) \, dW'_t,
\]

with

\[
\langle dW_t \, dW'_t \rangle = \rho \, dt.
\]

The drift term \(\alpha(m - Y_t)\) directs \(Y_t\) through the long-term value \(m\), while the second term behaves like a random noise. The term \(g(t, Y_t)\) represents a generic function of the time and the volatility process \(Y_t\).

One important feature of these models is that the stochastic element, the Brownian motion \(W'_t\), in the volatility process \(Y_t\) is not perfectly correlated with the Brownian motion \(W_t\) in the price process \(S_t\). Thus, there exist in the volatility process a casual element which is independent from the price of the underlying asset.

Many authors have extended the BS model to allow for SV. The difference in the proposed SV models lies in the formulation of the volatility process.
For example, Hull-White (1987) presented a model in which the volatility is not characterized by mean-reversion and the process is log-normal with correlation equal to zero.

Ball-Roma (1994), instead, described the volatility as a Cox-Ingersoll-Ross process with, again, zero correlation.

Nonetheless, the assumption that the volatility is uncorrelated with the spot return failing to capture crucial skewness effects. The innovation Heston (1993) introduced accounts for a correlation different than zero. In fact, the sign of this behavior can be traced down from financial markets. The sign and the intensity of the correlation term $\rho$ vary with respect to the nature of the stock. In Section 1.4.1 of this paper the Heston approach will be presented more in detail.

The principal element of differentiation between the Black-Scholes and a stochastic volatility model is the probability distribution of the prices. The latter replicates in a more similar way the implied volatility profile approximating better its structure in financial markets. In particular, the distribution is not longer log-normal but it can resemble other distributions, especially with fatter tails. Despite the improvements, they present more computational problems and introduce a new source of risk, the volatility risk, which might cause more hedging complexity.

1.4.1 Heston model

In 1993 Steven L. Heston found a closed-from solution for the price of European options introducing for the first time the correlation between the asset spot price and the volatility. According to the stock price behavior observed in financial markets, it seems feasible that the volatility is different with respect to different levels of price. For example, for stocks, the volatility usually grows when the price is decreasing, signaling a period of greater uncertainty. Instead, for commodities, the situation becomes more uncertain in case of high prices. The hypothesis of no correlation is, then, considered unrealistic. As a consequence, the correlation is adjusted to include all the possible real scenarios.

The underlying process for the stock price $S_t$ and its variance $v_t$ are

\[ dS_t = \mu S_t \, dt + \sqrt{v_t} S_t \, dW_t, \] \hspace{1cm} (1.30)

\[ dv_t = k(\theta - v_t) \, dt + \sigma \sqrt{v_t} \, dW'_t, \] \hspace{1cm} (1.31)
where \( \mu_t \) is the deterministic instantaneous drift of stock price returns and \( \sigma > 0 \) is the positive volatility of volatility. \( W_t \) and \( W'_t \) are two Wiener processes with correlation given by

\[
\langle dW_t dW'_t \rangle = \rho \, dt.
\]

A Cox-Ingersoll-Ross square-root process describes the behavior of the instantaneous volatility \( v_t \). It ensures mean-reversion through the long-run mean parameter \( \theta \), with reversion speed \( k > 0 \), and the variance to be positive by its standard deviation factor \( \sigma \sqrt{v} \) given the condition \( 2k\theta > \sigma^2 \).

### 1.4.2 Heston price of an European option

The Heston model presents a closed-form solution only for options of the European type. Given that the model has two sources of randomness, the bivariate Itô’s Lemma is used to derive the fundamental partial differential equation.

The derivation of the Heston equation is done constructing a riskless portfolio hedged both by the stock movements and the volatility risk. Assuming the non-arbitrage argument, like in the Black-Scholes model, it is possible to derive the Heston equation for the price of a derivative \( U(S_t, v_t, t) \), as:

\[
\frac{\partial U}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \rho \sigma v S \frac{\partial^2 U}{\partial v \partial S} + \frac{1}{2} \sigma^2 v \frac{\partial^2 U}{\partial v^2} + r S \frac{\partial U}{\partial S} - r U + [k(\theta - v_t) - \lambda(S_t, v_t, t)] \frac{\partial U}{\partial v} = 0
\]  
(1.32)

The term \( \lambda(S_t, v_t, t) \) represents the price of volatility risk and it is the part of the price \( U \) tied to the volatility process \( v_t \). An important assumption made by Heston is that the risk premium \( \lambda \) is proportional to \( v_t \), so that

\[
\lambda(S_t, v_t, t) = \lambda v_t.
\]  
(1.33)

In practice, it is not possible to estimate it. However, it is possible to derive it from the returns of a portfolio hedged by the movements in the stock price. By analogy with the Black-Scholes formula the solution for a call option must be of the type

\[
C(t, S_t, v_t, K, T) = S_t P_1 - K e^{-rt} P_2,
\]  
(1.34)
where \( \tau = T - t \) and \( P_1, P_2 \) are the risk-neutral probabilities that the option expires in-the-money. Both terms must satisfy the original PDE in (1.32).

In the Heston framework, semi-analytical formulas for the pricing of plain vanilla option can be derived. Unfortunately, these formulas require the evaluation of logarithms with complex arguments during the involved inverse Fourier integration step. Numerical instability arises when dealing with these methods and its implementation is not so straight-forward. Since the goal of this work is to explore volatility measures, going beyond a deep analysis in option pricing techniques, the complete formulas for the price of a European call option are given in Appendix A.

### 1.4.3 Advantages and disadvantages of the Heston model

The Heston model enhances for different aspects the considerations made by the models examined so far.

The greatest advantage is that it adapts closer to the price process. The mean-reverting feature of the volatility process is consistent with the reality. If this was not the case, the volatility could either go up to infinity, to zero or stay the same.

It also conveniently explains the properties of prices in terms of the underlying distribution of spot returns. Allowing for non-normality, it improves the considerations made by Black and Scholes adapting coherently to the empirical research.

A crucial distinction has to be made with respect to the parameters affecting the shape of the distribution. The correlation factor positively influences the skewness and the volatility of volatility \textit{per se} increases the kurtosis. As a matter of fact, when there is a negative correlation \( \rho < 0 \) between the volatility and the spot return, the right tail becomes thinner while the left tail becomes fatter. As a consequence, we observe an increase in the price of the in-the-money call options (out-of-the-money put options) and a decrease in the price of out-of-the-money call options (in-the-money put options) with respect to the BS comparable values. The opposite is true when \( \rho > 0 \).

The parameter \( \rho \) also accounts for the leverage effect, since there can be a negative relation between stock level and volatility.

A greater kurtosis given by a greater volatility of volatility \( \sigma \) increases the thickness of the tails of the implied distribution with respect to a normal distribution. As a consequence, far in-the-money and far out-of-the-money
option prices raise while near-the-money prices become lower with respect to the BS comparable values.

However, the BS option price formula gives prices which are virtually identical to those of the Heston model when dealing with near-the-money options. Since most of the options are traded near-the-money, this gives some empirical credibility to the BS model. Adopting a stochastic volatility model, instead, causes very large improvements for less traded, yet not less important, options.

Despite these clear advantages, the Heston model present some drawbacks. On the one hand, it presents a closed-form solution only for European type option contracts and parameters constant in time.

Benhamou et al. (2009) extended the Heston view deriving an analytical formula to price plain vanilla options that account for time dependent parameters. The same approach has been used by Langrenè et al. (2015). Besides this, they considered the volatility process to be driven by an Inverse Gamma process (which gives the name to the IGa model). The work have been supported by many researches which showed that the best fit for the volatility distribution was given by an Inverse Gamma distribution. On the other hand, they describe the asset dynamics introducing a new source of uncertainty, the volatility. Since the volatility cannot be directly traded, it might affect the completeness of the original Black-Scholes model.
Chapter 2

Model-free volatility

Nowadays, many volatility indexes base their measure of the market volatility on a model-free approach. As we have seen in Chapter 1, an option pricing model generally starts from the specification of the volatility process to derive the price of the options. Alternatively, a model-free approach reverses the traditional procedure. Its peculiarity is to derive an estimation of the volatility directly from the market prices of the options, without making any assumption neither on the underlying asset price process nor on its distribution.

This Chapter presents the construction of the model-free implied volatility and the theoretical framework behind it.

We start by considering the work of Breeden and Litzberger (1978). Assuming the non-arbitrage argument, they coherently extracted a risk-neutral density from the option prices, which is the market’s estimate of the probability distribution for every level of the underlying asset’s price. The risk-neutral density is the cornerstone upon which the model-free implied volatility is estimated.

The term “model-free” implied volatility was first coined by Britten-Jones and Neuberger (2000), who presented a method to derive the implied volatility from the options market without specifying \textit{ex ante} a model. Indeed, they discovered that the risk-neutral implied variance of the stock returns is entirely specified by a complete set of option prices, namely for all strikes.

Later in this Chapter, we explain how Demeterfi \textit{et al.} (1999) dug deeper in the matter, demonstrating that the concept of risk-neutral implied variance can be associated to the price of a variance swap. In particular, they found that it is possible to replicate the payoff of a variance swap by conveniently
assuming a position in a set of options with appropriate expiration, which is hedged from the variance risk. The price of constructing such a portfolio will be the so-called ‘fair value of future variance’, which is the market price the investors are willing to pay to receive the forecast quantity, the squared volatility.

The model-free implied volatility does not rely on the restrictive assumptions of both the Black-Scholes constant volatility and the Dupire’s deterministic volatility. Instead, it suggests a world of stochastic volatility but, given the forecast problems intrinsic in the latter approach, it proposes a way to fit correctly the option prices without suffering from these issues. Conceptually, the model-free implied volatility should be a superior estimation technique since it is based only on the aggregate previsions made by the investors.

2.1 Risk-neutral distribution

The concept of model-free implied volatility is built upon the risk-neutral distribution of the prices of the underlying asset. Breeden and Litzberger (1978) showed that it is possible to extract this density, given a complete set of option prices, composed by options of all maturities with strike price ranging from 0 to $\infty$. A complete set is needed because it means that there exist an exercise price for every possible future state of the world.

Unlike the models we presented in Chapter 1, this distribution does not depend on any assumption regarding neither the form of the distribution and the price process that generated it, nor the specification of a variance process. Apart from that, the assumptions made by Black and Scholes on a frictionless market are maintained.

The risk-neutral density is defined as the market’s estimate of the probability distribution for every level of the underlying asset’s price, on the option expiration. However, it is not equal to the objective probability\(^1\), given the fact that the risk-neutral density does not incorporate the risk premium associated to the variance risk.

The derivation of the risk-neutral measure for the volatility is made by using the non-arbitrage price of an elementary claim in terms of a European call option. An elementary claim of an asset is defined as a security which pays $1 at time $T$ if the value of the asset is $S$. Instead, if the value of the

\(^1\) An objective probability is defined as the probability that an event occur, based on an analysis in which each measure is derived from a recorded observation.
asset is not $S$ at that time, the security pays nothing. It is possible to derive
the payoff of such a security, creating a portfolio composed by long and short
positions in options having various exercise prices, with expiration $T$. In an
arbitrage-free world, the price of the portfolio must be equal to the price of
the elementary claim.

The possible values of the portfolio in $T$ periods are $S = \$1, \$2, \ldots, \$N$.
The payoff vector of a European option with strike price $K$ at expiration $T$
is denoted by $C(K, T)$. When the exercise price of the option grows from $K$
to $K + 1$, two things change: the sets of states with $S = K + 1$ have a payoff
that becomes 0 and the payoffs in states where $S > K + 1$ are reduced by
the change in exercise price. Table 2.1 sums up intuitively the payoffs of the
call option for different levels of $S$.

Table 2.1: Call option payoffs

<table>
<thead>
<tr>
<th>$S$</th>
<th>$C(0, T)$</th>
<th>$C(1, T)$</th>
<th>$C(2, T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S(T) = 1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$S(T) = 2$</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$S(T) = 3$</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$S(T) = N$</td>
<td>$N$</td>
<td>$N-1$</td>
<td>$N-2$</td>
</tr>
</tbody>
</table>

Given this setting, a portfolio composed by a long position in a call option
with strike price $K$ and a short position in a call option with strike price $K + 1$,
returns 1 in every state where $S > K$. In an equivalent way, a portfolio of a
long and short option with strike prices $K + 1$ and $K + 2$, respectively, gives
a payoff of 1 in every state where $S > K + 1$. The price of the total position,
namely the subtraction of the two portfolios, would have a payoff vector of:

$$ [C(K, T) - C(K + 1, T)] - [C(K + 1, T) - C(K + 2, T)] = $$

$$ = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. $$  \hspace{1cm} (2.1)
The last vector shows that it is possible to replicate the payoff of an elementary claim with a portfolio of call options. Given the no-arbitrage argument, the price of the claim must be equal the portfolio value.

Summing up, a portfolio which pays 1$ only when the value of the underlying asset is $S$ at time $T$ is given by a position of a long call with strike $K = S - 1$, two short calls with $K = S$ and a long call with $K = S + 1$. The overall position can be associated to a *Butterfly spread*, of which the payoff is presented in Figure 2.1.

![Butterfly Spread](image)

Figure 2.1: Payoff of a *Butterfly spread* strategy: a long call with $K = 4$, two short calls $K = 5$, a long call with $K = 6$.

From (2.1) is possible to obtain a generalized framework, in which the step size between the possible values of the portfolio is $\Delta S$. According to this, the price of a portfolio that pays $1 if and only if the value of the underlying asset in $T$ is $S$, is given by:

$$P(S, T; \Delta S) = \frac{[C(S - \Delta S, T) - C(S, T)] - [C(S, T) - C(S + \Delta S, T)]}{\Delta S}.$$  \hspace{1cm} (2.2)

Dividing the price of the portfolio $P(S, T; \Delta S)$ by the step size $\Delta S$, we obtain the price of the elementary claim in terms of call options, as:

$$\frac{P(S, T; \Delta S)}{\Delta S} = \frac{[C(S - \Delta S, T) - C(S, T)] - [C(S, T) - C(S + \Delta S, T)]}{(\Delta S)^2}.$$  \hspace{1cm} (2.3)
As the step size $\Delta S$ approaches 0, we have:

$$
\lim_{\Delta S \to 0} \frac{P(S, T; \Delta S)}{\Delta S} = \frac{\partial^2 C(S, T)}{\partial S^2}.
$$

(2.4)

The prices of elementary claims, which is actually the price of the option portfolio, can be seen as state prices. Since options are quoted in terms of strike price and not in terms of the price of the underlying asset, it is necessary to substitute $S$ with $K$ in (2.3) and (2.4). This is the reason why a complete set of options with strikes ranging from 0 to $\infty$ is needed. Consequently, the obtained results are:

$$
P(K, T; \Delta K) = \frac{[C(K - \Delta K, T) - C(K, T)] - [C(K, T) - C(K + \Delta K, T)]}{(\Delta K)^2},
$$

(2.5)

$$
\lim_{\Delta K \to 0} \frac{P(K, T; \Delta K)}{\Delta K} = \frac{\partial^2 C(K, T)}{\partial K^2}
$$

(2.6)

(2.5) and (2.6) can be interpreted as probability density functions for the future value of the underlying asset in a discrete and in a continuous framework, respectively. Hence, the risk-neutral distribution can be recovered from a complete set of option prices $C(K, T)$ with strike $K$ equal to all the possible values of the underlying asset price $S$.

Note that the risk-neutral distribution is independent of the price process of the underlying asset. It only requires that:

- there exist a complete set of European options,
- the price function $C(K, T)$ is twice differentiable,
- $C(K, T)$ is strictly convex in the strike price $K^2$.

At a certain point in time, the risk-neutral probability and the associated risk-neutral distribution contain an enormous amount of information about the market's expectation and risk preferences. Furthermore, their dynamics can be analyzed to reveal how information releases and events, that affect risk attitudes, impact the investor's preferences.

However, the estimation of the risk-density is hampered by two serious problems. First, the theory calls for a continuum of option prices for all strike

\[\text{In the opposite case, the state prices may not be strictly positive}\]
prices on a certain expiration. This is clearly an optimistic view of the reality, since option markets trade only a relatively small number of discrete strikes. The second major problem is that it can be extracted only over the range of possible strike prices, which usually does not extend very far in the tails of the distribution. All the issues of the risk-neutral probability distribution will be reflected in the estimation of the model-free implied volatility as well.

2.2 Model-free implied volatility

The standard approach in option pricing is to define a process for the price of the underlying security, and then derive the price of the option as a function of the process parameters. Clearly, one of the inputs is the specification of the volatility of the process. In the same way, the use of a specific option pricing model has been a prerequisite for the extraction of the implied volatility from the market price of the options.

However, this approach is subject to several shortcomings. Most importantly, the comparison between the realized volatility and the implied volatility is a joint test of the assumptions of the used model and the efficiency of the option markets.

The breakthrough made by Britten-Jones and Neuberger (2000) was to reverse the procedure. They based their work on the Breeden and Litzberger’s risk-neutral distribution, which in turn is built upon the non-arbitrage argument. Their fundamental contribution has been to extract as many information as possible about the underlying price process, given the market prices of the options. Accordingly to that, they derived a model-free measure of implied volatility finding that the risk-neutral return variance between two dates is fully specified by a complete set of option prices.

Intuitively, the model-free implied volatility (hereinafter, MFTV) is obtained from the market prices as the amount that investors are willing to pay in order to receive the sum of squared returns between the current date and an arbitrary date in the future. As a matter of fact, we can see this as the fair market’s prediction of the future volatility, since we use the squared returns as a measure of volatility.

In addition, we point out that such a value is a forward price, and must be discounted at the risk-free rate. Since model-free implied volatility is based on the risk-neutral density, the measure will differ from the value obtained using the objective probabilities, because it excludes the premium.
the investors require for variance risk. However, Andersen and Bondarenko (2007) showed that the MFTV is likely to be very closely related to the future realized volatility.

The implementation of the model by Britten-Jones and Neuberger (2000) considers a discrete setting for both stock prices and time. Despite this, the results can be also generalized in a continuous setting, taking the limits of the discrete setup as the interval size approaches zero.

Let us consider a finite set of equally-spaced points in time $h$. The set of times $T$ is ranging from 0 to $T$:

$$T = \{0, h, 2h, 3h, ..., T\}.$$

Let $S_0$ denote the initial price of the stock, and assume that it can take values in the set $K$, which is a finite geometric series of stock prices ranging from $S_0 u^{-M}$ to $S_0 u^M$:

$$K = \{K : K = S_0 u^i, i = \pm 1, \pm 2, ..., \pm M\},$$

where $u > 1$.

A grid consisting in nodes $(t, K)$, representing time-price events, is in this way created. The stock price can move upon the grid.

A continuous process for the stock price is required, meaning that it can go up or down by at most one level each period:

$$\text{if } |i - j| > 1, \text{ then } P[S_{t+h} = S_0 u^j | S_t = S_0 u^i] = 0.$$ 

Note that the stock price can only move up, or down, or stay the same over one period, but this does not impose any restriction on the volatility. Indeed, by making the time interval as smaller as possible, the volatility level can be raised without limit.

At time zero, there exists a complete set of European call options with prices $C(K, t)$, for time $t \in T$ and strike price $K \in K$. Dividends and interest rates are set to zero without loss of generality. In the case of nonzero interest rates, the price of the options and the underlying asset can be seen as forward prices. Hence, it is straightforward to convert back them to spot prices.

The absence of arbitrage in a complete market is necessary to derive a unique risk-neutral measure. This enables to use the risk-neutral valuation, which states that the market price of a security is the expected value of its
discounted future payoffs. Given this setting, it is possible to determine the risk-neutral probability of the stock price to reach level. However, the price path leading to a specific price is not determined, since the probability is only conditioned on the last price instead of the whole price history.

The risk-neutral probability of the stock price $K$ to reach any particular price level, say $K^*$, on any future date $t + h$ is only determined by the initial option prices $C(t, K)$. The intuition has been demonstrated assuming a position in a Calendar spread , namely a short position of one option with expiration $t$ and strike $K$ and a long position of one option with the same strike price $K$ but expiring one period later, at $t + h$. The overall cost of this position is, clearly, $C(t + h, K) - C(t, K)$. An hedging strategy is built in a manner that results in a payoff of zero unless the stock price at time $t$ is $K$ and the stock price at time $t + h$ is $Ku$, in which case the payoff is $K(u - 1)$. The defining feature of risk-neutral probabilities is that the expected payoff of the portfolio must equal its cost. From this, the risk-neutral probability of an upward movement of the stock price is derived.

Consequently, the condition that $S$ must be a martingale and continuous and the fact that the probabilities must sum to one, gives also the probabilities for the stock price to move down or stay at the same level, as it follows:

$$P[S = K, S_{t+h} = K^*] = \begin{cases} 
\frac{C(t+K) - C(t,K)}{K(u-1)} & K^* = Ku \\
\frac{C(t+K) - C(t+K) + u C(t,K/u)}{K(u-1)} & K^* = K \\
\frac{C(t+K) - C(t,K)}{K(1-1/u)} & K^* = K/u \\
0 & \text{otherwise}.
\end{cases}$$

The joint probability can be written as the product of a marginal and a conditional probability:

$$P[S_t = K, S_{t+h} = Ku] = P[S_{t+h} = Ku | S_t = K] P[S_t = K].$$

The joint probability of an up move in terms of the option prices can be obtained substituting (2.7) in (2.8), as follows:

$$P[S_{t+h} = Ku | S_t = K] = \frac{C(t+h, K) - C(t, K)}{C(t, Ku) - (1 + u) C(t, K) + u C(t, K/u)}.$$
Same conditions as before are applied to find the remaining probabilities of the stock price to stay the same or move down.

Now, it is possible to calculate the conditional expectation of squared returns, conditional on the stock price to reach a particular level \( K \) at time \( t \), which are true for any continuous risk-neutral process, as:

\[
E \left[ \left( \frac{S_{t+h} - S_t}{S_t} \right)^2 \mid S_t = K \right] = \frac{[C(t+h, K) - C(t, K)] (u-1)^2 (u+1)/u}{C(t, Ku) - (1+u) C(t, K) + u C(t, K/u)}. \tag{2.10}
\]

This equation can be interpreted as the volatility forecast conditional upon the stock price level.

Since we are interested in the implied volatility over some interval without conditioning, such forecast implied from the option prices is derived as follows.

The risk-neutral expected sum of squared returns between two arbitrary dates \( t_1 \) and \( t_2 \) is given from the set of option prices that expire on these two dates, as:

\[
E \left[ \sum_{t \in [t_1, t_2-h]} \left( \frac{S_{t+h} - S_t}{S_t} \right)^2 \right] = (u - 1/u) \sum_{K \in K} \frac{C(t_2, k) - C(t_1, K)}{K}. \tag{2.11}
\]

Substituting the time interval with 0 and \( T \) and taking the limit as both the time interval \( h \) and the jump size \( u-1 \) approach zero, we get the following formula:

\[
E \left[ \int_0^T \left( \frac{dS_t}{S_t} \right)^2 \right] = 2 \int_0^\infty \frac{C(T, K) - (S_0 - K)^+}{K^2} dK, \tag{2.12}
\]

where \((S_0 - K)^+\) is the intrinsic value of an option with strike \( K \).

The forecast squared volatility presented in (2.12) is a risk-neutral measure, which means that it can differ from the objective probabilities embedding the volatility risk. In addition, it is a forward squared volatility, since the interest rate in this model has been set to zero.

Given positive interest rates \( r \), (2.12) is defined by Jiang and Tian (2005) as follows:

\[
E^F \left[ \int_0^T \left( \frac{dF_t}{F_t} \right)^2 \right] = 2 \int_0^\infty \frac{C^F(T, K) - (F_0 - K)^+}{K^2} dK, \tag{2.13}
\]
where the asset price and the option price are both forward prices:

\[ F_t = S_t e^{rT}, \]
\[ C^F(T, K) = C(T, K) e^{rT}. \]

This equation represents the return variance between the current date 0 and a future date \( T \), which is fully specified by the set of prices of call options expiring on time \( T \). Instead, its square root is the MFIV.

Intuitively, the MFIV in (2.13) can be seen as the market price for the sum of squared returns, where the latter is used in practice as a proxy of the sample variance in volatility estimation.

The initial model of Britten-Jones and Neuberger (2000) envisaged a continuous asset price process, but this could represent a serious limitation since random jumps are an important aspect of the price dynamics of many financial assets. Jiang and Tian (2005) generalized the MFIV approach showing that first, it also holds when the price process contains jumps, and second, it is valid for all martingale processes. Hence, the formula presented in (2.13) is valid for a very wide class of processes.

### 2.2.1 Volatility and variance swaps

The most fundamental principle in security pricing, which applies to all financial instruments, not only options, is that a security market’s price should be the expected value of its future payoff, discounted back to the present at the appropriate rate. In order to calculate correctly the payoff, in case of option pricing, it becomes crucial to determine which is the expected value of the volatility. In a model-free approach, we can intend the implied volatility as the price the investors are willing to pay in order to receive the forecast quantity, the volatility itself. In other words, the final goal is to find the “fair value of future volatility”.

Although we are using the volatility as a measure of variability, in this section we will deal with the variance, or squared volatility. Indeed, Demeterfi et al. (1999) showed that it is possible to associate the concept of “fair value of future variance” to the cost of a variance swap.

As a matter of fact, the most correct way to value a swap is to value the portfolio that replicates it, and the swap that can be replicated more reliably is the a variance swap. However, variance swaps provide similar volatility exposure to straight volatility swaps. Given their fundamental
role, variance swaps can serve as the basic building block for constructing any other volatility-dependent derivative.

Volatility derivatives, such as variance swaps or options, are gaining popularity in the financial world as both hedge and speculation instruments. Volatility owns many features that makes it interesting: first, it reverts to its mean, high volatility will eventually decrease and low ones will likely rise; secondly, it is often negatively correlated with stocks or indexes level, and tends to stay high after large downward moves in the markets. Since volatility cannot be measured univocally, volatility related derivatives are highly customized contracts. As a consequence, they are typically traded OTC. Instead, derivatives based on volatility indexes, such as VIX derivatives, can be traded in regulated markets. They can be included in the investment strategies of many type of investors: for example, option traders that need to hedge against volatility or portfolio managers aiming to diversificare their investments. This work will cope with these instruments in a more profound way in Chapter 4.

Demeterfi et al. (1999) reproduced the replicating strategy of a variance swap both under the Black-Scholes model assumptions and a generalized way. Given its intuitive approach, in this section we will present the first method. However, it is possible to derive the same results under more a general setting, as long as the stock price process is continuous.

A variance swap is a security that provide exposure to variance, and only to variance. Basically, it is defined as a long position in a forward contract on annualized variance. Its payoff at expiration is equal to:

\[
(\sigma^2_R - K_{VAR}) \times L,
\]

where \(\sigma^2_R\) is the stock’s realized variance over the life of the contract quoted in annual terms, \(K_{VAR}\) is the annualized variance delivery price and \(L\) is the notional amount of the swap in dollars per annualized volatility point squared. At expiration, the holder of a variance swap receives \(L\) dollars for every point by which the realized variance \(\sigma^2_R\) overtakes the variance delivery price \(K_{VAR}\). Hence, the investor is swapping a fixed variance \(K_{VAR}\) for a floating future variance \(\sigma^2_R\). The “fair value of future variance” at any time will be the delivery price \(K_{VAR}\) that makes the swap’s current value zero.

The calculating procedure of the realized variance should be specified under these three terms:

- the observation source and frequency of stock, or index, prices,
• the annualization factor to translate from daily or hourly observations to annualized variances,

• whether the variance is calculated subtracting the mean or by assuming a zero mean, with the latter preferred since for frequently observed prices the difference is negligible.

Assuming a null risk-free interest rate, at time $t$, we suppose that the investor owns a standard European call option, whose value is given by the Black-Scholes formula as $C_{BS}(S, K, \sigma \sqrt{\tau})$. Here, the price depends on the strike $K$, the stock price $S$ and the annualized volatility $\sigma \sqrt{\tau}$. The annualized volatility is a combined function of $\sigma$ and the time to maturity $\tau = T - t$. Clearly, the total variance of the stock to expiration is $\sigma^2 \tau$.

The exposure of an option to a stock's variance is the so-called variance _vega_ $V$, which measures the value changes in the option with respect to the variance. The sensitivity $V$ is defined as:

$$V = \frac{\partial C_{BS}}{\partial \sigma^2} = \frac{S \sqrt{\tau} e^{-d_1^2}}{2\sigma} \frac{1}{\sqrt{2\pi}}, \quad (2.15)$$

where

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + \frac{(\sigma^2 \tau)}{2}}{\sigma \sqrt{\tau}}. \quad (2.16)$$

For each strike, the variance exposure $V$ is largest when the option is at-the-money, while falls off quickly when the stock price moves in-the-money or out-of-the-money. In other words, $V$ declines extremely rapidly as $S$ leaves the proximity of the strike $K$. Moreover, $V$ is closely related to the time decay of the option, since the option’s value is proportional to the total variance $\sigma^2 \tau$.

In order to build a long position in future realized variance, a single option is not enough. As soon as the price moves, the sensitivity of further changes is altered. Instead, the objective is a portfolio whose sensitivity to realized variance is independent of the stock price. The way to construct such a portfolio is to combine options characterized by different strike prices.

The intuition behind the construction of such a portfolio is graphically explained in Figure 2.2. The upper-right panel displays the variance exposure $V$ of the portfolio consisting of the three options in the upper-left panel. The portfolio is built with equally weighted options (dotted line) and with
options whose weights are inversely proportional to the square of the strike price \( K^2 \) (solid line). As the range of options included increases and the space between two consecutive strike prices decreases, the \textit{variance vega} \( V \) of the strike-weighted portfolio becomes flatter. This is shown in the lower-right panel, in which the solid line represents the strike-weighted portfolio’s \textit{vega}. The portfolio consists of the options in the lower-left panel, with increased strike range and diminished strike spacing.

![Graphs showing strike options and portfolio vega](image)

Figure 2.2: \( V \) as a function of \( S \). The two left panels show the \( V \) of call options for different \( K \) taken singularly, while the two right panels show the overall \( V \) of the portfolio. The sum of the contributions of the options in the portfolio is made in two ways: the dotted line represents the equally weighted portfolio, while the solid line represents the portfolio with weights inversely proportional to \( K^2 \). Source: Demeterfi et al. (1999), \textit{More than you ever wanted to know about volatility swaps}, The Journal of Derivatives.

As we can see, the portfolio with options weighted inversely proportional to the squared strike produces an exposition \( V \) virtually independent to the stock price \( S \), as long as \( S \) lies inside the range of available stock prices,
namely far from the edges of the range, and provided that the strike prices \(K\) are spaced uniformly in a very fine grid.

Demeterfi et al. (1999) derived this results mathematically but this can be understood also intuitively. As the stock price moves up to higher values, each additional option of higher strike in the portfolio will provide an additional contribution to \(V\) proportional to the strike. The height of the peaks in the left panels of Figure 2.2 are derived from (2.15). Hence, every option grants a contribution to \(V\) that grows accordingly to \(S\). Moreover, there is an overlapping contribution for every level \(S\). In order to produce a flat exposure it is necessary to weight the options inversely proportional to \(K^2\).

Consider now the portfolio \(\Pi(S, \sigma \sqrt{\tau})\) of options for all strikes \(K\) but a unique time to expiration \(\tau\), weighted inversely proportional to \(K^2\). Since out-of-the-money options are generally more liquid, we employ put options \(P(S, K, \sigma \sqrt{\tau})\) for strike prices ranging continuously from 0 to a reference price \(S^*\) and call options \(C(S, K, \sigma \sqrt{\tau})\) for strikes from \(S^*\) to infinity. \(S^*\) represents the at-the-money forward stock level that divides the options in calls and puts. Formally, the portfolio expression is:

\[
\Pi(S, \sigma \sqrt{\tau}) = \sum_{K<S^*} \frac{1}{K^2} P(S, K, \sigma \sqrt{\tau}) + \sum_{K>S^*} \frac{1}{K^2} C(S, K, \sigma \sqrt{\tau}). \quad (2.17)
\]

At expiration \(T\), the sum of all the payoffs of the options in the portfolio is:

\[
\Pi(S_T, 0) = \frac{S_T - S^*}{S^*} - \log\left(\frac{S_T}{S^*}\right). \quad (2.18)
\]

Similarly, summing up all the Black-Scholes option values, the total portfolio value at time \(t\) is:

\[
\Pi(S_t, \sigma \sqrt{\tau}) = \frac{S_t - S^*}{S^*} - \log\left(\frac{S_t}{S^*}\right) + \frac{\sigma^2 \tau}{2}. \quad (2.19)
\]

Note that the difference between the value of the portfolio at expiration \(T\) and the same value at a generic time \(t\) prior to expiration is only the half value of the total variance \(\sigma^2 \tau\).

The vega of the portfolio is:

\[
V = \frac{\tau}{2}. \quad (2.20)
\]

Obviously, if the objective is to obtain a variance exposure of $1 for every volatility point squared, the investor needs to hold \(2/T\) units of \(\Pi\). Thus,
the new portfolio $\Pi'$ becomes:

$$\Pi'(S, \sigma \sqrt{T}) = \frac{2}{T} \left[ \frac{S_t - S^*}{S^*} - \log \left( \frac{S_t}{S^*} \right) \right] + \frac{\sigma^2 \tau}{T}.$$  \hspace{1cm} (2.21)

The first term inside the brackets describes $1/S^*$ forward contracts on the stock with $S^*$ their delivery price. Instead, the second term can be associated to a short position in a log contract with reference price $S^*$. A log contract is an exotic option in which the payoff at expiration is proportional to the log of the stock at expiration. It is important to note that all the volatility sensitivity of the overall portfolio is contained in the log contract. Moreover, the content inside the brackets in (2.21) is then the same as in (2.18). Hence, the payoff at expiration of the sum of the options weighted inversely to the squared strike is equivalent to a long position in a forward and a short position in a log contract.

Considering a Black-Scholes world, the implied volatility $\sigma^2_I$ is the estimate of future realized volatility. By taking a position in the portfolio $\Pi'$, its fair value at time $t = 0$ should be (2.21) with the last term replaced by $\sigma^2_I$. Furthermore, if the realized volatility turns out to be $\sigma^2_R$, the fair value at expiration $T$, which has been captured by delta-hedging, should be, again, the same as (2.21) but with $\sigma^2_R$ in place of the last term.

The net payoff of the position, hedged to expiration, is then:

$$\sigma^2_R - \sigma^2_I,$$ \hspace{1cm} (2.22)

which is equivalent to the variance swap payoff in (2.21) with a face value of $\$1$. In here, the fair value for the strike $K_{VAR}$ is the implied volatility $\sigma^2_I$. Thus, the investor will profit if the realized variance is greater than the implied volatility, and lose if the reverse situation happens.

Thus, we showed that the “fair value of future variance” is equal to the cost of implementing a strategy in which call and put options are weighted inversely proportional to the squared strike price.

Although these results have been derived in a Black-Scholes world, it is possible to generalize them taking into account a wide range of processes. The only assumption needed for the derivation of the replicating strategy is that the evolution of the price is continuous, namely no jumps are allowed. Given this, the stock price process may take the form of:

$$dS_t = \mu(t, Y_t)S_t \, dt + \nu(t, Y_t) \, dW_t.$$ \hspace{1cm} (2.23)
The drift $\mu(t, Y_t)$ and the continuously-sampled volatility $v_t(t, Y_t)$, which are functions of time and any transformation $Y_t = g(v_t)$ of the volatility, do not restrict the range of possible processes. Demeterfi et al. (1999) demonstrated that, following this re-balancing strategy ensures that the variance is captured no matter which path the stock price takes, as long as it moves continuously. The generalized method provides a direct connection between the market cost of the options and the strategy that captures the variance, even when there is a volatility smile effect or the Black-Scholes formula is no longer valid.

Despite the intuitive results of this method, it presents evident replication drawbacks, especially when dealing with the liquidity of option markets. The strategy replicates the variance exactly, provided that two conditions are verified:

- the portfolio is composed by options with strikes ranging from zero to infinite in the appropriate weight,
- the stock price evolves continuously.

Clearly, the conditions cannot match the reality of the option markets. On the first hand, there is only a limited number of options strikes available at maturity. The spacing between the strikes is not uniform and the range is not infinite. As we have seen in Figure 2.2, the variance exposure is only hedged when the stock price lies inside of the range of available strikes.

On the other hand, the stock price may jump. This may cause hedging issues since the it makes the price to be more likely to fall outside of the range in which the variance exposure is correctly hedged.

Finally, variance replication needs a log contract, which is not traded in practice, so the method derives its payoff with standard options in a limited strike range. Since the issues stated above do not permit to capture exactly the variance, this causes a lower value of the log contract, thus underestimating the real value of the fair variance.

### 2.2.2 Bias in model-free implied volatility estimation

Given its theoretical foundation, the implied volatility extracted with the model-free approach should be a superior estimate of the volatility in the markets. A model-free forecast measure, indeed, possesses a number of clear advantages.
First, instead of relying on the restrictive assumptions of an option pricing model, it supplies an estimation which is not model-specific. As a matter of fact, it is a very general measure which comprehend a very wide class of processes. Moreover, the tests based on the model-free implied volatility are direct tests on markets efficiency instead of a joint test of market efficiency and the assumed pricing model. As a consequence, the model-free approach is not subject to model misspecification errors.

Secondly, unlike the Black-Scholes implied volatility, the model-free approach extracts its information from a set of options across all strikes, instead of a single option, and summing them up in a unique value. The view of many investors, who make trading decisions based on current information obtained from different sources, flow into the options markets and is expressed in option prices. By aggregating information coming from different prices, the model-free implied volatility provides a complete view about the investor's volatility expectations. Its informational content is considered to be, in fact, more efficient than the forecasts given by the models in Chapter 1.

These reason allow us to see the model-free implied volatility as an accurate estimate of the market’s forecast of future volatility. However, since it is calculated under the risk-neutral valuation, it is not a pure volatility forecast for the underlying asset price. The reason is that it does not incorporate the variance risk premium.

As expected, many studies confirmed that the variance risk premium is negative. Risk-averse investors normally ask for a positive risk premium for returns risk. In other words, they require that the stock will be appreciated more than average in presence of riskier stocks. On the contrary, a negative variance risk premium is required by investors if the variability is higher. Therefore, whereas higher average returns is regarded as the compensation for higher return risk, lower variance levels are regarded as a compensation for the variance risk. For example, investors are ready to pay more for a constant volatility asset than for a stochastic volatility asset.

Carr and Wu (2006) analyzed the difference between the ex post realized volatility and the volatility given by a variance swap. The latter can, in fact, be associated to the “fair value of future volatility” in a risk-neutral world and their difference should estimate the variance risk premium. They ended up confirming the presence of a consistent negative risk premium for the volatility. Moreover, they searched for correlation between the stock price variability and the volatility risk premium. Carr and Wu (2009) concluded that only a small portion of the volatility risk premium is correlated to the
underlying asset’s variation, instead the majority of the risk premium is generated by a risk factor which is totally independent from the stock price.

However, it is not clear if the model-free approach is relevant in the real world because, in presence of variance risk premium, the objective probabilities differ from the risk-neutral probabilities. Generally, the model-free implied volatility is an upwardly biased measure for the realized volatility. This happens because, from the perspective of variance swap investors, they are not willing to pay as much as the forecast suggests, since the market price of the volatility is negative.

Despite being a theoretically superior construct, the model-free approach presents also some disadvantages due to its limitations in the practice implementation. It is not so clear if the forecast performance of the model-free implied volatility is superior to the Black-Scholes implied volatility since the research provided mixed opinions. While Jiang and Tian (2005) considered the goodness of the forecast to be superior to the Black-Scholes forecast, Andersen and Bondarenko (2007) came up with opposite results.

One of the problems is that the model-free implied volatility inherits all the errors innate in the construction of the risk-neutral density function. A continuum of strike prices from zero to infinity is clearly impossible to be obtained in reality. Given that the lowest possible boundary for the stock price is necessary zero, there is no upward boundary since the price may rise, theoretically, up to infinity, thus making impossible for the strike price to be perfectly paired with the possible price levels. In fact, markets only trade options within a limited number of strikes at every expiration. As we move from at-the-money options to far out-of-the-money options, the available strikes become more and more scarce, as the distance between two consecutive strike prices increases. Furthermore, as generally happens in the tails of the distribution, some strikes can be missing. Even if options with far out-of-the-money strikes exist, often the prices may be not so reliable due to lack of liquidity. Trading typically take place at at-the-money level and these options may contain noise. This usually causes the volatility swap to capture less than the actual implied volatility, hence, the model-free implied volatility is usually underestimated.

Even though there is a sufficient set of options, jumps in the stock price may cause problems in the hedging strategy. Jiang and Tian (2005) demonstrated that the model-free implied volatility can be calculated also if jumps in the price process happen. But, this holds true as long as the jumps are quite small and stock price remains between the range of available options.
strike prices.

In conclusion, under the risk-neutral valuation, the model-free forecast is an unbiased estimate of the realized volatility if two conditions are met:

- there are available options in a continuum of strike prices,
- the volatility risk premium in the markets is not priced.

The empirical evidence suggests that both conditions are denied in real markets. As we will see, most model-free implied volatility indexes use stratagems to bypass the issues caused by the first condition. But, given a negative market price for the volatility, the bias in the second condition has to be kept in mind while analyzing the forecasting performance of the model-free implied volatility. Nevertheless, risk-neutral valuation is usually considered relevant in the real world. As stated by Britten-Jones and Neuberger (2000), the upward bias in the model-free implied volatility can be seen as negligible.
Chapter 3

The VIX index

Volatility indexes are innovative financial instruments, claiming to offer a unique measure to the implied volatility in the financial markets. Often referred as “fear index”, the VIX index is the most important volatility index. Introduced in 1993 by the Chicago Board of Options Exchange (CBOE), the index has widely become the de facto benchmark for stock market volatility. The actual VIX index is the aggregate’s market measure of the implied volatility being projected through the price of S&P500 options. Since the implied volatility is a measure of risk intrinsic in option pricing techniques, the VIX is considered as a cue of the market sentiment, or more specifically, a gauge of fear in the overall market.

Originally, the construction of the VIX index was based on options data on the S&P100 index. It was conceived as an average of the Black and Scholes option implied volatility with strike prices close to the current spot value of the index and maturities interpolated at about one month.

In 2003, the CBOE revamped both the definition and the calculation of the index. The S&P100 options has been replaced by options written on the S&P500, the core of U.S. equities, to resemble a wider representation of the overall market’s sentiment. Moreover, the new methodology calculates the index as a weighted average of options implied volatilities among all the available strike prices at two interpolated maturities surrounding one month. Hence, the VIX index can now be actually associated to the concept of “fair value of future variance”. As a consequence, it represents to all intents and purposes a model-free measure of the implied volatility.

After the acknowledgment of the VIX index as a valid proxy for implied volatility, derivatives based on it came in light. In 2004, the CBOE launched
a new exchange, the Chicago Futures Exchange, to start trading futures on the new VIX. Later on, in 2006, also options on the index were introduced.

In this Chapter, we present the interpretation and the use of the VIX index in financial markets. The implementation methodologies of the old and new VIX for the calculation of the implied volatility, along with the theoretical and practical motivations of the switch made by the CBOE, are explained in detail in the subsequent sections. Furthermore, we investigate the differences between the CBOE calculation procedure and a pure model-free measure of the implied volatility, since the adopted methodology may generate some errors while trying to capture the whole market implied volatility. In the remainder of this Chapter, we analyze the historical behavior of the new volatility index, exploring its interaction with stock markets and realized volatility.

3.1 The interpretation of the index

The VIX index is quoted in terms of percentage points and translates, on annual basis, the expected range of movement of the S&P500 index (hereinafter, SPX) in the next 30-days time horizon.

It is calculated in real-time but, unlike a stock index, which possess a meaning related to its history, the VIX has a probabilistic interpretation regarding the expected interval of the rate of return on the price of the underlying asset, in this case the SPX.

Whaley (2009) supplied a useful method\(^1\) to interpret the VIX level, which is summed up in Figure 3.1. To understand how it works, suppose the actual VIX level is 60. With respect to the “50%” blue line, we denote that the

\(^1\)The technique is built upon two assumptions:

- the rate of return of the SPX is distributed normally,
- its 30-day mean is null.

Neither of the assumptions is unreasonable.

The properties of the standardized normal distribution allow to derive the such probability figures for the SPX rate of return. For examples, the “50%” line has been derived using the following relation:

\[
\text{Expected range at 50\%} = 0.1947 \times VIX.
\]
plot returns the value of 11.50%, approximately. We can then deduce that, if the value of the VIX is 60, there is a 50% probability that the SPX rate of return over the next 30 days will lie inside the ±11.5% range. The red and the green line are respectively the 75% and the 90% probability that the rate of return will end up within the selected range. For example, at a VIX level of 60, the chances that the return will be in the ±20% or ±34%, are 75% and 95%, respectively.

In order to confirm the usefulness of the method, Whaley (2009) compared the calculated range with the realized returns of the SPX. The results of the test confirmed that the VIX works reasonably well as a predictor of expected returns of the stock index.

Since the VIX is in reality a standard deviation, it is possible to derive the expected volatility range for any length of time by multiplying the VIX figure by $\sqrt{\Delta t}$, where $\Delta t$ is the time period expressed in years. Hence, in order to obtain the monthly value it is necessary to divide its value by $\sqrt{12}$. Since the VIX is a forward-looking estimate, this could be regarded as a forecast of future volatility.

![Graph](image)

**Figure 3.1**: Expected range of SPX returns over the next 30-days conditional on current VIX level. Source: Whaley (2008), *Understanding VIX*.

The SPX option market has always been dominated by hedgers, who buy index put options for protection when they are concerned about a potential drop in the market. As a measure of the “fair value of future volatility”, the VIX index is an indicator for market’s level of insurance. There is a direct
relation between the cost of the options and the index value. In fact, the concept of “fair value of future volatility”, as we have seen in Section 2.2.1, is referred to the price of a portfolio of options, that is conveniently hedged by the volatility risk. The more risk is present in the market, the more expensive will be to ensure such a portfolio, which translates in a higher value of the VIX.

More generally, the index is an aggregate market’s sentiment indicator. It tracks down the two main drivers of financial market’s movements: “greed” and “fear”. A low value in the VIX index signals a calm market but, when it starts to grow, the sensation among the investors is that something bad is going to happen, even if there is no possibility to determine a priori the direction of the market movement. The investor’s sensations, with different values of the index, range from “satisfaction” for low values to “fear” and also “panic”, when the index reaches more extreme values. Even though the volatility should measure either the upward and the downward movements of the market, the relation between the VIX and the underlying is asymmetric. In fact, a period of growth in the SPX is often accompanied by a stable VIX. Inversely, a decrease in the SPX usually causes a great upper spike in the VIX.

Given its actual implementation, the calculation of the VIX does not rely on any particular option pricing model or volatility model. Its formula approximates the “fair value of future volatility”, developed by Demeterfi et al. (1999), and it is derived only by the combination of market prices of options available in the market. Indeed, it is based on the work by Breeden and Litzberger (1978), who showed that it is possible to derive the risk-neutral distribution of the prices of the underlying asset from a complete set of option prices.

The theoretical underpinning of the new CBOE’s VIX calculation is thus rooted within the broader context of model-free implied variance, discussed in Chapter 2. As a matter of fact, Jiang and Tian (2007) demonstrated that the concept of “fair value of future variance” is identical to the model-free implied volatility definition of Britten-Jones and Neuberger (2000).

Despite its theoretically superior construction, the measure of the volatility supplied by the VIX suffers all the implementation issues of a pure model-free measure of the implied volatility and, consequently, of the risk-neutral density. The CBOE’s calculation methodology accounted for these potential errors, creating an index which partially takes into consideration these discrepancies and aims to limit them.
3.2 Calculation methodology

3.2.1 Old procedure

In 1993, the CBOE started to compute and publish the VIX index on real-time basis. It was based on the findings of Robert Whaley and calculated on the options of the S&P100 index (OEX). The old VIX, which has been renamed to VXO after the introduction of the renewed methodology, is essentially an average estimate of the one-month Black and Scholes implied volatility of at-the-money options. The old VXO time series is still published by the CBOE, who gathers together all the quotations from 1986 to now.

More in detail, the old procedure encompassed 8 options with strike prices close to the actual prices of the index at the nearest maturities. Once the time to the nearest maturity is within 8 calendar days, the subsequent two nearest maturities are used instead. The 8-day limit is imposed because options with lower maturities are often characterized by high implied volatility.

The use of the OEX at-the-money options with a 8-day limit is mainly due to liquidity restrictions. The motivation beyond this choice was that, by the time of the index introduction, OEX options were by far the most actively traded options and, in particular, at-the-money options with shorter maturities. However, if the time to maturity is too short, this could cause distortion in the option prices given the high volatility that is usually quoted.

For each maturity, the CBOE used to choose two calls and two puts at the two strike prices that straddle the spot level and are nearest to it. Of these options, the Black and Scholes implied volatility is calculated. The procedure involves the calculation of an average and two interpolations. First, the average between the two put and call options of the same strike price is executed. Then, a linear interpolation between the two average implied volatilities is calculated, in order to derive the at-the-money spot implied volatility. Finally, the at-the-money volatilities at the two maturities are further interpolated along the time dimension to find a 22-trading-day volatility, which represents the VXO.

The VXO is characterized by an artificially induced upward bias. In fact, the Black and Scholes implied volatility is an annualized volatility. The annualization is based on an actual 365 day convention. Instead of using this implied volatility directly, the CBOE introduced an artificial "trading-day" conversion inside the calculation procedure. The at-the-money implied
volatility estimate is multiplied by the square root of the days in a month and divided by the square root of trading days in a month. Since each month has roughly 22 trading days, the VXO should represent the one-month implied volatility. However, the multiplication raises artificially its value, making it no longer comparable to annualized realized volatilities computed from index returns.

Academics and practitioners often use the Black and Scholes implied volatility as an approximate realized volatility forecast. Since the underlying model assumes constant volatility, the economic sense of using such a measure is absent. However, many researchers found that the at-the-money Black and Scholes implied volatility is an efficient, although biased, forecast of future realized volatility. Thus, the meaning of the implied volatility as an effective forecast is based more on empirical evidence than on any theoretical linkages.

3.2.2 New procedure

Over the years after the inception of the VIX in 1993, the structure of index option trading changed fundamentally in two ways. First, given the rising popularity of the SPX futures market, the SPX derivatives market volume seriously overtook the OEX volume. As a matter of fact, the SPX is considered by investors to represent the benchmark for the performance of the stock market in the United States. Trading activities between calls and put options used to be the same, and mostly at a at-the-money level. However, later on, the option market became dominated by portfolio insurers, who buy out-of-the-money and at-the-money put for hedge purposes. The need of an active option market results crucial to ensure the effectiveness and the meaningfulness of the VIX, therefore a modified methodology became necessary.

In 2003, the CBOE launched the new index. Despite the purpose of the CBOE’s index remained the same, the innovation introduced many new solutions in the calculation procedure of the VIX with respect to the old VXO. The most important changes are:

- shift from OEX options to SPX options;

- inclusion of out-of-the-money options, which contain important information regarding the demands of portfolio insurance and, hence, of the whole market volatility;
• shift from Black and Scholes implied volatilities to market prices: this translates in a model-free measure of the implied volatility, which does not suffer of model misspecification errors.

At time $t$, the CBOE’s formula for the variance in the new VIX calculation setting is:

$$\sigma_{VIX}^2 = \frac{2}{T-t} \sum_{i=1}^{n} \frac{\Delta K_i}{K_i^2} Q_i(K_i, T) e^{r(T-t)} - \frac{1}{T-t} \left[ \frac{F_t}{K_0} - 1 \right]^2, \quad (3.1)$$

where:

$T$ is the expiration date, and $T-t$ is the time to expiration, calculated in minutes;

$F_t$ is the forward level of the SPX index at time $t$;

$K_i$ is the strike price of the $i$-th out-of-the-money option: a call if $K_i < F_t$ or a put if $K_i > F_t$;

$K_0$ is the first strike price below the forward index level, such as $K_0 \leq F_t$;

$r_t$ is the instantaneous risk-free interest rate at time $t$ with maturity $T$, which are the bond-equivalent yields of the U.S. T-bill maturing closest to the expiration date;

$\Delta K_i$ is the half-difference of the two strike prices immediately below and above the strike price $K_i$, which is defined\(^2\) as:

$$\Delta K_i = \frac{K_{i+1} - K_{i-1}}{2}; \quad (3.2)$$

$Q(K_i, t)$ is the mid-point (average) at time $t$ of the bid-ask spread for the out-of-the-money option with strike price $K_i$.

\(^2\)In the case that $K_i$ is the lowest strike available, the interval $\Delta K_i$ is simply the difference between the lowest strike and the next higher strike. Likewise, for the highest strike price, $\Delta K_i$ is the difference between the highest price and the one immediately below.
In order to calculate the index value, only SPX out-of-the-money options are used, except one. In correspondence of \( K_0 \), the mid-point \( Q_t(K_0, T) \) represents the average price of the call and the put option at this strike level. Here, the call option results in-the-money, since \( K_0 \leq F_t \). This is the reason why the extra term in (3.1) has been added to account for the only in-the-money call option. Indeed, the latter has been transformed via the put-call parity into an out-of-the-money put.

Moreover, to determine the forward index level \( F_t \), the CBOE chooses the pair of put and call prices which have the narrowest difference. Then, the forward price is calculated using the put-call parity relation, as follows:

\[
F_t = K + e^{r(T-t)} \min [c_t(T, K) - p_t(T, K)],
\]

where \( p_t(K, T) \) and \( c_t(K, T) \) are, respectively the put and the call price.

The options included in the calculation are all the available calls with strike greater than \( F_t \) and all puts with strike lower than \( F_t \). Options with a bid price equal to zero are excluded. Furthermore, when two consecutive puts are found to have zero bid prices, no other options with lower strike prices are included in the calculation, causing a truncation in the option’s strike range. Similarly, no higher strike calls are included when two calls are found to have no bid.

As a consequence, as volatility rises and falls, both the strike ranges of considered options and options with a null bid price tend to expand and contract. This means that the number of options gathered together for the calculation may vary from day to day and possibly, even from minute to minute.

Since the VIX measures the 30-day implied volatility, and options with a precise 30-days expiration are not always available, the CBOE calculates the \( \sigma^2_{VIX} \) for two expirations, the so called near-term and next-term. These are, respectively, options with more than 23 days and less then 37 days to expiration, which include both options with “standard” 3\(^{rd}\) Friday expiration dates and newly developed “weekly” options, that expire every Friday, except the 3\(^{rd}\) Friday of the month. Once each week, the two expiration dates ‘roll’ to new contract maturities\(^3\).

\(^3\)Actually, the introduction of “weekly” options enabled this latest setup of the VIX calculation. Before 2014, in fact, the CBOE considered the near-term and the next-term maturities the two next expiration dates, which were “standard” monthly maturities. If the time to maturity was less than 8 days, the CBOE used to consider the option contracts
The VIX is quoted in terms of annualized volatility percentage points. In order to obtain its value calculated for a 30-day horizon, the CBOE interpolates the two variances found for near-term and next-term option. The interpolation makes use of an actual 365-days convention. Finally, once calculated the square root, the found value is multiplied by 100. Thus, the VIX final value at time \( t = 0 \) is:

\[
VIX_t = 100 \sqrt{\frac{N_{365}}{N_{30}} \left[ T_1 \sigma_1^2 \frac{N_{T_2} - N_{30}}{N_{T_2} - N_{T_1}} + T_2 \sigma_2^2 \frac{N_{30} - N_{T_1}}{N_{T_2} - N_{T_1}} \right]}, \tag{3.4}
\]

where \( N_x \) is the number of minutes contained in \( x \) days, \( T_1 \) and \( T_2 \) are the near-term and the next-term maturities, and \( \sigma_1^2 \) and \( \sigma_2^2 \) are the respective variances.

The current CBOE procedure enhances both the economic meaning and the effectiveness of the VIX. The old VIX represented a monotonic but non-linear transformation of at-the-money option prices. In contrast, the new one is the price of a linear portfolio of options, which has a very concrete economic interpretation. In fact, the VIX squared approximates the risk-neutral expectation of the annualized return variance over the next 30-day period. Hence, it is a representation of the “fair value of future variance”, obtained by the approximation of the 30-day variance swap rate. As we already discussed in Section 2.2.1, it is possible to decompose such a swap into a portfolio of options weighted inversely to the squared strike price.

Moreover, the current CBOE’s methodology fixed two annoying errors: it solves the upward bias induced by the day-counting convention and the latest introduction of “weekly” options into the calculation procedure helps to enhance the interpolation precision.

### 3.2.3 Replication of the VIX calculation procedure

As a part of this work, we implemented an algorithm that calculates the value of the VIX. By reproducing the CBOE’s calculation methodology, its aim is to provide a routine that allows to derive rapidly and efficiently the

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expiring on the \( 2^{nd} \) and \( 3^{rd} \) month. The goal was to minimize the price anomalies which verify when we get closer to the expiration date. However, the change is not dramatic, as it just approximates better the 30-day horizon estimate.
real-time value of the index starting from the available data on SPX options, downloaded from the CBOE's website.

In this Section, we will explain the principal steps of the procedure needed for the computation of the index, together with the main features of the routine.

The code has been written in MatLab language and it follows closely the CBOE's methodology described in White paper (2015). Since the options data has been downloaded directly from the website, we needed also to create a macro in Excel VBA, in order to make the prices readable from the MatLab environment. The entire code of the routine is reported in Appendix B.

Many refinements, that go beyond the purpose of this work, could be arranged to the code. The algorithm is neither fast nor immediate, meaning that some manual operations must be executed in order to make it work. The principal adjustment to smooth the routine should be to make the MatLab software and the Excel platform communicate, together with the downloading procedure from CBOE’s website.

Besides this considerations, the algorithm could serve as a starting point to calculate the VIX more precisely. Moreover, the code can be modified and customized to meet specific necessities. For example, like in our implementation, it is possible to reduce the time interval between the two maturities used to calculate the variance, which are interpolated to obtain a precise 30-day estimate of the volatility. As a matter of fact, the official procedure involves only “standard” SPX options, expiring on the 3rd Friday of the month, and normal “weeklys” SPX options, expiring on all the Fridays of the month, except the 3rd. Instead, our algorithm embraces also newly developed “weekly” SPX options, which expire on Mondays and Wednesdays.

Now, the CBOE gives the opportunity to invest in options with a very tight-spaced range of maturities. Since the VIX represents the estimation of the annualized volatility on a 30-day horizon, the CBOE’s procedure considers the options that expire closer before and after a 30-day length of time. Hence, the distance between the near-term and the next-term options is 7 days. For example, on the 6th October 2016, the VIX has been calculated using SPX options expiring 28 days after, on the 4th November, and 35 days after, on the 11th November. In contrast, our procedure reduces the interpolation error, which will be discussed in Section 3.3 of this Chapter, by shrinking the distance between the two maturities. In fact, the interpolation is made upon two narrower dates, reducing the range from 7 days to only 2 days.
In addition, the code could be shaped in order to customize the calculation procedure of the VIX. Possibly, one could even enhance the volatility estimation, using this greater estimate to exploit potential arbitrage opportunities.

As we have seen in the previous Chapters, applying a pure model-free approach to the estimation of the implied volatility in real markets is not an easy task. This could lead to some inaccuracies. Thus, also the CBOE’s procedure is imperfect, and relies on particular techniques to deal with these errors. These stratagems are, on one hand, the elimination from the calculation of the index of the options with null bid price and, on the other hand, the truncation of the range of considered options, once two consecutive null bid prices are found. The reason is that the prices of options in the tails of the distribution may be distorted. However, they contain relevant information regarding the market volatility. One could include these options in the calculation by first extending the considered range and second by interpolating the known option prices to derive the unknown prices (i.e. with a null bid).

Another possibility, since the volatilities are discounted from one month up to the actual date, is that one could change the interest rates accordingly to his preferences and forecasts. Maintaining all the other factors constant, in presence of decreasing rates, the value of the VIX will be higher in the future. Inversely, a surge in the interest rates would cause the VIX to shrink.

Our algorithm, which reproduces the CBOE’s calculation procedure of the VIX, can be resumed in the following 6 steps.

**Step 1: Collection of the useful data**

The information needed to calculate the VIX are all the bid and ask quotations of the out-of-the-money SPX call and put options. These are collected both for the near-term and the next-term maturity. Our routine starts from the download of a complete option chain from the CBOE’s website. The table contains the real-time bid and ask quotations for all the strikes and expirations available at that moment. Then, the table is refined by the macro and by the MatLab code, to extract the useful data. For every expiration, the equivalent risk-free interest rate is calculated, by deriving them from the 1-month yield of the U.S. T-bill. The price of every option is calculated averaging the bid and the ask quotations.

**Step 2: Computation of the time to maturity**

The time to maturity in (3.4) is calculated very precisely, in the order of
minutes. This is derived for both the near-term and the next-term maturities. The general formula is:

\[ T = \frac{M_{\text{today}} + M_{\text{sett}} + M_{\text{middle}}}{M_{\text{year}}}, \]  

(3.5)

where:

- \( M_{\text{today}} \) are the minutes until midnight of the current day;
- \( M_{\text{sett}} \) are the minutes from midnight until 8:30 a.m. for “standard” SPX expirations, or minutes from midnight until 3:00 p.m. for “weekly” SPX expirations;
- \( M_{\text{middle}} \) are the total minutes between the midnight of the current day and the expiration day;
- \( M_{\text{year}} \) are the minutes in a year.

The components in the numerator of (3.5) are varying depending on the type of the options considered for each maturity, on the moment of the day at which the valuation is made, and on the days missing until the expiration date, while the component in the denominator, \( M_{\text{year}} \), is constant and serves as a annualization factor.

**Step 3: Computation of the forward level of the index**

For each expiration, the at-the-money VIX option is extracted from the option chain. This is the option at which the absolute price difference between the price of the call option and the price of the put option is smallest. The call and the put prices are obtained averaging each option’s bid and ask quotation.

Once defined the at-the-money option, the forward level of the SPX is found, for each maturity \( m = 1, 2 \), via the formula:

\[ F_m = K_m + e^{r_mT_m} (c_m - p_m), \]  

(3.6)

where \( K_m \) is the at-the-money strike price and \( c_m \) and \( p_m \) are, respectively, the price of the at-the-money call and the at-the-money put.
Step 4: Identification of the options to be used in the VIX calculation

The CBOE’s includes in its calculation methodology only the price of out-of-the-money options. The selected options are puts and calls centered around the at-the-money strike price $K_0$, which is the strike price immediately under the forward level $F$. The out-of-the-money calls have strike prices $K_i$ greater or equal to $K_0$, while the out-of-the-money puts have strike prices $K_i$ lower or equal to $K_0$.

Starting from puts with the first strike price immediately below than $K_0$, the procedure moves successively to lower strike prices. The options with a bid price equal to zero are not included and, once two puts with consecutive strike prices are found to have null bid prices, the options with lower strike prices are discarded from the computation. The same conditions applied also to out-of-the-money call options, starting from the strike price immediately greater than $K_0$ and moving consequently to higher strike prices.

Finally, both the call and the put option with strike price $K_0$ are selected. Their price must be averaged to produce a single value.

Step 5: Computation of the implied volatility

The formula for the derivation of the implied volatility is:

$$\sigma^2 = 2 \sum_{i=1}^{n_m} \frac{\Delta K_{m,i}}{K_{m,i}^2} Q(K_{m,i}, T_m) e^{r_m T_m} - \frac{1}{T_m} \left[ \frac{F_m}{K_{m,0}} - 1 \right]^2. \quad (3.7)$$

Here, for each maturity $m = 1, 2$, $T_m$ is the time to expiration in years, $n_m$ is the number of options in the out-of-the-money series, $K_{m,i}$ is the strike of the $i$-th option, $r_m$ is the risk-free interest rate, $F_m$ is the forward level of the index, $Q$ is the bid-ask average price of the option with exercise price $K_{m,i}$, and $K_{m,0}$ is the strike price of the option immediately under the forward level.

Note that, for the calculation of the sum in (3.7), it is considered only one option for each strike price, except in the case of the at-the-money strike price. As we have seen before, the procedure selected both a call and put, whose values have been averaged to obtain a single price.

Step 6: Interpolation and calculation of the VIX level

The final value of the VIX is obtained through:

$$VIX_t = 100 \sqrt{\frac{N_{365}}{N_{30}} \left[ T_1 \sigma_1^2 \frac{N_{T_2} - N_{T_1}}{N_{T_2} - N_{T_1}} + T_2 \sigma_2^2 \frac{N_{30} - N_{T_1}}{N_{T_2} - N_{T_1}} \right]}, \quad (3.8)$$
where \( N_x \) is the number of minutes contained in \( x \) days, \( T_1 \) and \( T_2 \) are the near-term and the next-term maturities, and \( \sigma_1^2 \) and \( \sigma_2^2 \) are the respective variances.

Specifically, the variances found in step 5 are, first, interpolated between the near-term and next-term maturity to reflect the actual 30-day horizon of the VIX. The values of the maturities have been computed in step 2 and, since the inclusion of SPX Weeklys options determines that the near-term options will always have more than 23 days to expiration and the next-term options will always have less than 37 days to expiration, the resulting VIX value will always reflect an interpolation of \( \sigma_1^2 \) and \( \sigma_2^2 \), namely each individual weight is less than or equal to 1 and the sum of the weights equals 1. Second, the square root of the found value is taken and multiplied by 100 to reflect the standard deviation in the percentage form, which represents the percentage implied volatility.

### 3.3 Index biases

While the new VIX is more appealing than its predecessor, the actual calculation of the index is also more complex. Necessary steps must be taken to reduce the implementation errors of the estimation procedure of the implied volatility. However, the current methodology adopted by the CBOE accomplishes only partially to this objective, leading to substantial biases in the published value of the index. Any bias in the index calculation may cause direct consequences for the derivatives based on the index. For example, the overestimation (underestimation) of the VIX will produce a corresponding underpricing (overpricing) in the VIX-related variance swaps.

The formula for the calculation of the VIX is based on a discretization of the formula for the “fair value of future variance”, first proposed by Demeterfi et al. (1999). Jiang and Tian (2007) have shown the equivalence of this concept to the model-free implied volatility by Britten-Jones and Neuberger (2000).

As a matter of fact, assuming \( t = 0 \), the formula to calculate the implied volatility in (3.1) represents a discrete version of the formula for the model-free implied volatility in (2.12). The latter can be rewritten in terms of
forward prices, obtaining:

\[
\sigma^2_{MFIV} = \frac{2}{T} \int_0^\infty e^{rT} C(T, K) - (S_0 e^{rT} - K)^+ \frac{dK}{K^2}.
\]  (3.9)

If we split this variance between call and put options at the forward level \(F_0\) using the call-parity relation, we obtain:

\[
\sigma^2_{MFIV} = \frac{2e^{rT}}{T} \left[ \int_0^{F_0} \frac{P(T, K)}{K^2} dK + \int_{F_0}^\infty \frac{C(T, K)}{K^2} dK \right].
\]  (3.10)

The integrals in the equation above are restated in three integrals in order to include the arbitrary price \(S^*\), chosen to be closer to the forward price, as:

\[
\sigma^2_{MFIV} = \frac{2e^{rT}}{T} \left[ \int_0^{S^*} \frac{P(T, K)}{K^2} dK + \int_{S^*}^\infty \frac{C(T, K)}{K^2} dK + \int_{F_0}^{S^*} \frac{P(T, K) - C(T, K)}{K^2} dK \right].
\]  (3.11)

The third integral inside the bracket is reformulated using, once more, the put-call parity, and integrating it out. In this way it is possible to obtain an equivalent definition for the model-free implied variance, as:

\[
\sigma^2_{MFIV} = \frac{2}{T} \left\{ rT - \left[ \frac{S_0}{S^*} e^{rT} - 1 \right] - \ln \left( \frac{S^*}{S_0} \right) \right. \\
+ \left. e^{rT} \int_0^{S^*} \frac{P(T, K)}{K^2} dK + e^{rT} \int_{S^*}^\infty \frac{C(T, K)}{K^2} dK \right\}. \]  (3.12)

Conceptually, the above equation is equal to the definition of “fair value of future variance”. Thus, the discrete formula of the VIX can be regarded as an actual model-free implied volatility estimate, which in turn is an approximation of the variance swap rate up to the discretization error.

The purpose of the CBOE’s methodology, described in Section 3.2.3, is to derive a model-free estimation of the implied volatility from the option prices. The model-free implied volatility is defined as an integral of option prices over an infinite range of strike prices, from 0 to \(\infty\). If options prices are available for all strike prices, the required integral is calculated straightforward using
numerical integration techniques. Clearly, only a limited number of strike prices is actually traded in the marketplace.

Given the limited availability of the options, the CBOE’s procedure may lead to inaccuracies in the calculation of the actual model-free implied volatility. Many types of approximation errors, such as truncation, discretization, expansion and interpolation errors, are present in the actual implementation. Some of these are unlikely to be economically significant, however all of them can impact the estimation procedure, causing a distorted value of the VIX.

The first error originated by the CBOE’s implementation of the VIX calculation is the truncation error and it is caused by the limited extension of available strike prices of the options in the market. In contrast, the model-free implied variance requests an infinite range of strike prices.

Let $K_U$ and $K_L$ be, respectively, the lowest and the highest strike prices listed for a certain maturity. The most extreme strike prices can be associated to very unlikely events in the behavior of the underlying asset and usually are not listed. Thus, a truncation of the strike range $[0, \infty]$ is introduced, causing the VIX to be calculated only on the finite range $[K_U, K_L]$. The equivalence between the range of the model-free implied variance and the range of the options in the VIX calculation is in this way not respected:

$$
\int_0^{K_0} \frac{P(T, K)}{K^2} dK + \int_{K_0}^{\infty} \frac{C(T, K)}{K^2} dK = \int_{K_L}^{K_0} \frac{P(T, K)}{K^2} dK + \int_{K_0}^{K_U} \frac{C(T, K)}{K^2} dK.
$$

The size of the truncation error $\delta_T$ is given by:

$$
\delta_T = -\frac{2}{T} e^{\frac{\tau}{T}} \left[ \int_0^{K_L} \frac{P(T, K)}{K^2} dK + \int_{K_U}^{\infty} \frac{C(T, K)}{K^2} dK \right].
$$

In addition, the CBOE interrupts the inclusion of further options, in the lower extremity for the put options and in the upper extremity for the call options, when two consecutive bid values are found to be null. Such a methodology may, eventually, expand the size of the truncation error.

The range of considered options is not fixed and can change over time. It can vary from day to day and, possibly, also from minute to minute. During periods of rapid market movement, this behavior can be both frequent and consistent.

The reason why the CBOE’s introduced such an artifice are essentially two: the need to accommodate the existing strike prices to the movements
of the underlying asset and the mispricing of some options, especially at extreme strike prices.

Indeed, the CBOE lists additional strike prices when the SPX moves beyond the existing range of listed strikes, causing the considered interval to enlarge. Furthermore, the price of the options in the tails of the risk-neutral distribution are usually associated to poor liquidity. Hence, they may be distorted. The same reasoning can be applied to options with a bid quotation equal to zero, which are considered mispriced and, thus, excluded from the calculation by the CBOE.

Hypothetically, if all the options have been priced correctly, the truncation error leads to a downward bias in the obtained variance, traducing in a lower value of the VIX. The negative sign in (3.14) indicates this effect.

As confirmed by Jiang and Tian (2005), the truncation error, in general, underestimates the true volatility and decreases monotonically as the truncation bounds, $K_L$ and $K_U$, move away from the forward level $F_0$. However, when the volatility is relatively low, it can be seen as negligible if the two truncation points are far more than three standard deviations from $F_0$. Inversely, the truncation error rises drastically as volatility increases. In fact, a larger mass of options is pushed to the tails beyond the truncation interval, causing a significant underestimation of the true volatility. It is important to highlight this issue, because it means that the error is likely to rise sharply when the volatility experiences a sudden surge, precisely when a more accurate measure of the VIX is warranted. Despite the additional strike prices cover a larger range of strike prices, the truncation error due to a volatility spike may be substantial.

The second type of approximation error in the VIX calculation is the discretization error, which is due to the numerical integration adopted in the CBOE’s procedure:

$$
\int_{K_L}^{K_0} \frac{P(T, K)}{K^2} dK + \int_{K_0}^{K_U} \frac{C(T, K)}{K^2} dK \approx \sum_{i=1}^{n} \frac{\Delta K_i}{K^2} Q(T, K_i). \quad (3.15)
$$

The size of the discretization error $\delta_D$ is, then:

$$
\delta_D = \frac{2}{T} e^{-\gamma T} \left\{ \sum_{i=1}^{n} \frac{\Delta K_i}{K^2} Q(T, K_i) \right. \\
- \left[ \int_{K_L}^{K_0} \frac{P(T, K)}{K^2} dK + \int_{K_0}^{K_U} \frac{C(T, K)}{K^2} dK \right]. \quad (3.16)
$$
The theory requires for a complete set of options prices for a given maturity, meaning that there exists an available strike price for every possible future state of the world. Clearly, this is impossible to obtain in the marketplace. However, even if the the error can be minimized by using a sufficiently fine partition of strike prices, the actual grid used by the CBOE is based on listed strike prices and it is quite coarse. Usually options on the SPX are quoted spaced 5 points apart around the at-the-money level, while the distance between two consecutive strike prices grows considerably moving to further strikes. For example, in the proximity of the most extreme strike prices, the interval can be even 50 points. Also this issue causes the model-free implied variance to be underestimated and, consequently, a lower value of the VIX.

Another type of error in the CBOE’s procedure is the expansion error. It is caused by the Taylor series expansion of the log function used in the VIX computation. Substituting $K_0$ to the arbitrary value $S^*$, the terms preceding the integrals in (3.12) can be restated as:

$$\frac{2}{T} \left\{ rT - \left[ \frac{S_0}{K_0} e^{rT} - 1 \right] - \ln \left( \frac{K_0}{S_0} \right) \right\} = \frac{2}{T} \left[ \ln \left( \frac{F_0}{K_0} \right) - \left( \frac{F_0}{K_0} - 1 \right) \right]. \quad (3.17)$$

Ignoring terms higher than the second, the Taylor series expansion of the log function is:

$$\ln \left( \frac{F_0}{K_0} \right) \approx \left( \frac{F_0}{K_0} - 1 \right) - \frac{1}{2} \left( \frac{F_0}{K_0} - 1 \right)^2. \quad (3.18)$$

Thus, the resulting approximation is given by:

$$\frac{2}{T} \left\{ rT - \left[ \frac{S_0}{K_0} e^{rT} - 1 \right] - \ln \left( \frac{K_0}{S_0} \right) \right\} \approx -\frac{1}{T} \left( \frac{F_0}{K_0} - 1 \right)^2. \quad (3.19)$$

The right-hand side of the equation above is the actual second term in the VIX formula (3.7).

The size of the expansion error $\delta_E$ is:

$$\delta_E = \frac{2}{T} \left\{ \left( \frac{F_0}{K_0} - 1 \right) - \frac{1}{2} \left( \frac{F_0}{K_0} - 1 \right)^2 \right\} - \ln \left( \frac{F_0}{K_0} \right). \quad (3.20)$$

The expansion error is negative because, since $F_0 \geq K_0$ by definition, the right-hand side of (3.18) is negative. However, the size of the error is
negligible, as long as \( K_0 \) is chosen as closest as possible, but not exceeding, \( F_0 \).

The last type of approximation error is given by the interpolation error. While the model-free estimation of the variance is based on a fixed-30 day maturity, usually there are no options expiring exactly 30 calendar days after the actual date. Hence, the solution is to interpolate the variances between two maturities, found at the nearest proximity of the 30-day horizon.

Indeed, the CBOE first calculates the variances for the two maturities, the near-term and the-next term, and then linearly interpolates them using (3.4). The interpolation may induce an approximation error if the relationship between the model-free implied variance and the maturity is a nonlinear function. The size of this error, \( \delta_I \), is given by:

\[
\delta_I = \sigma^2_{VIX} - \sigma^2_{MFIV},
\]

where \( \sigma^2_{VIX} \) is the term under the square root in (3.4), and \( \sigma^2_{MFIV} \) is the true model-free implied variance on a time horizon of 30 days.

As well documented in literature\(^4\), the implied variance term structure is neither linear nor monotonic. The interpolation error was problematic since the VIX index was calculated with its original model-free implementation (2003), in which the next two monthly maturities were considered. Nevertheless, the latest introduction in 2014 of “weekly” SPX options in the calculation of the index, shrinks the time interval between the maturities considered from 1 month to 1 week, enhancing the interpolation precision of the index.

In general, the CBOE’s procedure for the calculation of the VIX leads to positive discretization errors and negative truncation errors. Jiang and Tian (2005, 2007) found that the two errors interact with each other differently according to different levels of volatility. At a modest volatility level (e.g. 20\%), the two errors appear to offset each other leading to a negligible mismeasurement of the index. At lower volatility levels (e.g. 20\%), the discretization errors overtook the truncation error and causes the CBOE method to overestimate the model-free implied volatility. Instead, in a situation of high volatility (e.g. 30\%), the truncation error dominates producing a downward biased estimation of the VIX. In general, the CBOE’s methodology for the calculation of the VIX underestimates the true model-free implied volatility. The reason is that the magnitude of the truncation error dominates the size

\(^4\) See, for example, Xu and Taylor (1994).
of the discretization error. Since the calculation of the VIX index is the cornerstone for the evaluation of VIX related derivatives, both errors can be economically significant.

Many alternatives have been presented to avoid the miscalculation of the model-free implied volatility. Jiang and Tian (2005) proposed an interpolation-extrapolation scheme in order to obtain a sufficiently large set of available strike prices, which covers all the price distribution. First, the missing prices of the options are estimated using a curve-fitting technique. Then, the prices in the tails are derived assuming the same Black and Scholes implied volatility as the options in the endpoint strike prices. In a alternative implementation setup, Jiang and Tian (2007) derived the prices in the tails by maintaining a linear extrapolation technique, and adjusting the slope of the extrapolation segment on both sides to the corresponding slope of the interior segment at the two endpoints. The authors claim that the two methods are in general more accurate than the standard truncation method. However, their work is supported by non-verifiable hypothesis regarding the tails of the option's price distribution.

3.4 Historical behavior and forecasting performance

The VIX is published in real-time by the CBOE during the whole trading window. The index is closely followed by market participants and financial media. Despite its new version, calculated with a model-free approach based on out-of-the-money SPX options, is officially available by 2003, the CBOE computed back the index value since 1990.

The index reflects supply and demand of options. Investors flow in and out of the options markets based on their expectations regarding the events occurring in financial markets. Generally speaking, a higher demand of options due to an increase of the investor's fear traduces in higher values of the VIX. It is possible to state that almost every event can potentially have an impact on the index, as the SPX is the most important equity index in the financial world.

There are many type of events affecting volatility. Some of them are reflected almost instantaneously according to an exact timetable that can be known in advance. Example of “exact” volatility events are the announce-
ments of government economic data, corporate earnings announcements or speeches by important economic authorities. Others, instead, unveil themselves progressively in a sparse way, such as geopolitical and financial crises or natural disasters.

Manifold economic and financial forces act on the VIX, interacting each other in different ways. Macroeconomic trends and geopolitical tensions, as well as systemic risks or monetary and fiscal policies, are reflected in the VIX through investor’s expectations. More strict financial questions also play their part in the VIX determination, such as liquidity and credit issues, or availability and convenience of hedging substitutes.

Contagion is another important aspect of volatility events. Sector contagion, geographical contagion or even entity-specific problems may spread and cause issues to subjects apparently not directly touched by the original problem.

Crisis have always had the greatest influence on the volatility in financial markets, causing the the fear among the investors to grow dangerously. Crises have many different causes: during the pre-VIX era, a mix of geopolitical issues and stock market crashes prevailed. However, in the VIX era, the importance of technical and psychological factors has increased. The rise of quantitative trading has put more emphasis on the technical data than the fundamental data. Furthermore, the greater investment accessibility by retail-level investors added a new layer of psychology to the markets. The Flash Crash of 6th May 2010 and the Dot-Com Bubble of the 2nd millennium are two sparkling examples of the mentioned aspects.

Looking at the events that most impacted the volatility, the global financial crisis of 2007-2009 dominates the scene. In times of dramatic market weakness, often the VIX will rise by a magnitude that is greater than the drop in the market. In correspondence of the heaviest day of the crisis, the 24th October 2008, the VIX reached its historical maximum value, which has been of 89.53 (intraday). On this day, called “Bloody Friday”, many markets worldwide experienced the worst declines in their history, dropping as many as 10%, amidst growing fear among investors that deep global recession was imminent if not already in act.

Other major spikes in the index as a consequence of financial crises can be found in correspondence of the Asian currency crises in 1997 and the First and Second European Debt crisis in 2010 and 2011. Sometimes the panic among investors is also caused by external factors, apparently without any link with the financial markets, as in the 9/11 World Trade Center attacks.
in 2001, the Fukushima nuclear meltdown in 2011 or the Hurricane Katrina in 2005. Furthermore, the Long Term Capital Management case of 1998 and the Dot-Com Crash in 2002 are two major collapses that heavily impacted the investor’s concerns. Even tough the effect this events had on the level of the VIX has been huge, it is very poor if compared to that of the subprime crisis. In fact, during the above mentioned cases, the index did not surpass the 50 points threshold.

Instead of finding a precise meaning to the index, the real benefit of the VIX comes from comparing its level with other historical benchmarks. Hence, to gauge the behavior of the index we should inspect its history.

Figure 3.2 shows the relationship between the SPX and the VIX from 1995 to 2016. Although the VIX is basically flat and the SPX has risen tremendously, the shorter-term relationship may be used to predict the future direction of stock prices. It is possible to denote the well-known inverse behavior of the VIX and the SPX, as the VIX appears to spike during bearish markets.

![Figure 3.2: End-of-day closing values of the SPX (right axis) and the VIX (left axis) from 1st January 1995 to 31st December 2016.](image)

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Despite volatility should rise during both upward and downward movements of the market, the relationship is asymmetrical. The reason behind this inverse relationship is portfolio insurance: when markets plunge, there is a rush to purchase protective put options, while there are rarely dramatically higher options purchasing, compared to option selling, when the market is rallying. The effect is amplified by the put-call parity, pushing the implied volatility up. In fact, the VIX index increases by a greater amount when the SPX drops with respect to when the SPX surges. However, there are also times when a run-up in stock prices is accompanied by a run-up in the volatility index. For example, in January 1999, both the level of the VIX and the SPX were rising. The pattern appeared also in other occasions. Clearly, investors could become nervous even during bullish markets. As a matter of fact, the VIX index moved in the opposite direction of the SPX in 79.78% of the times. It is important to note that, despite VIX returns often show a strong negative correlation with the SPX returns, this relationship is extremely dynamic: it varies among different SPX return ranges and may vary extremely in time.

Similarly to volatility, the VIX follows a mean-reverting process, as Whaley (2009) points out. If we regress the rate of change in the SPX to the rate of change of the SPX, we obtain an intercept term that is not statistically different from zero, signaling a non-deterministic growth of the VIX. This means that, if the SPX does not change during the day, the rate of change of the VIX should be negligible. As a matter of fact, when the latter is high, it tends to be pulled back down to its long-run mean and, when it is too low, it tends to be pulled back up.

A curious feature of the VIX is tied to a sort of "weekend effect". The index decreases its magnitude moving from the first days to the last days of the week. In fact, the levels of the index registered on Mondays are, on average, 0.5 points higher than the Friday levels.

As the aggregate behavior of the investors in the option markets testifies the aggregate consensus towards future volatility levels, the implied volatility in the markets serves also as a forecasting tool. Based on historical data on daily closing prices of the SPX and the VIX on the period that goes from the 1st January of 1995 to the 31st December of 2016, we analyzed the relationship between the realized volatility of the SPX and the implied volatility, represented by the VIX. As the VIX is calculated as a 30-day forecast of the market implied volatility, its value today should represent that of the realized volatility 30 days later, which has been calculated back on the
previous 30 days. For each day $t$, we computed the \textit{ex-post} realized volatility during the next 30 calendar days according to the following equation:

$$RVOL_{t,t+30} = 100 \sqrt{\frac{365}{30} \sum_{j=1}^{30} \left[ \ln \left( \frac{S_{t+j}}{S_{t+j-1}} \right) \right]^2}, \quad (3.22)$$

where $S_t$ is the price of the underlying asset, in this case the SPX. We followed the industry standard by computing the return squared without de-meaning the returns and by annualizing the volatility using an actual 365-day accounting convention. These calculations allow us to determine to which extent the VIX index accurately predicted the realized volatility occurred during the past 30 calendar days.

Figure 3.3 shows the VIX levels, coupled with the realized volatility translated in the future by 30 days.

![VIX vs. Realized Volatility](image)

Figure 3.3: Comparison between realized volatility and VIX from 1\textsuperscript{st} January 1995 to 31\textsuperscript{st} December 2016.

At first glance, we can see that the two measures’ behavior is quite similar but the VIX usually overestimates the historical volatility. However, during
great spikes in volatility the index underestimates the realized volatility. Furthermore, we observe that the VIX anticipates the movement of the volatility in the markets. Investors usually reacts instantaneously to the news affecting financial markets, thus suddenly provoking a surge in the implied volatility level.

The difference between the level of the VIX and the 30-days translated realized volatility is the so-called volatility risk premium (VRP). In order to earn higher returns than the risk-free rate investors must, by definition, take on some risks. VRP is the premium that an investor in some asset pays to reduce exposure to the volatility of the future returns of that asset, in our case the SPX. Basically, the price hedgers are prepared to pay some premium to speculators to offload price risk. The VRP is reflected in SPX options by options being overpriced. Sellers of these options will receive the VRP through these higher prices. Since the VIX is a measure of the implied volatility of these options, its value is higher compared to where it should be if it were an unbiased predictor of future SPX volatility. In a nutshell, the VIX overpredicts the SPX volatility and the difference is represented by the VRP. Furthermore, in Figure 3.3 it is possible to denote that the VRP might be either positive or negative. When volatility reaches extremely high values, the VRP is likely to be negative.

Table 3.1 displays the summary statistics for the VIX and the realized volatility, computed both on the end-of-day levels and the first differences.

Table 3.1: Summary statistics of VIX and realized volatility of the SPX, both levels and daily differences. Data from 1st January 1995 to 31st December 2016.

<table>
<thead>
<tr>
<th>MOMENTS</th>
<th>VIX_{lev}</th>
<th>RVOL_{lev}</th>
<th>VIX_{diff}</th>
<th>RVOL_{diff}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>20.31</td>
<td>16.35</td>
<td>0.0003</td>
<td>0.0002</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>8.24</td>
<td>9.64</td>
<td>1.33</td>
<td>0.94</td>
</tr>
<tr>
<td>Skewness</td>
<td>2.05</td>
<td>2.86</td>
<td>0.75</td>
<td>0.70</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>7.25</td>
<td>12.96</td>
<td>26.47</td>
<td>30.54</td>
</tr>
</tbody>
</table>

Comparing the index with the corresponding realized volatility, we find that the VIX level is roughly 4 percentage points greater, on average, than the realized volatility of the SPX. Instead, looking at the first differences, we find that both have a null mean, confirming the well-known mean reverting
behavior of the volatility.

Volatility levels show a positive skewness and a large kurtosis. However, the kurtosis of the first differences is even greater, showing potential discontinuous index return volatility movements. Indeed, many researchers found that both VIX and volatility dynamics are characterized by jumps in the process.

The VIX shows a valuable correspondence with the volatility of the SPX, as it is positively correlated with the subsequent realized volatility, but the correlation becomes smaller when measured in daily changes. Table 3.2 shows the cross-correlation between the two measures.

Table 3.2: Cross-correlations between VIX and realized volatility of the SPX, both levels and daily differences. Data from 1st January 1995 to 31st December 2016.

<table>
<thead>
<tr>
<th>Correlation</th>
<th>VIX_{lev}</th>
<th>RVOL_{lev}</th>
<th>VIX_{diff}</th>
<th>RVOL_{diff}</th>
</tr>
</thead>
<tbody>
<tr>
<td>VIX_{lev/diff}</td>
<td>1.00</td>
<td>0.89</td>
<td>1.00</td>
<td>0.12</td>
</tr>
</tbody>
</table>

The leverage effect in the volatility is documented by Carr and Wu (2006). By looking at the relationship between SPX returns and daily changes in the VIX at different lags, we observe a strongly negative instantaneous correlation, but the correlation at further lags is much smaller. However, lagged returns of the SPX, within a week, show a marginally significant positive correlation with daily changes in the VIX. This indicates that index returns may predict future movements in the volatility index. Nevertheless, index returns with negative lags do not display a significant correlation with the daily changes in the volatility index. Hence, the inverse relationship is not verified, as VIX movements do not predict SPX returns.

In finance literature, the predictive performance of the implied volatility, and consequently of the VIX, is debated. However, evidence of the power of the VIX as a forecasting tool can be retraced in many works. As pointed out by Christensen and Prabhala (1998), current option prices as reflected in implied volatilities have information about future market volatility. This motivates using the VIX as a forecasting tool, as confirmed by Busch et al. (2011), who embedded option’s implied volatility into forecasting models for bonds, currency and stock markets. An important point has to be clear: since the VIX includes a risk premium, it will not be an unbiased predictor.
of future realized volatility. Furthermore, Kanas (2012) examined the predictive ability of GARCH models with the addition of the squared VIX as an exogenous variable. Compared with the GARCH model without the squared VIX and the squared VIX itself, the model showed a better forecasting performance for the variance.

Starting from 2012, the CBOE began to calculate an index to measure the implied volatility in the VIX, the VVIX. The aim of the CBOE is to provide volatility traders the possibility to formulate new strategies based on the relationship between the VIX index and its volatility. The VVIX is calculated using the same methodology of the VIX, but using out-of-the-money options written on the VIX itself. Basically, the new index measures the volatility of the volatility. It reflects the market’s consensus of expected volatility of the 30-day forward price of the VIX index, since the options on the VIX are based on forward prices.

![VIX vs. VVIX](image)

Figure 3.4: Comparison between the VIX and the VVIX from 1st January of 2007 to 31st December of 2016.

The CBOE calculated back the VVIX until 2007. The time series of the VIX and the VVIX from 2007 to 2016 is presented in Figure 3.4. At
a first glance, we observe that a spike in the VIX is usually accompanied by a spike in the VVIX too. However, the opposite behavior is not always true, as the VVIX may reach very high values even if the VIX is quite low. Further interpretations about the relationship between the two indexes are more complicated, as their joint behavior is not straightforward. However, as the CBOE’s VVIX White Paper (2012) highlights, we can deduce some features of their interaction from their historical behavior.

First, the range of values of the VVIX are significantly higher than those of the VIX. The VVIX has ranged in the examined period between 168 and 59 points, with an average of 87 points. Instead, the VIX ranged between 9 and 80 points, with an average of roughly 21 points. This means that the volatility of volatility is remarkably higher than the volatility itself.

Second, the correlation between the variations in the VIX and the variations in the VVIX is poor, except at high values of the VIX. For example, both indexes reached local maximums in correspondence of the credit crisis of 2008 and after the Flash Crash in 2010. However, their relationship is weaker if compared to that of the VIX and the SPX.

Furthermore, the VVIX reverts to its mean, a very well-know feature of volatility measures.

Finally, when events impacting severely the markets happen, the VVIX still remains at high levels for a long period of time. For example, after the Flash Crash, the VVIX never dropped below 70 points for the consecutive two years. This suggests that, after an extreme event, a new volatility regime appears and market participants seem to become more uncertain about the future value of the VIX.

3.5 Other volatility indexes

In addition to the index based on S&P500 options implied volatility, the CBOE has developed a handful of other volatility measures based on other equity indexes in the U.S., which make use of the respective index options as the underlying options. Examples are the VXO, based on the index options relative to the Dow Jones Industrial Average, and the RVX, calculated among Russell 2000 options. The first, as the VIX, reflects an overall view on the stock market volatility, while the second, given the composition of the index, is based more on the volatility of the small-companies operating mainly within domestic markets.
The majority of stock markets in Europe possess their own volatility index. For example, on the German market the Deutsche Börse computes the VDAX. In France, the MONEP (Marché des Options Negotiable de Paris) calculates the volatility indexes VX1 and VX6 based on the CAC-40 index options. The FTSE group publish the index of the Italian market, the IVI (Italian Volatility Index) based on FTSE MIB options, and the VFTSE for the London Stock Exchange.

Around the world, there are also many volatility indexes that are not published in real time, for example the Australian VIX and the India VIX.

All the volatility indexed are based on different methodologies, aiming to improve the precision of the estimated implied volatility. Among all, two indexes are interesting for our purposes, as they can be associated to the CBOE’s calculation methodology of the VIX: the FTSE methodology, on which both the IVI and the VFTSE indexes are based, and the VSSI volatility index on the Swiss market.

The methodology adopted by the FTSE to compute the IVI on the Italian stock index FTSE MIB is equivalent to that of the CBOE. Indeed, the procedure is the same but there is a slight difference in the calculation formula and the integration technique. The formula of the IVI replies exactly the definition of model-free implied variance by Demeterfi et al. (1999) in (3.12), in which the terms outside of the integrals have been restated according to (3.17).

With respect to the standard trapezoidal rule\footnote{The trapezoidal rule works by approximating the region under the graph of the function \( f(x) \) as a trapezoid and calculating its area. It follows that:

\[
\int_a^b f(x) \, dx \approx (b - a) \left[ \frac{f(a) + f(b)}{2} \right].
\]

\footnote{Suppose there is a function \( f(K) \), whose values are known at \( K_0, K_0 + \Delta K_1, K_0 + \Delta K_1 + \Delta K_2 \) and there is no need that the distances are equal (\( \Delta K_1 = \Delta K_2 \)). The integral of \( f(K) \) in the interval \([K_0, K_0 + \Delta K_1 + \Delta K_2]\) can be found through the Unequal-Interval}}

5

6

...the possibility of different distance between two consecutive strike prices. In here, the option data is partitioned into contiguous groups of three strike prices and a final overlapping set of two points. The latter are integrated using linear interpolation in the normal way for trapezoidal integration.
Instead, the VSMI is calculated on options based on the SMI (Swiss Market Index) quoted on the Eurex and by taking into account all the available options on all expirations. Eight sub-indices are computed among eight expiry months within a time to expiration of maximum two years. The methodology follows the same procedure as the model-free implied volatility adopted by the CBOE. Finally, the two indices that are closest to the required time of expiration of the main index VSMI (i.e. 30 days) are linearly interpolated.

\[
S(K_0, \triangle K_1, \triangle K_2) = \int_{K_0}^{K_0 + \triangle K_1 + \triangle K_2} f(K) \, dK \\
\approx \frac{\triangle K_1 + \triangle K_2}{6 \triangle K_1 \triangle K_2} \left[ (2\triangle K_1 - \triangle K_2)\triangle K_2 f(K_0) \\
+ (\triangle K_1 + \triangle K_2)^2 f(K_0 + \triangle K_1) \\
+ (\triangle K_1 + 2\triangle K_2)\triangle K_1 f(K_0 + \triangle K_1 + \triangle K_2) \right]. \quad (3.23)
\]
Chapter 4

Volatility derivatives

The development of financial markets over the last years have caused the role of volatility to transform: instead of being just a component in pricing theory, or an \textit{ex post} measure of the market’s variability, volatility has evolved into an asset class of its own. Since one can not directly invest in volatility, an investor can take a position in volatility derivatives, which are specific financial tools giving the opportunity to trade in volatility.

In the past, traders would normally use a delta-hedged position to trade in volatility. This, however, does not perfectly match with volatility alone, because the return is also dependent of the underlying stock price. Since the late 1990s, securities such as variance and volatility swaps as well as futures and options have provided a pure exposure on volatility, allowing investors to trade future realized variance or volatility against the current implied volatility. Unfortunately, these instruments have been exchanged exclusively OTC.

Given the increasing requirement by market participants to trade in liquid volatility-based derivatives, the CBOE introduced in 2004 futures based on the VIX index, which are standard futures contracts on forward 30-day implied volatility. Later on, in 2006, European-style options on the VIX index were also launched by the CBOE. These are standard options considering the forward level of the VIX index as the underlying asset and, essentially, they can be regarded as options on VIX futures.

The main reason to invest in volatility is that, likewise stock investors suppose to predict the future movements of the stock market or bond investors believe they can forecast the direction of interest rates, one may think to know in advance something about the future volatility levels. If the investor
suggests that future volatility will be higher, it is possible to assume a position in these securities which will lead to a profit if the volatility increases. Standardized contracts in official market contribute to the liquidity of these instruments, by allowing to find easily the proper counterpart.

Volatility as a trading tool is characterized by several features which makes trading attractive. Although it is technically possible to achieve a variance swap payoff exclusively with plain vanilla options, a natural question becomes why should one be encouraged to trade in volatility compared to other financial tools. The reason is that these innovative securities supply direct exposure to volatility, removing the path dependency issues that comes when delta-hedging an option. By trading in regular options, the investor has to both be aware of the underlying price as well as the volatility behavior. Furthermore, volatility is likely to grow when uncertainty increases and it appears to revert to its mean: high volatilities will eventually decrease, while low ones will likely rise. Finally, volatility is usually negatively correlated with stock market’s levels.

Given these considerations, volatility derivatives are traded mostly for hedging and diversification purposes, but they are also used in a more speculative manner. Recently, there has been indeed a growing acceptance of VIX-related products as risk management tools. VIX options and VIX futures enable investors to trade volatility independent of the direction or the level of stock prices. Whether an investor’s outlook on the market is bullish, bearish or somewhere in between, VIX options and futures can provide the ability to diversify a portfolio or hedge, mitigate or capitalize on broad market volatility. As a matter of fact, research have found that such strategies do provide significant protection, especially in downturns.

In this Chapter, we present an overview of the two securities quoted by the CBOE in order to invest in volatility: the VIX futures and the VIX options. Variance swaps have been described yet in Chapter 2 while discussing about model-free implied volatility and “fair value of future volatility”, however their construction is very similar to a volatility futures contract.

## 4.1 VIX futures

Since implied volatility has a tendency to revert to its long-run mean, future behavior in spot VIX is extremely predictable. Investors would be able to invest in the spot VIX when it is low and selling it when is high: a circumstance
that would never happen in an efficient market. In order to make the investment in volatility feasible, CBOE launched in 2004 the first exchange-traded VIX futures contract.

From their introduction, the CBOE Futures Exchange has experienced steady growth in trading activity on VIX futures contracts, which has been the result of the acceptance of volatility as a trading vehicle and asset class by a wide variety of market participants. Figure 4.1 depicts the escalation of VIX futures trading. If we leave apart the first 6 years from their introduction, the average volume of VIX futures started increasing dramatically from 2010 until nowadays: volumes in 2016 have been indeed 13 times greater than in 2010. The open interest followed almost the same trend.

![Average daily volume of VIX futures](image)

Figure 4.1: Average daily volume on annual basis of VIX futures from their introduction in 2004.

VIX futures are defined as standard future contracts on forward 30-day implied volatility of the S&P500 index, namely the VIX index. The VIX futures price reflects the market participants’ best collective guess as to what the VIX index will be at settlement. Contrary to futures on commodities, which require physical delivery of the underlying asset, VIX futures are settled in cash. In fact, the investor receives, or pay, the difference between their purchase (or sale) price and agreed prices on the settlement date. The payoff at maturity of a long position in VIX futures is:

$$1000 (VIX_T - P_{VIX}),$$  \hfill (4.1)
where $VIX_T$ represents the price of the underlying asset at maturity, namely the value of the VIX, $P_{VIX}$ is the agreed price at the begin of the contract and $1000$ is the contract multiplier. As a consequence, an investor purchasing (selling) a VIX futures contract will gain (lose) if the agreed price is lower (greater) than the value of the VIX at maturity.

The minimum variation of the contract is $0.05$ points in the index, equal to $50$ for each contract. VIX futures contracts can be listed for up to nine consecutive months. The settlement date is on the Wednesday that is $30$ days prior to the $3^{rd}$ Friday of the calendar month immediately following the month in which the contract expires. The final settlement value for the VIX futures is based on a Special Opening Quotation (SOQ) of the VIX index, calculated using opening prices of the constituent SPX options, that expire $30$ days after the relevant VIX expiration date. For example, the final settlement value for VIX derivatives expiring on January 19, 2017 will be calculated using SPX options that expire $30$ days later on February 18, 2017.

VIX futures own a number of unique features if compared to traditional futures contracts. While spot VIX represents a measure of the expected volatility of the S&P500 over the next 30-days, the price of VIX futures is based on the current expectation of how much the 30-day volatility will be at a specific time in the future, namely the expiration date. As a consequence, theoretical spot VIX returns are driven by changes in the level of the implied volatilities. Conversely, the returns of VIX futures are determined by the changes in the implied volatility future expectations. Hence, despite VIX futures should converge to the spot at maturity, there might be a considerable mismatching between spot VIX and VIX futures prior to expiration.

VIX futures price is a non linear function of VIX levels. The VIX futures trade in anticipation of the future price of the VIX index and at times may trade at premium to the index and at times may trade at a discount to the index. As a matter of fact, a theoretical investment in spot VIX compared to an investment in VIX futures leads to totally different outcomes. If the expectation of the market have been correct, VIX futures will usually lead to a lower return in the same period since VIX futures are likely to trade at a premium.

Another interesting aspect of VIX futures is the fact that they tend to exhibit significantly lower volatility than spot VIX. This is partially due to the mean-reverting behavior of the VIX. Consequently, pricing of this derivatives must take account carefully of this feature.

The relationship between spot VIX and VIX futures is further compli-
cated by the fact that it is not characterized by a typical cost of carry model, since there is no inherent cost of carry arbitrage between them, which means that:

$$F_{t,T} \neq VIX_t e^{r(T-t)},$$

where $F_{t,T}$ is a futures contract purchased in $t$ and expiring in $T$, and $VIX_t$ is the actual level of the VIX at time $t$.

Brenner et al. (2007) studied the relation occurring between the VIX futures and the underlying index, and derived three conclusions. First, the VIX futures are highly correlated with the VIX, thus negatively correlated to the S&P500. VIX futures returns are not normally distributed, but they exhibit positive skewness and high excess kurtosis. This testifies the opposite relationship of volatility and stock index: the latter is more likely to produce negative jumps while the former is more likely to experience positive jumps. Second, the longer the maturity of the VIX futures, the higher their price. In fact, the term structure of futures prices is often upward sloping. However, short-term volatility was so elevated during the financial crisis of 2008, that caused the curve to become downward sloping. Eventually, after 2009 the term structure returned to the normal upward-sloping behavior. Utilizing the term structure of VIX futures may also result in an indicator for the overall future direction of the stock market. Third, volatility of the VIX futures is very likely to decrease as time to maturity of the contracts increases, showing that shorter maturities contain more noise as they are heavily influenced by small changes in the investor’s sentiment.

4.2 VIX options

In 2006 the CBOE introduced options based on the VIX index. The introduction of VIX options came about two years after VIX futures started trading at the CFE, but they have rapidly gained popularity and, based on volume, have surpassed the futures contracts: the daily volume of VIX options is now as nearly as three times the daily volume of VIX futures. Financial institutions have found that at times VIX options offer the ability to hedge or diversify a portfolio better than other index option products. This use of VIX options as a cheap hedging vehicle has led to the quick growth in trading volume, however the speculative power of this derivatives is a component that must not be undervalued.
Figure 4.2 exhibits the average daily volume of VIX options, divided in calls and puts. After their peak in 2014, the volume remained almost stable on the last two years. Call options on the index are more popular than put options, as they are more efficient by means of hedging and diversification. The reason accounting for this phenomenon is investors’ typical reaction to market downturns and upturns: the majority looks for gaining upside exposure while hedging against any risks caused by unexpected recessions. By assuming a long position in the option, traders are able to benefit from unlimited upside exposure and simultaneously risking to lose only the premium paid for the call option in case of economic crash. Nonetheless, the ratio between the calls and the puts have changed between the years: when volatility reaches high values, it is likely that investors rush into put options. By the way, given the mean-reverting behavior of the index, this occurs only rarely.

![Average daily volume of VIX options](image)

Figure 4.2: Average daily volume of VIX puts, VIX calls and total VIX options on annual basis from their introduction in 2006.

VIX options are based on the forward level of the index: this is the reason why they are sometimes called “options on VIX futures”. If standard options expire on the 3rd Friday of the month, VIX options expire on Wednesdays, at standard expiration dates (i.e. on the 3rd Wednesday of the month) and also weekly expirations in the other Wednesdays of the month, except the 3rd. The liquidity of weekly VIX options is generally lower with respect to
standard-expiration options. The CBOE lists up to 6 weekly expirations and up to 9 standard monthly expirations.

The payoff at maturity of a long position in a VIX call option is:

$$100 \left( VIX_T - K_{VIX} \right)^+, \quad (4.3)$$
in which $VIX_T$ is the value of the index at maturity and $K_{VIX}$ is the strike price. The multiplier for a VIX option contract is $100$, hence a VIX option contract represents 100 times the quoted price of the option. Instead, the VIX futures contracts have a multiplier of $1000$, thus a VIX option contract actually represents one tenth of the value of a VIX futures contract.

VIX options are cash-settled instruments. The settlement for an in-the-money option will result in a cash transfer between the short position holder and the long position holder. The settlement price for VIX options is determined using the SOQ discussed in Section 4.1. If the option will expire in-the-money every difference point leads to a gain of $100$. The minimum variation of the exercise price is 0.05 VIX points for the options with value lower than $3$ and 0.10 points for options with value greater than $3$.

The CBOE lists a wide range of strike prices, varying from in-the-money options as well as out-of-the-money options. When the VIX index changes considerably, new listed strike prices are usually added.

VIX options are European-style options and, since the underlying of the VIX option is the forward price of the VIX index, the potential disconnection between the price of the spot VIX and the price of the options prior to expiration, which has been discussed in Section 4.1, is still valid. At times, VIX option contracts may appear to be mispriced relative to the VIX index as we observe quoted prices that are lower than the intrinsic value. This phenomenon is the result of the anticipatory nature of forward implied volatility relative to the current level of the VIX index. Although VIX options settle to the VIX index at expiration, prior to expiration prices of VIX options are based on the current expectations of what the VIX index will be at that date.

If there is a market expectation of a lower VIX, then the VIX futures contract that expires on the same date as the VIX option would be trading at a discount to the VIX index. This VIX future would also be the proper underlying instrument to value VIX options that have the same expiration date since both the option and futures prices that share expiration are based on the same market expectation of the direction of volatility.

Another unique feature that differentiate VIX options from most index options is the fact they are characterized by extremely high implied volatili-
ties. As a matter of fact, the VIX index exhibits greater volatility, which in turn is the volatility of volatility, if compared to most stock indexes, or even many individual equities. This is confirmed by the behavior of the VTVIX, discussed in Section 3.4.

Pricing of VIX-related products is particularly challenging because it has to take account of the mean-reverting nature of volatility. Furthermore, the issue is complicated by the fact that the VIX shows considerable persistence, which produces large and lasting deviations of the index from its long-run mean.

Recently, the most used models are the square-root model with the addition of jumps, analysed by Grumbichler and Longstaff (1996) and the log-normal Ornstein-Uhlenbeck\(^{3}\) (log-OU), considered by Detemple and Osakwe (2000). These approaches follow a traditional procedure that starts from the specification of a stochastic process for the volatility, the VIX, and estimate the parameters of the model by making use of historical prices. The implicit assumption made by these models is that the volatility reverts at a simple, non negative exponential rate, failing to capture the strong persistence occurring after volatility shocks. Such a limitation becomes particularly sizeable during bearish stock markets, in which the volatility typically exhibits abundant growth and remains at high levels for long periods.

Mencia and Sentana (2013) argue that an important feature of the VIX is the fact that it rapidly mean-reverts to a central tendency, which in turn reverts more slowly to a long-run constant mean. Thus, he generalised the log-OU model by considering several empirically relevant extensions: a central tendency in the mean, jumps, and stochastic volatility. While jumps have been yet considered in literature, the central tendency allows the average volatility level to be time-varying and stochastic volatility permits a changing dispersion of the log-VIX. Finally, he concludes that a generalised log-OU model that combines a time varying central tendency with stochastic volatility is needed to obtain a good pricing performance during both bull and bear markets.

A more innovative approach has been followed by Carr and Lee (2007),\(^{4}\)

\(^{3}\)In a log-normal Ornstein-Uhlenbeck process, the volatility \(v_t\) satisfies the following stochastic differential equation:

\[
dv_t = \theta(\mu - x_t) \, dt + \sigma \, dW_t.
\]
which derived the price of VIX options without knowing anything about the process of the underlying asset. The pricing model does not require any estimate since the inputs are recovered easily from the market. The model proposed is a Black and Scholes-type model, and utilizes associated variance and volatility swaps in order to find the VIX options prices.

VIX derivatives allow investors to assume a position on volatility, also if volatility itself is not investable. The main features of VIX futures and VIX options described in this Chapter suggest that a long volatility position may be exploited in order to protect traditional investments from the downturns in the market. In the next Chapter we will explore more in detail the relationship between the behavior of the markets and the VIX option’s prices, discovering in which way it is possible to mitigate efficiently the risk such traditional portfolios.
Chapter 5

Risk diversification with VIX call options: an empirical analysis

The main purpose of diversification is to reduce the risk of a portfolio by investing in different asset classes. Diversification opportunities are extremely various, as it is possible to invest in stocks, bonds, commodities, real estate, private equity and hedge funds. Correlation among these asset classes is usually very low, ensuring a limited variability in the overall portfolio since the single asset class movements are frequently offset by the opposite behavior in the other asset classes.

During the global financial crisis of 2008, the S&P500 faced a dramatic collapse, dropping almost 47%. Also the majority of other asset classes exhibited consistent losses and the usually low correlation between them grew sharply: asset classes once believed efficient diversifiers revealed their inability to provide protection against market crashes. Even hedge funds and commodities were not immune to decline, too.

Conversely, there has been one asset class which performed well during the financial crises: volatility. While many investments plummeted, the VIX index skyrocketed to a new high of 81 points since 1995 and the negative correlation that exists between the S&P500 and the VIX index has been exploited to obtain the needed diversification. The integration between worldwide financial markets, coupled with the extreme uncertainty in periods of crises, contributed to the diffusion of such alternative investment strategies based on volatility derivatives.

During turbulent periods in the markets, assuming a long position in volatility efficiently acts as a diversification tool since it reduces both the
losses and the volatility of traditional portfolios. A review of the literature reinforces this belief, as many studies\textsuperscript{1} have underlined the power of investing in spot VIX in order to provide significant diversification benefits to different portfolios. However, spot VIX is not directly investable and it is necessary to invest in derivatives based on the index.

Szado (2009) found that the addition of a portion of VIX options and VIX futures to traditional portfolios provided several benefits during the financial crisis of 2008, by simultaneously reducing the losses and, marginally, the volatility.

These observations suggest that the opposite relationship between the VIX and the other asset prices may be exploited efficiently for diversification purposes when financial markets are characterized by intense volatility, caused by the drop in the value of the S&P500. However, when markets rise, the relationship between the VIX and the S&P500 is more challenging to gauge. Furthermore, in periods when the market grows constantly and the VIX is calm, unexpected news may generate sudden spikes in the VIX, caused by the dramatic increasing of the price of underlying options.

Previous literature only focused on periods characterized by heavily bearish markets, failing to consider bullish ones. Given their considerable leverage and the sensitivity of the VIX index to the uncertainty in the markets, VIX derivatives may also lead to some undesirable effects.

The aim of this study is to assess the extent to which an investment in VIX options can impact the risk-return profile of traditional portfolios during bullish markets. To do so, a composite strategy is built by including to the base portfolio, invested in traditional and alternative asset classes, a percentage of at-the-money and out-of-the-money VIX calls. The research extends previous literature by considering not only the potential benefits of these strategies but also the drawbacks, occurring especially during calm and bullish markets. Given the complex nature of the relationship between the price of an asset and the market’s volatility, it is borderline to refer to this strategy as a pure hedge. Instead, we will define it as a diversification strategy, which exploits the negative correlation between the two to mitigate the risk of the overall portfolio.

Next Section will explain the main motivations behind this analysis. The base portfolios features and the respective asset classes are presented, along with a focus on the interaction between the VIX call options and the markets.

\textsuperscript{1}See, for example, Daigler and Rossi (2006), Dash and Moran (2005) and Black (2006).
Then, the methodology that has been carried out to perform this analysis will be explained in detail. Finally, we will discuss the obtained results and we compare the impact of the different strategies on the portfolio's risk-performance profile by performing a sensitivity analysis in a mean-variance framework.

5.1 Motivation

This work\(^2\) analyzes the benefits and the risks of investing a small portion of a portfolio in VIX options. It is carried out on the period 2015-2016, which has been characterized by a moderate growth in the markets and by a relatively calm volatility interrupted by violent spikes. As discussed in previous Chapters, the impact caused by upward and downward movements of the SPX index on the volatility are opposite: unpleasant economic issues impact rapidly making the VIX to surge, while positive news have the effect to calm down the VIX. Figure 5.1 displays the comparison between the behavior of the SPX and the VIX in the above mentioned period. Here, we witness four main volatility peaks, accompanied by four corresponding declines in the stock market. This confirms the very well-know negative correlation between the SPX and the VIX: as soon as the first starts to drop, the latter surges immediately.

The greatest volatility spike occurred in August 2015, caused by the interaction of two sudden news. First, China devalued the Yuan for two consecutive days in order to increase exportation: while Chinese and American indexes plummeted, the fear that Chinese economy was not as healthy as it was described grew among investors. Second, on the 24\(^{th}\) of August, an ETF Flash Crash hit the Dow Jones Industrial Average index, which opened as many as 1000 points below the previous close, a 6.6\% drop. Already mined by previous days sell offs, many stock remained without bid prices and consequently some ETFs were valued erroneously, causing the panic among investors. The same days, the VIX reached 53 points, experiencing a growth of 113\%, the largest weekly gain ever.

\(^2\)All the plots, the tables and, in general, the data presented in this analysis have been elaborated for the purpose of this work. The used data have been downloaded from Bloomberg.
By the end of 2015 and the begin of 2016, several events impacted the financial markets, causing the VIX to reach nervous values for almost three months. Starting from multiple terrorist attacks in France in November 2015, market’s volatility remained high while the FED hiked rates for the first since the financial crisis and the ECB disappointed investors amidst further economic stimulus. Simultaneously, oil price dropped to extremely low levels.

In 2016, the VIX reached its greatest peaks in correspondence of two controversial political events: the totally unexpected decision by the United Kingdom voters to leave the European Union, in June, and the election of Trump as President of the United States, in November.

During financial crises, VIX derivatives may be exploited to protect traditional portfolios from significant losses. Szado (2009) studied how the addition of a small quantity of VIX options and futures served as a diversifier for traditional portfolios during the financial crisis of 2008. His paper will represent the starting point of our analysis. Research suggested that, over the long term, a static long position in volatility is expected to result in negative excess returns. Hence, an appropriately rebalanced strategy that selectively invests in long volatility exposure may be effective for diversifica-
tion purposes. By implementing a strategy composed by a small investment in VIX futures and at-the-money and out-of-the-money VIX calls, Szado (2009) verified the impact that those alternative strategies had on the risk-return profile of a set of base portfolios, composed by equity, bonds, hedge funds and many others alternative assets. He found that both VIX futures and ATM VIX calls reduced the risk and cut the losses of the portfolio, while OTM VIX calls improved the returns by simultaneously increasing the risk. He thus concluded that an appropriately constructed long volatility position may result in an effective protection during turmoil in financial markets. These composite strategies suggested even a more efficient mean of diversification than “standard” protective strategies involving put options on the S&P500.

In order to understand the possible benefits that a long position in VIX options will provide to the risk-return profile of traditional portfolios, it is important to investigate the main features of these derivatives. The VIX index tends to anticipate the actual volatility by overreacting to financial news and, since the VIX options are based on the forward prices of the VIX, namely the projected price of the VIX, option prices are likely to be heavily influenced by the moves in the index. This happens especially when the news come suddenly and unexpectedly.

Moreover, the interaction between the movements in the VIX and the movements in the SPX is extremely variegated and dynamic. The correlation of the two indexes is, indeed, conditional and time-varying. Despite it is well known that volatility during financial crises moves in the exact opposite direction as the market, the relationship between the two in bullish periods is not crystal clear. Figure 5.2 and Figure 5.3 show the correlation in time and the conditional correlation. Even though the VIX is considered to move in the opposite direction of the SPX, sometimes their correlation may also be positive, signaling that volatility may grow considerably also during market’s expansion. Focusing on 2015-2016, their correlation has always been negative. However, the negative interaction results less pronounced while markets experience great contractions.
Figure 5.2: 30-day rolling correlation between the SPX and the VIX from 1st January 2015 to 31st December 2016.

Figure 5.3: Conditional correlation between the SPX and the VIX, given the rate of return of the SPX, from 1st January 2015 to 31st December 2016.
The examination of the conditional correlation denotes different behaviors of the volatility with respect to different range of returns in the SPX index, and the well-know negative correlation is detected. However, if we move to more extreme returns of the SPX, we note that the negative correlation increases (greater absolute value) for extreme negative returns of the SPX, while decreases (lower absolute value) if extreme positive returns in the stock index occur. We can thus conclude that: the VIX reaches almost perfect inverse correlation with the SPX during very large drops in the market, and the volatility raises more when the equity market is bearish than when it is bullish.

As already discussed in Section 4.2, VIX futures and VIX options share the same market’s expectation of what value the VIX index will have at maturity. Hence, it is important to understand the relationship that exists between the level of the VIX and the VIX futures price. Figure 5.4 shows the behavior of the spot VIX related to the VIX futures price in 2015-2016, while Figure 5.5 displays the 30-day rolling realized volatility of the two. From the analysis of the Figures, we can retrieve some considerations. First, the upward movements in the VIX are accompanied by the same movements also in the VIX futures but the latter reacts afterwards and the spikes are smaller. After a VIX’s hike occurs, the VIX futures stays longer at higher levels than spot VIX. The reason is that VIX futures is the projected price of the spot VIX and, after a spike, the forecasted value will be lower. In addition, VIX futures usually trade at a premium, which means that an higher price is requested from the seller of the futures to carry on the volatility risk. Second, the 30-day rolling realized volatility of the spot VIX is drastically greater than the volatility of the VIX futures. Due to the mean-reverting nature of the VIX, the forecasted value of the VIX is clearly less volatile because extreme values of the spot VIX will eventually revert to the mean. We also confirm the fact that volatility of both spot VIX and VIX futures is clearly above the volatility of the S&P500, in line with the higher implied volatility of VIX options. These issues will have an impact both on the price and the behavior of VIX options, since the volatility premium is translated into the options’ price, which will exhibit higher volatility if compared to “standard” equity or index options.
Figure 5.4: Behavior of the spot VIX and the VIX futures price in 2015-2016.

Figure 5.5: 30-day rolling historical volatility of the spot VIX and the VIX futures price in 2015-2016.
Diversification benefits arise when assets are uncorrelated but, during periods of market stress, the asset’s cross correlation surges. As a consequence, portfolios considered well diversified based on historical data were proven to be not diversified at all. As research highlights\(^3\), we should find a greater correlation between assets during financial crises than in relatively calm periods. The comparison between these correlations is displayed in Table 5.1, which exhibits the correlation of various financial assets during the crisis of 2008 and the period 2015-2016. The results we obtain are mixed: while some asset’s correlation increases drastically during bearish markets, others denote a greater correlation during bullish markets. Analyzing the volatility, represented by the VIX index, the correlation is negative for all assets, except for bonds. However, we cannot determine a clear relationship since the VIX in bullish markets tends to be more negatively (i.e. greater absolute value) correlated to equity, high yield bonds and hedge funds, with respect to what occurs in bearish markets. Conversely, the correlation between VIX and CTAs, private equity and commodities tends to decrease (i.e. lower absolute value).

Table 5.1: Cross-correlations between assets during 2015-2016 and August-December 2008 (in brackets).

<table>
<thead>
<tr>
<th></th>
<th>E</th>
<th>B</th>
<th>HYB</th>
<th>HF</th>
<th>CTA</th>
<th>RE</th>
<th>PE</th>
<th>C</th>
<th>VIX</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity</td>
<td>1</td>
<td>(-0.18)</td>
<td>(-0.17)</td>
<td>(0.62)</td>
<td>(0.12)</td>
<td>(0.83)</td>
<td>(0.82)</td>
<td>(0.42)</td>
<td>(-0.87)</td>
</tr>
<tr>
<td>Bonds</td>
<td>-0.29</td>
<td>1</td>
<td>(0.2)</td>
<td>(-0.05)</td>
<td>(0.53)</td>
<td>(-0.17)</td>
<td>(-0.07)</td>
<td>(-0.18)</td>
<td>(0.11)</td>
</tr>
<tr>
<td>High yield bonds</td>
<td>0.48</td>
<td>-0.12</td>
<td>1</td>
<td>(0.45)</td>
<td>(-0.02)</td>
<td>(-0.08)</td>
<td>(0.38)</td>
<td>(0.37)</td>
<td>(-0.20)</td>
</tr>
<tr>
<td>Hedge Funds</td>
<td>0.79</td>
<td>-0.27</td>
<td>0.65</td>
<td>1</td>
<td>(0.07)</td>
<td>(0.33)</td>
<td>(0.68)</td>
<td>(0.48)</td>
<td>(-0.67)</td>
</tr>
<tr>
<td>CTAs</td>
<td>-0.08</td>
<td>0.43</td>
<td>-0.19</td>
<td>0.03</td>
<td>1</td>
<td>(0.46)</td>
<td>(0.02)</td>
<td>(0.21)</td>
<td>(-0.18)</td>
</tr>
<tr>
<td>Real estate</td>
<td>0.63</td>
<td>0.13</td>
<td>0.30</td>
<td>0.46</td>
<td>0.11</td>
<td>1</td>
<td>(0.47)</td>
<td>(0.22)</td>
<td>(-0.65)</td>
</tr>
<tr>
<td>Private equity</td>
<td>0.75</td>
<td>-0.30</td>
<td>0.70</td>
<td>0.81</td>
<td>-0.24</td>
<td>0.44</td>
<td>1</td>
<td>(0.69)</td>
<td>(-0.75)</td>
</tr>
<tr>
<td>Commodities</td>
<td>0.37</td>
<td>-0.28</td>
<td>0.35</td>
<td>0.37</td>
<td>-0.45</td>
<td>0.10</td>
<td>0.45</td>
<td>1</td>
<td>(-0.44)</td>
</tr>
<tr>
<td>VIX</td>
<td>-0.84</td>
<td>0.25</td>
<td>-0.39</td>
<td>-0.74</td>
<td>-0.03</td>
<td>-0.54</td>
<td>-0.64</td>
<td>-0.31</td>
<td>1</td>
</tr>
</tbody>
</table>

The difficulty to determine a clear behavior between the volatility and “standard” asset classes during periods of moderate growth suggests that the diversification benefit of VIX derivatives may be ineffective, or at least moderate, during these periods.

\(^3\)See, for example, Chow et al. (1999)
Great care should be applied to the interpretation of the results of this analysis. If the negative impact prevails, that impact is likely to be also more pronounced given the considerable leverage VIX derivatives include. One further point must be stressed: these instruments, and in particular the VIX options analyzed in our work, are not considered as a long equity hedge, but as a diversifying tool. As already discussed, the relationship between the VIX and other asset classes is extremely challenging to define, thus the use of strict rules for hedging a position with VIX derivatives may not be useful.

5.2 Data

The data utilized by this analysis comprehends a range of two-years daily prices\(^4\), that goes from the beginning of 2015 to late 2016. For sake of convenience, the strategy starts on the 17th of December 2014 and ends on 21st of December 2016, since the monthly VIX options expire on those dates.

In order to investigate the effect of the introduction of VIX options into traditional allocation strategies, we employ two traditional portfolios as well as a portfolio composed by alternative assets:

- Base portfolio 1: 100% stocks;
- Base portfolio 2: 60% stocks and 40% bonds;
- Base portfolio 3: 60.5% stocks, 30.5% bonds, 1.3% high yield bonds, 1.2% hedge funds, 0.1% managed futures, 0.3% commodities, 1.6% private equity, 4.5% real estate\(^5\).

In Table 5.2 are summarized the indexes used in our analysis as proxies for different investments\(^6\).

\(^4\) Daily indexes closing prices and daily options prices downloaded from Bloomberg.

\(^5\) Asset allocation based on average allocation of Pensions and Investments US Institutional Tax-Exempt Assets.

\(^6\) The majority of the considered indexes are Total Return indexes, in order to account for the overall investment, since those incorporate also the dividends. Furthermore, all indexes are quoted in USD.
Table 5.2: Range of possible investable assets, and respective indexes.

<table>
<thead>
<tr>
<th>Asset</th>
<th>Index</th>
<th>Bloomberg ticker</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity</td>
<td>S&amp;P500 Total Return Index</td>
<td>SPXT Index</td>
</tr>
<tr>
<td>Bonds</td>
<td>Barclays Capital U.S. Aggregate Total Return Index</td>
<td>LBUSTRUU Index</td>
</tr>
<tr>
<td>High yield bonds</td>
<td>Barclays Capital U.S. High Yield total return index</td>
<td>LF98STRUU Index</td>
</tr>
<tr>
<td>Hedge funds</td>
<td>HFRX Global Hedge Fund Index</td>
<td>HFRXG1 Index</td>
</tr>
<tr>
<td>Managed futures</td>
<td>SG CTA [formerly Newedge CTA] Index</td>
<td>NEIXCTA Index</td>
</tr>
<tr>
<td>Real estate</td>
<td>First Trust U.S. REIT Total Return Index Fund</td>
<td>FRI US Equity</td>
</tr>
<tr>
<td>Private equity</td>
<td>S&amp;P Listed Private Equity Index</td>
<td>SPLPEQTY Index</td>
</tr>
<tr>
<td>Commodities</td>
<td>S&amp;P GSCI Total Return Index</td>
<td>SPGSCTIR Index</td>
</tr>
</tbody>
</table>

In Table 5.3 the features of the different asset classes for the considered period are summarized. Returns and volatility among different asset classes are extremely variable, which means that an effective diversification effect could eventually be obtained by investing in these asset classes. Except from commodities, which have been heavily hit by the tremendous drop in oil price, all the other assets showed positive returns during 2015 and 2016. Equity and high yield bonds displayed the best risk-return profile, highlighted by a largely positive Sharpe ratio, while commodities and private equity have experienced the largest drawdowns.

Table 5.3: Descriptive statistics of asset classes in 2015-2016.

<table>
<thead>
<tr>
<th>Asset</th>
<th>Equity</th>
<th>Bonds</th>
<th>High yield bonds</th>
<th>Hedge funds</th>
<th>CTA</th>
<th>Real estate</th>
<th>Private equity</th>
<th>Commodities</th>
</tr>
</thead>
<tbody>
<tr>
<td>skewness</td>
<td>9.25%</td>
<td>6.46%</td>
<td>4.04%</td>
<td>2.19%</td>
<td>5.22%</td>
<td>2.19%</td>
<td>5.22%</td>
<td>2.19%</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>4.48%</td>
<td>2.94%</td>
<td>6.00%</td>
<td>5.43%</td>
<td>5.38</td>
<td>2.70%</td>
<td>6.08%</td>
<td>1.07%</td>
</tr>
<tr>
<td>Maximum W drawdown</td>
<td>2.96%</td>
<td>4.28%</td>
<td>3.15%</td>
<td>5.12%</td>
<td>5.82%</td>
<td>5.20%</td>
<td>5.82%</td>
<td>5.20%</td>
</tr>
<tr>
<td>% Up days</td>
<td>51.25%</td>
<td>51.25%</td>
<td>52.75%</td>
<td>50.00%</td>
<td>55.00%</td>
<td>54.25%</td>
<td>55.00%</td>
<td>54.25%</td>
</tr>
<tr>
<td>% Down days</td>
<td>48.75%</td>
<td>48.75%</td>
<td>48.75%</td>
<td>45.83%</td>
<td>45.83%</td>
<td>45.83%</td>
<td>45.83%</td>
<td>45.83%</td>
</tr>
<tr>
<td>Annualized Sharpe ratio</td>
<td>0.25%</td>
<td>0.25%</td>
<td>0.25%</td>
<td>0.25%</td>
<td>0.25%</td>
<td>0.25%</td>
<td>0.25%</td>
<td>0.25%</td>
</tr>
</tbody>
</table>

Furthermore, we can observe that all assets, except commodities, are characterized by negative skewness and high kurtosis. The skewness of a returns distribution determines its asymmetry, while the kurtosis is related to the height of the peak and the fatness of the tails. On the one hand,
a distribution characterized by a negative skew indicates that the tail on the left side of the probability density function is longer or fatter than the right side. Instead, a positive skew suggests that the tail on the right side is longer or fatter than the left side one. On the other hand, if the value of the kurtosis is greater than 3, the distribution will exhibit fatter tails with respect to the normal distribution. Usually investors prefer positive skewness and low kurtosis: this means that they prefer investments characterized by a moderate probability of extreme returns and returns more concentrated on the right side of the distribution. In particular, high yield bonds and private equity display large negative skewness and high excess kurtosis, thus the probability of observing large negative returns is substantial.

![Portfolio growth](image)

**Figure 5.6**: $100 portfolio growth of the three base portfolios.

The overall growth, scaled to a normalizing factor of $100, of the three base portfolios is reported graphically in Figure 5.6. We can observe that the 100% equity portfolio (BP1) has the greatest performance, but exhibits a considerable variability, especially during the drops in the stock market. The diversification benefit of BP2 and BP3, which have been obtained by adding bonds and alternative investments to stocks, is quite similar. On the one hand, we observe that both offer protection during downturns in the
market. On the other side, they reduce significantly the overall return of the portfolio. However, while the returns are almost equal, the BP2, composed by 60% equity and 40% bonds, is more efficient by means of diversification than BP3, since the drawdowns during market shocks are less pronounced while maintaining a similar performance.

5.3 Methodology

The methodology that has been carried out in this work utilizes one-month to expiration VIX call options with different degrees of moneyness. These are “standard” VIX options, expiring on the 3rd Friday of each month, that have been chosen among other weekly maturities, which expire on the other Fridays of the month, for their greater liquidity. At-the-money options as well as 10% and 20% out-of-the-money options are included in the base portfolio in the measure of 0.5% and 1%, reshaping the original weights of the other components accordingly to the original proportions. The chosen weights are completely arbitrary since, as we discussed earlier, determining a precise composition is a challenging task.

Initially, we started considering options weights in the proportion of 1% and 3%, accordingly to Szado (2009). After realizing such investments in VIX calls led to extreme results, we decided to diminish the weight of the portfolio allocated in options. This does not change the results of this analysis, since greater weights have the effect to amplify the effects observed from smaller option’s weights. Even though we will provide a sensitivity analysis of the composite portfolios to mean and variance at the end of this Chapter, the purpose of this work has never been to provide a perfect and strict allocation proportion of VIX options, given the variegate nature of the correlation between these derivatives and traditional asset classes.

To recreate the composite strategy, the one-month expiration options are rolled monthly into the new options with proper strike price, by rolling out the options at the day previous to expiration and rolling in the new option at the expiration date of the previous option. The overall portfolio is rebalanced monthly, in correspondence of the expiration date, to reflect

\footnote{Initially, we investigated the possibility to recreate a synthetic index investing in the S&P 500 and in one VIX call option, in order to resemble the methodology used by strategy indexes such as the CBOE Buy Write Index, which is made of a long position in the S&P 500 portfolio and a short position in one index call option. Since the VIX options are based on...}
the initial weights\textsuperscript{8}. For sake of consistency, also the base portfolios are rebalanced monthly to the target weights. Then, the daily returns of the various composite portfolios are computed. To capture transaction costs, the VIX calls are rolled at the ask price, and rolled out at the expiration date at the intrinsic value. The returns in the days between the purchase date and the maturity are calculated using the mid point price (i.e. the average of the bid and the ask price).

The results are then compared by means of traditional risk-performance measures as well as more developed measures, accounting for the investor's risk preferences. In particular, we first compute the first four moments of the returns distribution: the annualized daily return and standard deviation, the skewness and the kurtosis. In addition, we calculate the maximum drawdown\textsuperscript{9}, and the percentage of days in which the strategy increases or decreases the value of the overall portfolio. Then, we compare the different strategies by making use of more developed risk-adjusted indicators considering for kurtosis and skewness, such as the Sortino ratio and the Leland's alpha and beta.

In the results we also report the Sharpe's ratio for convention, however this indicator can be easily manipulated and can generate misleading interpretations, especially when dealing with portfolios containing options. As a matter of fact, investors tend to prefer positive skewness and low values of kurtosis. As long as the returns distribution is not normal, the Sharpe ratio becomes useless since it accounts only for the first two moments of the distribution, the mean and the standard deviation. Given the importance that the third and the fourth moment of a distribution have on the risk profile of a strategy, we also compare the strategies based on indicators that are robust for non-normality. These indicators are the Sortino ratio and the Leland's alpha and beta, which are suitable for hedge funds and pension funds. More in general, they are useful for all the strategies investing in derivatives.

\textsuperscript{8}A different underlying asset (i.e. the volatility), the difficulty to find effective proportions made us think to switch to rebalancing strategies, which reflect also in a better way the cost of implementing option's strategies.

\textsuperscript{9}VIX options are settled based on a special opening quotation of the VIX on the day of expiration. Following Szado (2009), in our analysis the options are rolled out at the close of the day before the maturity date. This may introduce a weak inaccuracy if the close of previous differs consistently with the special quotation. Nevertheless, it should not produce a systematic error.

\textsuperscript{9}The maximum drawdown is the maximum percentage loss from the peak to bottom, before a new peak is attained.
Sortino (1994) aimed to correct the Sharpe ratio by differentiating the harmful volatility from the total overall volatility, which is obtained by using the downside deviation (i.e., the standard deviation of only the negative returns of an asset) instead of the classic standard deviation. It is curious to highlight that even Markowitz himself, who created the Modern Portfolio Theory (1959) in a mean-variance framework, recognized that using downside deviation to measure risk would be more appropriate than using the standard deviation. The Sortino ratio $I_S$ is defined as:

$$I_S = \frac{\bar{r}_p - r_f}{\sigma_D},$$

where $\bar{r}_p$ is the mean return of the portfolio, $r_f$ is the risk-free rate\(^\text{10}\) and $\sigma_D$ is the downside deviation (or downside risk), which is calculated from:

$$\sigma_D = \sqrt{\frac{1}{T} \sum_{t=1}^{T} \left( Min(0, r_t - r_f) \right)^2},$$

where $r_t$ is the portfolio return at time $t$, in a time window of length $T$.

The Sortino ratio enhances the Sharpe ratio by accounting for investor’s preference of positive volatility (i.e., standard deviation of the positive returns). In fact, it accounts for the returns distribution’s shape via the downside deviation and, in particular, it improves the valuation of strategies producing a skewed returns distribution. For example, given the same rate of return, positively skewed strategies will achieve that return with less risk than negatively skewed ones, thus the latter will be characterized by a lower Sortino ratio and the former is preferred.

Leland (1999) argued that portfolios displaying low kurtosis and positive skewness are incorrectly underrated. The risk of owning such position results in a decreased downside risk, which is not identified by traditional measures based on the Capital Asset Pricing Model. It is, in fact, likely that this kind of portfolios generate non-linear exposures to standard asset classes. Leland’s measures are then indicators that aim to correct the CAPM by allowing non-normality in the returns. They are defined by means of the standard CAPM, as:

$$E(r_p) = r_f + \beta_L(\bar{r}_m - r_f),$$

\(^{10}\)Without loss of generality the risk-free rate is set equal to zero, both in the calculation of the Sortino ratio and the Leland’s alpha and beta.
where $E(r_p)$ is the expected return of the portfolio given this relationship and $r_m$ is the average return of the market.

The difference with respect of the CAPM is determined by the Leland’s Beta, which is defined as:

$$\beta_L = \frac{Cov[r_p, -(1 + r_m)^{-b}]}{Cov[r_m, -(1 + r_m)^{-b}]} \quad (5.4)$$

with the parameter $b$ equal to:

$$b = \frac{ln(1 + r_m) - ln(1 + r_f)}{Var[ln(1 + r_m)]}. \quad (5.5)$$

The Leland’s alpha is then obtained, in a manner equal to the CAPM, as the excess return of the portfolio, as:

$$\alpha_L = r_p - \beta_L(r_m - r_f) - r_f. \quad (5.6)$$

where $r_p$ is the actual average return of the portfolio and $r_m$ is the average return of the market.

The BP1 is totally invested in stocks, thus it represents also the market portfolio. It is characterized by a Leland’s alpha of 0 and a beta of 1. A positive beta implies that the portfolio will follow the market: if the market moves upwards, the portfolio will likely follow. Conversely, a negative beta means that the portfolio will behave in the opposite way of the market: if the market goes up (down), the portfolio will move downwards (upwards). When the portfolio reacts more than the market, the beta of that portfolio will be greater than 1.

A Leland’s alpha which is greater than 0 means that the portfolio leads to superior returns with respect to the market, while a negative alpha suggest an underperformance. The overall effect of a positive Leland’s alpha with a Leland’s beta means that the portfolio will overperform the market based on its risk adjusted for the positive skewness and low kurtosis.

In our work, the indexes are calculated using the S&P500 as a proxy for the market benchmark. Higher values of Leland’s beta signal greater levels of risk if compared with low values of beta. Strategies containing VIX options are likely to present lower values of beta since the VIX index has a negative correlation with the market, which has a beta of 1. The analysis must be
coupled with the Leland’s alpha, which is often used as a measure of the ability of active portfolio managers to outperform the market. Instead, the Sortino ratio represents the strategy’s risk-adjusted performance, penalizing the rankings given by the Sharpe ratio for a negative skewness.

5.4 Results

5.4.1 ATM VIX call options

This section considers the investment in VIX call options besides the base portfolios. The three base portfolios (BP1, BP2 and BP3) are combined with the allocation of 0.5% and 1% of at-the-money VIX calls expiring one month later, and the weights of the other assets are consequently scaled down to the original composition: the remaining part of the portfolio (99.5% and 99%) is thus divided between the portfolio’s original assets accordingly to the initial proportions. Portfolios with a greater allocation of VIX calls have been also analyzed but they amplify the effect provided by an investment of 1% in options, since the leverage of VIX calls is sufficient to determine a valuable effect also by investing such small quantities.

Table 5.4 displays the summary statistics for the strategies including ATM VIX calls in the considered period.

Table 5.4: Summary statistics of the base portfolios alone and base portfolios with the addition of 0.5% and 1% ATM VIX calls, between 17th December 2014 and 21st December 2016.
Although in minor percentages, the addition of an investment in ATM VIX call options to the base portfolios provided astonishing results. The inclusion of such derivatives provided an overall diversification effect since the standard deviation has been reduced in all the strategies. There seems to be the an inverse relation between the percentage of ATM VIX calls in BP1, which is totally invested in stocks, and the volatility. BP1’s volatility reduces from 14.40% annually, to 11.99% and 11.13%. This effect is not true for both BP2, formed by 60% stocks and 40% bonds, and BP3, invested in a variety of traditional and alternative asset classes. In fact, if we just add 0.5% of options to the portfolios, the volatility decreases, but eventually increases after we added another 0.5% of VIX calls.

The overall contraction in portfolio's risk comes obviously at a cost. For all the portfolios, further investments of 0.5% in options seems to cut the returns every time halfway, reaching negative returns for both BP2 and BP3, from 5.48% and 5.52% annually to -0.77% and -0.72%, respectively.

The influence on the skewness and the kurtosis of the base portfolios is clear: when we add a small quantity of ATM VIX calls to the original investment we obtain a consistent reduction of both negative skewness and kurtosis. However, if we include 1% of options, both the negative skewness and the kurtosis enlarge dramatically. This effect is particularly evident when we add 1% of VIX ATM calls to the portfolio formed by stocks and bonds, which is characterized initially by a skewness of -0.20 and a kurtosis of 4.32. The two measures exceptionally surge to -0.57 and 25.47, thus evidencing a very large probability of extreme outcomes concentrated in the left tail, the negative part of the return’s distribution.

The maximum drawdown gets larger as we add further fractions of VIX call options, and so does the percentage of down days (i.e. the days of the year in which we observe negative returns).

From the interpretation of the Sharpe index, we can deduce the overall effect the inclusion of ATM VIX calls have on the risk-return profile of the base portfolios. Nevertheless, as already discussed, the Sharpe ratio is a biased indicator under non-normality of returns. Since these strategies are likely to present sizeable skewness and kurtosis, the Sharpe index would not account for the investor’s risk preferences, which tend to prefer positive skewness and low kurtosis. Furthermore, negative values of the Sharpe ratio provide useless information. All the option strategies deteriorate the Sharpe index of the original portfolio. A less dramatic contribution is done by the addition of 0.5% of VIX options to the 100% of stocks, which is initially 0.56
and passes to 0.41. Conversely, both BP2 and BP3 enhance the risk-return profile of the portfolio.

In order to account for the strong departure from the normality of the returns distribution of these composite portfolios, we turn our attention on measures that are robust under non-normality, such as the Leland’s alpha, the Leland’s beta and the Sortino ratio. The Sortino ratio exhibit a pattern which is identical to the Sharpe ratio, thus the departure from the original skewness do not generate a significative enhancement in the overall risk-return profile of the strategies, such as to overcome the smaller gains. It is interesting to see that the change from 0.5% to 1% allocation of ATM VIX calls to the BP1 translates in a worsened skewness but in a lower downside deviation.

Leland’s beta is similar to the CAPM’s beta and represent a measure of the risk of the portfolio compared to the risk of the market, but also penalizes negative skewness and excess kurtosis. The Leland’s beta of the market is 1, in fact the BP1 composed only by stocks has a value of 1. All the portfolios cause the Leland’s beta to decrease: by increasing the VIX options’ weights in the base portfolios we observe a correspondent decline in the overall risk. We would then conclude that the addition of ATM VIX options reduces efficiently the total risk of the base portfolios, however we must take into account also the Leland’s alpha in order to give a more useful interpretation. Identical to the CAPM’s alpha, the Leland’s alpha is a measure of the overperformance of the market. Given the excess performance of the market to be, clearly, zero, the addition of VIX call options induces negative values of the index, highlighting a general underperformance in the range between -2.89% and -0.54%, on annual basis. This means that the strategies containing ATM VIX calls are likely to produce negative overperformance, also by considering the benefits of greater skewness and reduced kurtosis. The importance of the Leland’s alpha can be derived from the fact that, even though their performance compared to that of the market is smaller, BP2 and BP3 both have a value of alpha which is greater than zero. Given the slight decrease in kurtosis and negative skewness provided by these two portfolios, investors will prefer it against the equity portfolio, since they lead to an “adjusted” overperformance of 0.94% and 0.46%, annually.
To gauge intuitively the diversification power of VIX options, we plotted in Figure 5.7 the growth of a $100 portfolio during the two-year period 2015-2016, considering the BP1 and the same portfolio with the inclusion of 0.5% and 1% ATM VIX calls. Even though we did not take it into account in the previous analysis, we report also the portfolio composed by 98% of stocks and 2% of VIX options.

From the plot, we are immediately able to detect the benefits of an investment in VIX options besides an investment in traditional assets. As a matter of fact, the options are more valuable during periods of high volatility, especially negative volatility, occurring when a sudden drop in the market happens. The evidence can be traced back in particular in the peak of August 2008 of the BP1+2% of options, represented by the black dashed line, and in the other portfolios’ lines too. Indeed, the increasing price of the VIX options during periods characterized by high volatility allows to offset the decline in equity’s value and the leverage of these options make the height of the peak to be exponentially higher while increasing the percentage of the portfolio invested in options. Instead, we can observe that the same behavior has not happened during the market’s depression occurred between the end
of 2015 and the first months of 2016. More specifically, the drops in the
market are barely offset by the upper movement in option’s value.

From these considerations, we could draw some conclusions. First, since
the market’s turmoil of August 2015 has been caused by the combination
of two extremely unexpected events, such as the devaluation of the Yuan
and the Dow Jones’ ETF crash, the investor’s were not aware of the future
possible volatility and priced options accordingly to this perceived sentiment.
The rebalancing strategy has indeed purchased a sizeable quantity of options
since they were “underpriced”\textsuperscript{11}. Two important aspects have to be kept
firmly in mind: VIX options’ quotation is based on the future’s price of the
VIX, which is the aggregate investor’s predicted value of the index, and VIX
option’s price increases when the volatility of the VIX future’s price increases.
If the market is aware that the volatility is likely to rise in the future, it
will price options accordingly, thus leading to poor gains the investors who
had, correctly, forecasted the rise in volatility. However, if the market’s
expectation is that the volatility will stay at the same level, or increase a
little, the outcome of the investors who had, again correctly, foresaw the
volatility hike would be larger, if one had the possibility to sell the options
on the peak.

In contrast, when market’s turmoil can be anticipated easily, the VIX’s
volatility will likely stay calm. As a consequence, the probability that the
price of VIX options will exhibit huge variability is lower. The confirmation
of this statement might be retraced by the market drops starting at the
begin of 2016 and continued along the same year. The volatility caused by
events such as central bank’s economic decisions and political elections is
usually discounted earlier by the market. Of course, the Brexit was totally
unexpected but, in some sense, the volatility that one decision or another
might have generated was efficiently predicted ahead of time by investors
and market-makers.

In conclusion, we argue that, the more unexpected the market’s crashes
are, the more likely is that the options volatility, and consequently their
price, will show an appreciable variability. Furthermore, the addition of
ATM VIX call options to traditional portfolios during bullish markets has
provided little diversification benefits. In fact, by allocating just 0.5% of

\textsuperscript{11}Actually, the options were “underpriced” \textit{a posteriori}, given the extremely high realized
volatility observed later in the market. We are not making a statement about the errors
in option’s pricing.
the base portfolios in ATM VIX calls, the volatility has been cut by only almost 1/5 but dividing also the returns by more than 2/5. The analysis of the indicators that account for the investor’s returns preferences led us to the same conclusions. The reduction of the losses has been evident only when the market was seriously hit by extreme volatility upper spikes. If one could possibly sell the options deeply in-the-money and realize the gains, the convenience would have been even greater, nevertheless, a strategy that systematically purchases one-month VIX option and maintain them until expiration can only partially benefit from these market’s movements.

5.4.2 10% OTM VIX call options

In this Section, we consider the cases in which we add to our base portfolios the 0.5% and 1% of 10% OTM VIX call options. The VIX options with a strike price that is 10% less than the current value of the VIX are purchased at the beginning of the period, and substituted at every new expiration with the new options, characterized by the proper 10% OTM strike price and the correct weight, accordingly to the value reached by the overall portfolio at the day before maturity.

Table 5.5 exhibits the summary statistics for the strategies including 10% OTM VIX calls in the considered period.

Table 5.5: Summary statistics of the base portfolios alone and base portfolios with the 0.5% and 1% addition of 10% OTM VIX calls, between 17th December 2014 and 21st December 2016.
As we could have imagined, the results provided by the addition of 10% OTM calls are more extreme if compared to ATM calls obtained in Section 5.4.1, since they are characterized by different payoffs and different levels of volatility.

The first thing we note is that the returns are cut faster. Previously, the 1% addition of ATM calls to the BP1 has not led to negative returns (1.79%), while now it does (-1.62%). Furthermore, the returns of BP2 and BP3 became largely negative (-0.77% and -0.72% against -4.16% and -4.12%, respectively) with the inclusion of just 1% of 10% OTM calls.

Conversely, the effect that 10% OTM calls have on the risk of the portfolios is similar. If we compare the volatility of the base portfolios with the addition of 1% of ATM and 10% OTM calls, we found that the values are just slightly greater (11.13%, 7.56% and 7.93% versus 11.25%, 7.89% and 8.22%, respectively).

In this period of moderate growth in the markets, small quantities, such as 0.5%, of VIX options contribute to reduce the negative skew of the returns distribution. However, if we just increase the weight to 1% we observe a dramatically significant negative skewness. A similar pattern can be recognized also for the kurtosis: while the smaller addition leads to thinner tails, a bigger allocation proportion translates in a considerable enlargement of the tail's fatness. The value of the skewness reaches -0.89 in the portfolio composed by stocks and bonds (BP2) with 1% of VIX calls and the kurtosis of the strategy including alternative assets (BP3) and 1% VIX calls gets close to 30. This means that a small inclusion of OTM VIX call options is likely to enhance the portfolio in terms of less negative skewness and diminished kurtosis. As a consequence, the probability of observing large negative returns is reduced. However, further portions of the portfolio invested in VIX OTM calls expand enormously the probability of those adverse returns.

In line with the previous results we find that the strategies containing options increase the maximum drawdown and the percentage of down days. The combined effect on the risk and the performance traduces in much reduced values of both the Sharpe ratio and the Sortino ratio for all the composite strategies. Also the diminished Leland’s beta and the largely negative values of the Leland’s alpha testify that the effect these derivatives have on traditional portfolios during bullish markets is unfavorable: they do decrease the overall risk of the portfolio but they cut the losses heavily as well.

Figure 5.8 displays the $100 portfolio growth of the BP1 combined with the same portfolio containing 0.5% and 1% of 10% OTM VIX calls. The
behavior we witness is similar to what observed above with ATM options. By turning our attention to the BP1 plus 1% of VIX calls (green line) and by considering the drop in the index occurred in the latest days of August 2015, we can deduce some considerations. The downward movement in the stock market is formed by two subsequent drops in the S&P 500 (blue line), however while in the first of the two drops the options provided a superior benefit to the strategy by enhancing the recovery, in the second they did not. The options have been, in fact, appreciated initially due to the growing volatility. After that the 10% OTM call options ended up deeply in-the-money, the strategy has been necessarily rebalanced because of the monthly expiration. Thus, the new options’ price has been determined using the new level of the forecasted volatility, namely the price of VIX futures, causing the subsequent options’ gains to diminish.

![Portfolio growth](image.png)

Figure 5.8: 100$ portfolio growth of BP1 with the addition of 0.5% and 1% of 10% OTM VIX call options.

In conclusion, we state that the addition of a small quantity of 10% OTM option calls provided less diversification benefit than the ATM VIX calls, since the OTM calls cut the returns much more and decreased the overall volatility in the same manner ATM calls do. Indeed, in the portfolio with
0.5% allocation in OTM VIX call options, the drop in returns has been in the order of 3/5 or more while the respective drop in volatility has been just around 1/5.

5.4.3 20% OTM VIX call options

The inclusion of VIX call options which are 20% out-of-the-money is considered in this Section. Not surprisingly, the results we obtain with 20% OTM VIX calls are worse than the ones obtained with 10% OTM VIX calls, since deeper OTM options are accompanied by an even greater degree of leverage.

The results we obtained adding 20% OTM VIX calls in the considered period are reported in Table 5.6.

Table 5.6: Summary statistics of the base portfolios alone and base portfolios with the 0.5% and 1% addition of 20% OTM VIX calls, between 17th December 2014 and 21st December 2016.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>BP1</th>
<th>BP1 + 0.5%</th>
<th>BP1 + 1%</th>
<th>BP2</th>
<th>BP2 + 0.5%</th>
<th>BP2 + 1%</th>
<th>BP3</th>
<th>BP3 + 0.5%</th>
<th>BP3 + 1%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annual daily return (%)</td>
<td>0.90%</td>
<td>2.06%</td>
<td>2.20%</td>
<td>3.40%</td>
<td>4.24%</td>
<td>5.22%</td>
<td>9.43%</td>
<td>9.27%</td>
<td>9.26%</td>
</tr>
<tr>
<td>Annual standard dev. (%)</td>
<td>33.40%</td>
<td>31.07%</td>
<td>28.91%</td>
<td>30.09%</td>
<td>32.53%</td>
<td>30.28%</td>
<td>43.54%</td>
<td>37.25%</td>
<td>36.07%</td>
</tr>
<tr>
<td>Win %</td>
<td>52.36%</td>
<td>60.84%</td>
<td>60.23%</td>
<td>51.18%</td>
<td>49.83%</td>
<td>51.20%</td>
<td>51.43%</td>
<td>51.36%</td>
<td>51.24%</td>
</tr>
<tr>
<td>% Up days</td>
<td>51.28%</td>
<td>46.40%</td>
<td>47.20%</td>
<td>51.28%</td>
<td>49.20%</td>
<td>51.43%</td>
<td>51.43%</td>
<td>51.36%</td>
<td>51.24%</td>
</tr>
<tr>
<td>% Down days</td>
<td>48.61%</td>
<td>53.16%</td>
<td>52.77%</td>
<td>48.72%</td>
<td>50.75%</td>
<td>48.59%</td>
<td>48.59%</td>
<td>48.64%</td>
<td>53.06%</td>
</tr>
<tr>
<td>Annualized Sharpe ratio</td>
<td>0.56%</td>
<td>0.60%</td>
<td>0.60%</td>
<td>0.60%</td>
<td>0.60%</td>
<td>0.60%</td>
<td>0.60%</td>
<td>0.60%</td>
<td>0.60%</td>
</tr>
<tr>
<td>Annualized Sortino ratio</td>
<td>0.23%</td>
<td>0.22%</td>
<td>0.22%</td>
<td>0.22%</td>
<td>0.22%</td>
<td>0.22%</td>
<td>0.22%</td>
<td>0.22%</td>
<td>0.22%</td>
</tr>
<tr>
<td>Leland’s Beta</td>
<td>1.00%</td>
<td>0.77%</td>
<td>0.77%</td>
<td>0.77%</td>
<td>0.77%</td>
<td>0.77%</td>
<td>0.77%</td>
<td>0.77%</td>
<td>0.77%</td>
</tr>
<tr>
<td>Annual Alpha</td>
<td>0.00%</td>
<td>-2.28%</td>
<td>-2.28%</td>
<td>-2.28%</td>
<td>-2.28%</td>
<td>-2.28%</td>
<td>-2.28%</td>
<td>-2.28%</td>
<td>-2.28%</td>
</tr>
</tbody>
</table>

The risk-return profile that the strategies involving 20% OTM calls provide is deteriorated: while maintaining a similar volatility with respect to 10% OTM calls, the returns are reduced more consistently. The inclusion of 20% OTM VIX call options cause the returns of the base portfolios to drop by as much as 10% annually. The effect on the shape of the return’s distribution exhibit the same pattern for the Sharpe ratio and the Sortino ratio as the 10% OTM calls, and the same happens by considering the Leland’s alpha and beta: even though in all the cases the inclusion of options led to
a lower Leland’s beta, the result is obtained through a drastically worsened Leland’s alpha.

We do not show the graphical behavior of the strategies since they are very similar to those obtained with 10% OTM VIX calls in Figure 5.8, except for the fact that they cause greater losses. The cost of implementing the strategy is sizeable and the probability that such deeply out-of-the-money options would end up in-the-money is limited. This translates in a diminished appreciation of the options’ value during the drops in the market.

Finally, we can conclude by stating that the diversification power of VIX call options during bullish markets is lower as we move to out-of-the-money strikes. Indeed, we found that the reduction in the volatility obtained by investing a portion of the base portfolio in VIX calls is very similar between the ATM, 10% OTM and 20% OTM VIX call options. Nonetheless, deeply out-of-the-money calls cut the returns more drastically than at-the-money VIX calls and simultaneously provide less upside potential during volatility hikes.

5.4.4 Sensitivity analysis

Given the previous results, neither of the strategies including VIX call options provided the enhancement of the risk-return profile the portfolios. The addition of VIX calls do indeed decrease the overall risk but it simultaneously make the returns to drop excessively. The achievement of a fully efficient diversification effect would have lead to a reduction in the volatility which is greater than the correspondent reduction in the returns. In other words, the benefits must be greater than the cost of implementing the strategy. In order to determine the combined effect the analyzed VIX call options have on the risk-return profile of the base portfolios, we report a sensitivity analysis to the mean and the standard deviation with respect to different levels of allocation in VIX calls.

In order to do this, we plot the efficient frontier VIX calls by moneyness in a mean-standard deviation framework\textsuperscript{12}. More specifically, we investigate the changes in the annualized daily returns and standard deviations by varying

\textsuperscript{12}In order to account for the adverse effect negative skewness have on the risk profile of the portfolio, we also performed a sensitivity analysis based on a mean-downside deviation criteria. Despite these strategies impact heavily on the returns distribution’s skewness, the results were almost identical and hence we do not show the efficient frontier plot. In this case, this means that the skewness have a small effect on the overall risk.
the allocated portion of the portfolio in VIX calls. The efficient frontier describes the portfolio that minimizes the volatility, for each level of return. The two assets in which the portfolio invests are stocks (BP1) and the three considered types of VIX calls: ATM, 10% OTM and 20% OTM. This allows us to quantify graphically and quantitatively the features of VIX calls with different degrees of moneyness. Note that we are not generating a portfolio optimization, instead what we want to do is to gauge the magnitude of the impact these options have on the risk-return profile of traditional portfolios.

Figure 5.9 exhibits the efficient frontier of the BP1 coupled with the allocation in ATM, 10% OTM and 20% OTM VIX call options. We considered VIX option weights up to 2% of the portfolio, hence the upper end of each line is invested 100% in equity and the lower end is divided between 2% in VIX calls and 98% in equity.

![Efficient frontier](image)

Figure 5.9: Efficient frontier of the different VIX options strategies in a mean-standard deviation framework.

The best portfolios are positioned in the upper-left side of the plot and they are characterized by a greater risk-performance profile, since they provide larger returns coupled with lower levels of volatility.

All the strategies cause the returns to decrease rapidly. The volatility, instead, is initially reduced with the allocation of small quantities of the

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13The BP1 is chosen among the others for the clarity of the results.
portfolio in VIX calls and enlarged by a greater allocation. The less risky portfolio is obtained by investing 1.28% in ATM VIX calls and 98.72% in stocks: the corresponding standard deviation is 11.13% and the mean return is 1.79%. Besides this, the lowest volatility of the strategies including 10% OTM calls is reached with a 0.93% allocation, and leads to a negative return of -0.94%. Moreover, we obtain the lowest risk with 20% OTM options by investing in them the 0.83% of the portfolio while achieving a volatility of 11.49%, accompanied by negative correspondent mean of -0.48%.

Besides this, the returns are cut dramatically. The strategies erase all the gains just by allocating 1.28% (ATM), 0.93% (10% OTM) and 0.83% (20% OTM) of the portfolio in VIX calls, and thus confirming the fact that out-of-the-money VIX options cause more extreme payoffs.

As we already discussed in previous Sections, the risk-adjusted performance of the strategies deteriorate that of the original portfolio, both in a mean-standard deviation and in a mean-downside deviation. This is testified by the lower values of both the Sharpe and Sortino ratio, respectively. Given the different curvature, the deeper the moneyness, the faster is deteriorated the risk-return profile of the portfolio. This means that, for a given level of standard deviation, the returns are lower with deeply out-of-the-money VIX calls. To put it in other words, more risk has to be taken in order to reach the same level of return.

Given the magnitude of the possible losses, these results show how much the portfolio’s risk and return is sensitive to very small allocations in VIX options and give us an insight about the sizeable leverage of these instruments. As a matter of fact, by investing just the 2% of the portfolio in VIX calls, we obtain an average annualized return of -4.50% (ATM), -11.32% (10% OTM) and -12.56% (20% OTM). Instead, if one would have invested in a portfolio composed totally by stocks it would have achieved a return of 8%, on yearly basis.

This analysis confirmed the greater benefits that ATM VIX calls provide to the portfolio, if compared to 10% OTM and 20% OTM VIX call options. During bullish markets, it seems that a diversification effect may be reached by investing a quantity in the order of 1% of the portfolio in VIX options, especially ATM calls. On the one hand, such a strategy decreases the risk but, on the other hand, it dangerously sacrifices the returns, eventually leading to large losses.
Conclusions

Throughout the thesis, we have presented the main steps of volatility modeling in financial markets. We have begun with the introduction of model-base approaches and then focused on model-free approaches, coupled with the description of the advantages and disadvantages of using such techniques. The central interest of this work has been on the VIX index and the VIX derivatives. In particular, we have presented the theoretical foundation beyond the calculation methodology of the VIX and the interaction between the VIX derivatives and the financial markets. Finally, we investigated the diversification effect of strategies investing in VIX call options, which aim is to mitigate the risk of traditional portfolios.

The complexity of finding a superior volatility model has been overtaken by financial markets with the development of volatility indexes, such as the renowned VIX, based on model-free approaches. The theoretical foundation of these methodologies is solid, nonetheless some practical issues arise when dealing with the liquidity of options markets. Unlike hard sciences, volatility modeling, and more in general financial modeling, will be always accompanied by some degree of flaw, since they try to model human behavior, which can not be modeled flawlessly. As Derman (2013) points out: “All models sweep dirt under the rug. A good model makes the absence of dirt visible”. With this regard, we believe that the model-free implied volatility is a clear and robust model to measure the volatility in financial markets. Clear, since it establishes a fair relationship between a set of conveniently selected call and put prices and the implied volatility. Even tough the VIX is usually referred in financial media as the gauge of volatility, the index is far from perfect, but it is robust, because an intelligent investor can recognize its omissions and adjust them.

The core of this thesis is represented by an empirical analysis which assesses the potential diversification effects made possible by the inclusion of VIX call options in a portfolio of traditional and alternative assets during markets characterized by a discontinuous growth. This analysis confirms only partially the results achieved by Szado (2009). During the financial crisis of 2008, he demonstrated that rebalancing strategies investing in at-the-money VIX calls provided an effective protection from the markets downturns, by simultaneously decreasing the risk and cutting the losses. Instead, out-of-the-money VIX calls improved the portfolio’s negative returns but coupled with a remarkable growth in the portfolio’s volatility.
Our work has detected that, during bullish markets, the appreciation of VIX options generated by spikes in the VIX index has not always been able to offset the drops occurred in the markets. In particular, we suggest that sudden volatility events are more likely to induce the needed option’s appreciation, while expected volatility events, such as political elections, are discounted in advance by the market through the price of VIX futures. As a matter of fact, the VIX options’ underlying asset is not exactly the VIX but indeed the forward price of the index.

Both ATM and OTM VIX call options helped to reduce the risk, but also cut sharply the portfolio’s gains. The sensitivity analysis also confirmed that OTM options heavily amplify the potential of the strategy, both in a positive and in a negative sense. By considering the preference of the investor for portfolios characterized by low kurtosis and positive skewness, we found that allocating a modest percentage of the portfolio in VIX calls may generate less negatively skewed and less fat-tailed returns distributions. Nonetheless, this enhancement does not translate in relevant benefits on the overall risk profile of the strategy.

The impact caused by the addition of VIX calls options to a portfolio composed by a set of traditional and alternative assets is sizeable, thus a drastic allocation in VIX options is not recommended. The cost of implementing such option strategies during bullish markets has largely overtaken the benefits obtained in terms of risk mitigation. Clearly, the purpose of this strategies is that to reach an efficient diversification effect both during rising and declining markets thus further efforts must be taken in order to improve their profitability and risk. In particular, we may suggest the appeal to other financial instruments especially when volatility patterns are easily predictable.

In conclusion, we highlight that the intricate relationship existing between the VIX options and the financial markets, coupled with their extreme leverage, may cause large losses to a naïve investor. Therefore, it is strikingly important to investigate the behavior of the VIX derivatives in order to detect and recognize both their potential strengths and, mainly, their drawbacks.
Appendix A. Heston option pricing formula

Defining \( F(t, T) \) as the forward price of the underlying asset and

\[ x_t = \ln \left( \frac{F(t, T)}{K} \right), \quad (5.7) \]

we have for \( j = 1, 2 \)

\[ P_j(x_t, v_t; x_T, lnK) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re \left( \frac{e^{-\phi lnK} f_j(\phi; t, x, v)}{i\phi} \right) d\phi \quad (5.8) \]

\[ f_j(\phi; v_t, x_t) = e^{[C_j(\tau, \phi) + D_j(\tau, \phi)v_t + i\phi x_1]} \]

and

\[ C_j(\tau, \phi) = ri\phi\tau + \frac{a}{\sigma^2} \left[ (b_j - \rho\sigma i\phi + d_j)\tau - 2\ln \left( \frac{1 - g_j e^{d_j\tau}}{1 - g_j} \right) \right], \]

\[ D_j(\tau, \phi) = \frac{b_j - \rho\sigma i\phi + d_j}{\sigma^2} \left( \frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}} \right), \]

where

\[ g_j = \frac{b_j - \rho\sigma i\phi + d_j}{b_j - \rho\sigma i\phi - d_j}, \]

\[ d_j = \sqrt{(b_j - \rho\sigma i\phi)^2 - \sigma^2(2iu_j\phi - \phi^2)}, \]

\[ u_1 = \frac{1}{2}, \quad u_2 = -\frac{1}{2}, \quad a = k\theta, \quad b_1 = k + \lambda - \rho\sigma, \quad b_2 = k + \lambda, \quad \tau^2 = -1. \]

Substituting the values found in (5.8) into (1.34), an exact closed-form solution will be obtained. Its integration could be computed only by numerical approximation. As a consequence, the price of a European option is in reality approximated.
Appendix B. Routine for the VIX calculation

The aim of this routine is to provide a procedure that allows to calculate rapidly the value of the VIX index starting from a table of options prices downloaded from the CBOE website. In here, we follow closely the White paper (2015). This Appendix will focus on the computational aspects, for a more detailed explanation please refer to the original White paper.

The starting point of our routine is an SPX options table that can be downloaded for free from the CBOE website\textsuperscript{14}. It contains all the options that are quoted at that moment on the SPX, for all the available time horizons, that are not only “standard” options that expire on the 3rd Friday of the month but also “standard” weekly options, that expire every Friday of the month except the 3rd, and newly developed weekly options that expire on Mondays and Wednesdays.

The official procedure for the calculation of the index involves only “standard” weeklys and “standard” options. The CBOE VIX index proposes to replicate the volatility over a time horizon of one month. Since we must consider the options that expire closer before and after these 30 days horizon, the “time accuracy” is 7 days. For example, on the 6th October 2016, the VIX will be calculated using SPX options expiring 28 days after, on the 4th November (called “near-term” options), and 35 days after, on the 11th November (called “next-term” options). So, there is a kind of “inaccuracy” of 7 days, 2 before and 5 after.

In our procedure, we embrace also newly developed weeklys, that reduce this effect and decreases the interval of interpolation from 7 days to only 2 days.

The downloaded table must be arranged to be read by the *MatLab* software. The options are divided in put and calls, listed by strike price and then grouped by date of expiration. We implement a *macro* in *Visual Basic* language that allows us to format correctly the table and extract all the values that are not explicitly indicated in the original table. In this way, we end up with a table summing up all the important values needed for our computations, which are, for every couple of calls and put with same strike price: the bid and ask prices, the time and date of expiration and the indication of the weekly type. In addition, we obtain the time and date of download, the value of the SPX at that moment and the number of rows and columns of the final table.

Finally, the *MatLab* routine elaborates the table and the final result is the real-time value of the VIX.

To conclude, we summarize the operations to execute in order to get the final value of the VIX index, which are:

- go to http://www.cboe.com/delayedquote/quotetabledownload.aspx, digit “spx” in the research bar and then push the “download” button;

- start Microsoft Excel and open the downloaded file “quotedata” by selecting the proper folder and from the dropdown menu “all the files”;

- a window will automatically appear to format the data: select “delimited” then “next”, de-select “tab” and select “comma”, then “next” again, finally push “end”;

- save the file as an Excel worksheet with the name “options_file”;

- open the file “options_macro” that contains the VBA code to correctly format the table;

- return on the previous file, open the macro window, select “macro_option” and then push “run”: you will end with a table containing all the options strikes, type, price and time of expiration;

- save the Excel file and close it;

- open Matlab and run the script “vix_calc”: the result will be the value of the VIX index at the time of download.
The Visual Basic and the MatLab code are listed below.

**Visual Basic code**

```vba
Sub macro_option()
    'declare variables
    Dim i, j As Integer
    Dim year As Integer
    Dim month As String

    'eliminates double options, recognised by "E"
    Range("a1").Select
    Do Until ActiveCell.Value = ""
        stringa = Right(ActiveCell.Value, 2)
        If stringa = "E" Then
            row = ActiveCell.row
            Rows(row).Select
            Selection.Delete Shift:=xlUp
        End If
    ActiveCell.Offset(1, 0).Select
    Loop

    ' transform the text in numbers
    For Each WS In Sheets
        On Error Resume Next
        For Each r In WS.UsedRange.SpecialCells(xlCellTypeConstants)
            If IsNumeric(r) Then r.Value = Val(r.Value)
        Next
        Next

    'select the text columns and substitute the dot with the comma
    Range("a10000:b4410000").Select
    Selection.Replace What:="." , Replacement:="", LookAt:=xlPart, _
    SearchOrder:=xlByRows, MatchCase:=False, SearchFormat:=True, _
    ReplaceFormat:=False

    'write the headers in the proper cells
    Range("a1").Select
    ActiveCell.FormulaR1C1 = "Hour"
    Range("a2").Select
    ActiveCell.FormulaR1C1 = "Date"
    Range("a1").Select
    ActiveCell.FormulaR1C1 = "Rows"
    Range("a2").Select
    ActiveCell.FormulaR1C1 = "Columns"
    Range("a3").Select
    ActiveCell.FormulaR1C1 = "Day"
    Range("a4").Select
    ActiveCell.FormulaR1C1 = "Month"
    Range("a5").Select
    ActiveCell.FormulaR1C1 = "Year"
    Range("a6").Select
    ActiveCell.FormulaR1C1 = "strike"
    Range("a7").Select
    ActiveCell.FormulaR1C1 = "weekly"

    'extract the day from the code with the function mid(stringa,strai)
    'transform in number with numbervalue
```
Range("ol"), Select
ActiveCell.FormulaR1C1 = "=IF(LEN([R[1-14]]<30,NUMBERVALUE(MID([R[1-14]],23,2))),
IF(LEN([R[1-14]])<30,NUMBERVALUE(MID([R[1-14]],23,2))),
IF(LEN([R[1-14]])<28,NUMBERVALUE(MID([R[1-14]],22,2))),
IF(LEN([R[1-14]])<27,NUMBERVALUE(MID([R[1-14]],21,2))))"
Selection.AutoFill Destination:=Range("O4:O100000")
'extract the month e transform in numbers
Range("p1"), Select
ActiveCell.FormulaR1C1 = "=IF(LEN([R[1-15]])<30,(MID([R[1-15]],25,1)),IF(LEN([R[1-15]])<29,(MID([R[1-15]],24,1)),
IF(LEN([R[1-15]])<28,(MID([R[1-15]],24,1)),IF(LEN([R[1-15]])<27,(MID([R[1-15]],23,1))))))"
Selection.AutoFill Destination:=Range("p1:p100000")
Range("p2"), Select
Do Until ActiveCell.Offset(0, -1) = ""
    month = ActiveCell.Value
    If month = "A" Then
        ActiveCell.Value = 1
    ElseIf month = "B" Then
        ActiveCell.Value = 2
    ElseIf month = "C" Then
        ActiveCell.Value = 3
    ElseIf month = "D" Then
        ActiveCell.Value = 4
    ElseIf month = "E" Then
        ActiveCell.Value = 5
    ElseIf month = "F" Then
        ActiveCell.Value = 6
    ElseIf month = "G" Then
        ActiveCell.Value = 7
    ElseIf month = "H" Then
        ActiveCell.Value = 8
    ElseIf month = "I" Then
        ActiveCell.Value = 9
    ElseIf month = "J" Then
        ActiveCell.Value = 10
    ElseIf month = "K" Then
        ActiveCell.Value = 11
    Else
        ActiveCell.Value = 12
    End If
    ActiveCell.Offset(1, 0).Select
Loop
'extract the year
Range("q2"), Select
ActiveCell.FormulaR1C1 = "=NUMBERVALUE(MID([R[1-16]],8,4))"
Range("q1"), Select
ActiveCell.FormulaR1C1 = "=2000+IF(LEN([R[1-16]])<30,NUMBERVALUE(MID([R[1-16]],21,2)),
IF(LEN([R[1-16]])<29,NUMBERVALUE(MID([R[1-16]],20,2)),IF(LEN([R[1-16]])<28,NUMBERVALUE(MID([R[1-16]],19,2))),
IF(LEN([R[1-16]])<27,NUMBERVALUE(MID([R[1-16]],18,2))))"
Selection.AutoFill Destination:=Range("q1:q100000")
'extract the strike price with the function mid(string,offset)
.Range("r-4"), Select
ActiveCell.FormulaR1C1 = "=numbervalue(IF(LEN([R[1-17]])<30,MID([R[1-17]],8,7),MID([R[1-17]],8,6))"
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Selection.AutoComplete Destination:=Range("xl:10000")
'extract the download time and date
Range("o1"), Select
ActiveCell.FormulaR1C1 = "=numbervalue(mid(r1|c[-14].15,2))"
Range("o1"), Select
ActiveCell.FormulaR1C1 = "=numbervalue(mid(r1|c[-15],18,2))"
Range("o2"), Select
ActiveCell.FormulaR1C1 = "=numbervalue(MID(rc[-14],5,2))"
Range("o2"), Select
ActiveCell.FormulaR1C1 = "=if(mid(rc[-15],1,3)=""Jan"",1,if(mid(rc[-15],1,3)=""Feb",2,
if(mid(rc[-15],1,3)=""Mar",3,if(mid(rc[-15],1,3)=""Apr",4,if(mid(rc[-15],1,3)=""May",5,
if(mid(rc[-15],1,3)=""Jun",6,if(mid(rc[-15],1,3)=""Jul",7,if(mid(rc[-15],1,3)=""Aug",8,
if(mid(rc[-15],1,3)=""Sep",9,if(mid(rc[-15],1,3)=""Oct",10,if(mid(rc[-15],1,3)=""Nov",11,
if(mid(rc[-15],1,3)=""Dec",12,0))))))))
'decide whether the option is weekly or not (different time of expiration)
'weekly marker=insert 1 if the option is weekly, 0 in the opposite case
Range("s4"), Select
ActiveCell.FormulaR1C1 = "=IF/OR(MID(RC[18],20,1)=""W",MID(RC[18],19,1)=""W""),1,0"
Selection.AutoComplete Destination:=Range("s4a10000")
'count the final rows and the columns of the table
Range("l1"), Select
ActiveCell.FormulaR1C1 = "=numbervalue(COUNTA(R[2|C[-8]|R[9999]|C[8]))"
i = ActiveCell.Value
Range("l2"), Select
j = ActiveCell.Value
'format column and put titles in bold
Columns("A:"), Select
Columns("A:"), EntireColumn.AutoFit
Range("A1:3") Select
Selection.Font.Bold = True
'delete the excess values
Range(Cells(i - 3, j - 4), Cells(i + 3, j)).Select
Range(Selection, Selection.End(xDown)).Select
Selection.ClearContents
Range("a11") Select
End Sub

Matlab code

%% load the data
filename = 'Options_file.xla';
save = xlsread(filename, 1, 'B1'); %sp index level
sp_hours = xlsread(filename, 1, 'O1'); %date and hour of the download
sp_min = xlsread(filename, 1, 'P1');
sp_day = xlsread(filename, 1, 'O2');
sp_month = xlsread(filename, 1, 'P2');
sp_year = xlsread(filename, 1, 'Q2');
rows = xlsread(filename, 1, 'I2')
cols = xlsread(filename, 1, 'I1')
table = xlsread(filename, 1)
op_table = table(1:rows + 2, 1:cols - 1); %options table (except 1st column)
r_near = 0; %risk free interest rates

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r_next=r_near;
% derive all the times to expiration in minutes
mph=60; %minutes per hour
mpd=24*mph; %minutes per day
mpy=365*mpd; %minutes per year(365 days)
dpm=365/12; %days per month
mpm=mpd*dpm; %minutes per month
term_time=30*mpd; %term minutes (30 days)
time=NaN(length(op_tab),3);
exp_hour=8; %hour of expiration
exp_min=30; %minutes of expiration
deeriv=0; %deriv
base_year=sp_year-1; %base year for calculation (in 2016 & 2015)
down_min=sp_min+sp_hour*mpy+sp_day*mpy+sp_month*mpm+(sp_year-base_year)*mpy;
%download moment in minutes from 2015
a=1;
for n=1:length(op_tab)
time(a,1)=exp_min+exp_hour*mpy+op_tab(a,14)*mpy+op_tab(a,15)*mpm+(op_tab(a,16)-base_year)*mpy; %expiration moment
time(a,2)=time(a,1)-down_min; %difference in min
time(a,3)=time(a,2)-term_time; %difference with 30 days minutes
end
% find the near term and next term options
c=1;
b=time(1,3);
while b<0
b=time(c,3);
c=c+1;
end
% construct a (semifinal) table for near term and next term options
nec_next=c-1; %position of the first next term op
mar_next=time(nec_next,3); %option 'marker'
dim_next=sum(time(:,3))==mar_next; %counter
tte_next=time(nec_next,2); %time to expiration in minutes
week_next=op_tab(nec_next,18); %1 if the options are weeklys, 0 if not
exp_next=(time(nec_next,2)+week_next*(6*mpy+30))/mpy; %expiration in years (weeklys expire 6/30 months later)
dim_near=sum(time(:,3))==mar_near; %counter
nec_near=nec_next-dim_near; %position of the first near term option
tte_near=time(nec_near,2); %time to expiration in minutes
week_near=op_tab(nec_near,18);
exp_near=(time(nec_next,2)+week_near*(6*mpy+30))/mpy; %expiration in years
max_tab=max([dim_next dim_near]); %dimension of the bigger table
sf_tab=NaN(max_tab,10); %preallocate space (semifinal table)
e=1;
%insert only the next term options and the features we need
while time(nec_next,3)==mar_next
sf_tab(e,1)=op_tab(nec_next,17); %strike price
sf_tab(e,2)=op_tab(nec_next,13); %bid call
sf_tab(e,3)=op_tab(nec_next,4); %ask call
sf_tab(e,4)=op_tab(nec_next,10); %bid put
sf_tab(e,5)=op_tab(nec_next,11); %call put
e=e+1;
end
nec_next=nec_next+1;
end
% insert only the near term options and the features we need
e=1;
while time(pce_near,3)==mar_near
    sf_tab(e,6)=op_tab(pce_near,17); %strike price
    sf_tab(e,7)=op_tab(pce_near,3); %bid call
    sf_tab(e,8)=op_tab(pce_near,4); %ask call
    sf_tab(e,9)=op_tab(pce_near,10); %bid put
    sf_tab(e,10)=op_tab(pce_near,11); %bid put
    e=e+1;
pce_near=pce_near+1;
end

% construct a (final) table with aligned strike prices
x=1; %index of the final table
y=1; %index of the left column (col 1)
z=1; %index of the right column (col 6)
tab=zeros(1,25); % preallocate columns
while y<=max_tab & x<=max_tab
    if sf_tab(y,1)<=sf_tab(x,6)
        tab(x,1)=sf_tab(x,6);
        tab(x,11)=sf_tab(x,6);
        tab(x,12)=sf_tab(x,7);
        tab(x,13)=sf_tab(x,8);
        tab(x,14)=sf_tab(x,9);
        tab(x,15)=sf_tab(x,10);
        z=z+1;
    elseif sf_tab(y,1)<sf_tab(x,6)
        tab(x,1)=sf_tab(y,1);
        tab(x,2)=sf_tab(y,2);
        tab(x,3)=sf_tab(y,3);
        tab(x,4)=sf_tab(y,4);
        tab(x,5)=sf_tab(y,5);
        tab(x,11)=sf_tab(y,1);
        y=y+1;
    else
        tab(x,11)=sf_tab(x,6);
        tab(x,12)=sf_tab(x,7);
        tab(x,13)=sf_tab(x,8);
        tab(x,14)=sf_tab(x,9);
        tab(x,15)=sf_tab(x,10);
        tab(x,1)=sf_tab(y,1);
        tab(x,2)=sf_tab(y,2);
        tab(x,3)=sf_tab(y,3);
        tab(x,4)=sf_tab(y,4);
        tab(x,5)=sf_tab(y,5);
        x=x+1;
        y=y+1;
    end
end

% find the forward spx level
% near term options
    tab(:,6)=(tab(:,2)+tab(:,3))/2; %call (avg bid/ask price)
    tab(:,7)=(tab(:,4)+tab(:,5))/2; %put
% next term options
    tab(:,16)=(tab(:,12)+tab(:,13))/2; %call (avg bid/ask price)
    tab(:,17)=(tab(:,14)+tab(:,15))/2; %call (avg bid/ask price)
% absolute difference, find the minimum, extract and calculate F

```
tab(:,8)=abs(tab(:,6)-tab(:,7));
tab(:,18)=abs(tab(:,16)-tab(:,17));
tab(tab==0)=NaN; % convert the zeros in NaNs
min_near=find(tab(:,8)==min(tab(:,8))); % find the position
min_next=find(tab(:,18)==min(tab(:,18)));
```

```
f_near=tab(min_near,1)+exp(r_near*exp_near)*(tab(min_near,6)-tab(min_near,7)); % forward
f_next=tab(min_next,11)+exp(r_next*exp_next)*(tab(min_next,16)-tab(min_next,17));
k_near=tab(find(tab(:,1)<f_near,1,'last'),1); % strike below forward
k_next=tab(find(tab(:,11)<f_next,1,'last'),11);
```

```
% % select the options to include (1=include, 0 or NaN =don’t include)
% column 9
% in the money call options (near)
g=min_near;
while g<length(tab) && isnan(tab(g,2))=0 || isnan(tab(g+1,2))=0
  if isnan(tab(g,2))=0
    tab(g,9)=1;
  else
    tab(g,9)=0;
  end
  g=g+1;
end
% column 10
% out of the money put options (near)
f=min_near;
while f<=0 && isnan(tab(f,4))=0 || isnan(tab(f+1,4))=0
  if isnan(tab(f,4))=0
    tab(f,10)=1;
  else
    tab(f,10)=0;
  end
  f=f+1;
end
avg_near= (tab(min_near,6)+tab(min_near,7))/2; %near term options put call average
% column 19
% in the money call options (next)
i=min_next;
while i<length(tab) && isnan(tab(i,12))=0 || isnan(tab(i+1,12))=0
  if isnan(tab(i,12))=0
    tab(i,19)=1;
  else
    tab(i,19)=0;
  end
  i=i+1;
end
% column 20
% out of the money put options (next)
h=min_next;
while h<=0 && isnan(tab(h,14))=0 || isnan(tab(h+1,14))=0
  if isnan(tab(h,14))=0
    tab(h,20)=1;
  else
    tab(h,20)=0;
  end
  h=h+1;
end
```

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end
avg_next = (tab(min_near,16)+tab(min_near,17))/2;  % near term options put call average

% calculate delta k
for i=2:length(tab)-1
tab(i,21)=tab(2,11)+tab(i,11);
tab(length(tab),21)=tab(length(tab),11)-tab((length(tab)-1),11);
tab(1,21)=tab(1+1,11)-tab(l-1,11))/2;
end

% calculate mid-quote price
% near term (column 22)
n=1;
for n=1:length(tab)
if tab(n,9)==1 & & tab(n,10)==1
tab(n,22)=(tab(n,6)+tab(n,7))/2;
else
if tab(n,9)==1
    tab(n,22)=tab(n,6);
else
    tab(n,10)==1
    tab(n,22)=tab(n,7);
else
    tab(n,22)=0;
end
end

% next term (column 23)
o=1;
for o=1:length(tab)
if tab(o,19)==1 & & tab(o,20)==1
tab(o,23)=(tab(o,16)+tab(o,17))/2;
else
if tab(o,19)==1
    tab(o,23)=tab(o,16);
else
    tab(o,20)==1
    tab(o,23)=tab(o,17);
else
    tab(o,23)=0;
end
end

% calculate sigma square
for m=1:length(tab)
tab(m,24)=[(tab(m,21))+(tab(m,1)^2)*exp(r_near*exp_near)*tab(m,22)];  % near term single contribution
end
fp_next=min(tab(:,24),omax)*2/exp_near;  % first part of the formula for near
sp_near=1/exp_near*exp_near;  % second part
sp_near=fp_next*sp_near;  % sigma square
fp_next=min(tab(:,25),omax)*2*exp_next;  % next
sp_next=exp_next*exp_next;  % sigma square
sp_next=fp_next*sp_next;

% interpolate the two maturities and compute VIX value
weight_1=([exp_next*exp_near]*tab(:,1)+exp_next*exp_near*tab(:,2))*mpy/mpm;
weight_2=([exp_near*exp_near]*tab(:,1)+exp_near*exp_near*tab(:,2))*mpy/mpm;

% display the real-time VIX
final=sprintf('At %d:%d of the %d/%d/%d the value of the VIX is %4f, sp_hour,sp_min,sp_day,sp_month,sp_year,vix);
disp(final)
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